On the Application of Mellin Transforms in the Theory of Option Pricing

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List of Papers

This thesis is based on the following four papers:

I Frontczak R., and R. Schöbel (2008): Pricing American Options with Mellin Transforms, Tübinger Diskussionsbeitrag Nr. 319, submitted for publication.

II Frontczak R., and R. Schöbel (2010): On modified Mellin transforms, Gauss-Laguerre quadrature, and the valuation of American call options, *Journal of Computational and Applied Mathematics, 234 (5), pp. 1559-1571.*

III Frontczak R. (2010): Valuing options in Heston's stochastic volatility model: Another analytical approach. An early draft of the paper appeared as Tübinger Diskussionsbeitrag Nr. 326, (2009). Submitted for publication.

IV Frontczak R. (2010): Simple Analytical Approximations for the Critical Stock Price of American Options, Working Paper, University of Tuebingen.

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Chapter 1

Introduction

1.1 Preliminaries

This thesis is concerned with financial mathematics in continuous time. It is devoted to the study of problems from the theory of option pricing. Options are financial instruments that are defined in terms of other underlying quantities such as stocks, indices, currencies, interest rates or volatilities. Option prices are usually determined as discounted expected values of the underlying variables. These expected values, however, solve parabolic partial differential equations. In this thesis, we study the applicability of the Mellin integral transform to solve these equations.

An option is an derivative security that grants its holder the right, but not the obligation, to buy or to sell the underlying asset, at or before some maturity date T, for a prespecified price X, called the strike or the exercise price. The act of making this transaction is referred to as exercising the option. The price, also called the options' premium, will generally be denoted by F. A call option gives its holder the right to buy the underlying asset, whereas the put option gives the right to sell. For a call option, the payoff at maturity is $max(S_T - X, 0) = (S_T - X)^+$, and for a put option the exercise payoff becomes $max(X - S_T, 0) = (X - S_T)^+$. Here and in what follows, $S = S_t$

denotes the price of the underlaying asset, and subscripts are used to place emphasis on the evolution of the underlying asset process through time. Due to the time dependency we will write for the price of an option $F = F(S, t)$ or $F = F(S, \tau)$ where $\tau = T - t$ is the remaining time to maturity.

Options which can only be exercised at the maturity date $(t = T)$ are called European. In contrast, American options give the holder the right to exercise the contract at any time until or at the option's exercise date $t \leq T$. Theoretically, the number of possible exercise dates offered by American-style derivatives tends to infinity. Similarly, a contract granting only a finite number of exercise dates between the starting point of the contract at time t and maturity T is called Bermudian. The terms "European", "American", and "Bermudian" describe different exercise rights and are not geographical classifications[1]. The simple additional feature of early exercise makes the valuation of American-style contracts substantially more complicated. Furthermore, many American options are written on assets that pay dividends, either at discrete times or continuously. The presence of dividends complicates the analysis of American options further. Since there are more exercise opportunities in an American option than in the European counterpart, it is obvious that the price of an American option will be greater than or equal to that of an European option. The prices coincide only in special cases. The American option pricing problem is an exciting and challenging issue in mathematical finance, and therefore an active field of research.

Options can also be separated into the following two main groups:

- Standard options

- Exotic options.

Standard options, or plain vanilla options, are subject to certain regularity and standardization conditions. They are actively traded on organized

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[1]Other option types with misleading names are for instance Asian, Russian, Israeli, or Parisian options.

exchanges. The Chicago Board of Options Exchange (CBOE) started to trade calls in an organized and standardized manner in 1973. The trading with put options started four years later. Since then, the growth of options (derivatives) has been explosive. They are now traded in huge volumes on all major world exchanges like the Chicago Board of Trade (CBOT), the London International Financial Futures Exchange (LIFFE), and the EUREX, which was created in 1998 with the merger of Deutsche Terminbörse (DTB) and the Swiss Options and Financial Futures Exchange (SOFFEX). Exotic options have more complicated payoffs. They were created to fill the needs of various types of investors. They include path-dependent options, such as barrier options, Asian options, and lookback options, or multi-asset options, such as baskets. All these products, as well as even more exotic types, are mostly traded in the over-the-counter market. Leading players are commercial and investment banks, such as Goldman Sachs, Merrill Lynch, Citibank or Deutsche Bank. The more complicated payoff, however, introduces much greater complexity to the valuation problem, thus demanding a sophisticated mathematical machinery.

1.2 Motivation and Structure

Analytical pricing formulae for a wide class of standard and exotic options are often derived by solving partial differential equations. These backward-in-time equations are of parabolic type and must be solved with payoff-specific boundary conditions. Although a solution can be derived straightforwardly in some cases, many contracts have corresponding partial differential equations that are too complex to allow for a standard solution. Advanced mathematics is needed to provide a solution or an accurate approximation of the solution. Classical examples are American options and European options in stochastic volatility and/or stochastic interest rate models. The first class of options has partial differential equations of free-boundary type, whereas for

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the second class the resulting equations become two or higher dimensional depending on the number of state variables[2]. In both cases, an application of integral transforms facilitates the analysis since the use of a specific transform reduces the complexity by reducing the dimensionality inherent in the valuation problem.

Although the history of integral transforms can be traced back to the 18th century to the works of d'Alembert and Euler, their permanent use in financial economics began two decades ago with the articles of Stein and Stein (1991) and Heston (1993), where Fourier transforms were used for an analytical valuation of European options on stocks with stochastic volatility[3]. It quickly became clear how powerful the new tool was and it became standard in the theory of option pricing. In his survey article, Carr (2003) lists 76 articles applying integral transforms to option pricing, most of them focusing on Laplace and Fourier transforms. Up to the current date the number of applications has certainly increased.

Besides Fourier and Laplace transforms, there are other interesting integral transforms used in theoretical and applied mathematics: Mellin transforms, Hankel transforms, Hilbert transforms, Stieltjes transforms or finite sine and cosine transforms, among others. Their importance in applied sciences comes from the fact that they provide powerful tools for solving initial value and initial-boundary value problems for differential and integral equations arising in applied mathematics, physics and engineering. Specifically, the Mellin integral transform gained great popularity in complex analysis and analytic number theory for its applications to problems related to the Gamma function, the Riemann zeta function and other Dirichlet series. The transform is also commonly used in applications to summation of infinite series. The key conceptual difference between the Fourier and the Mellin transform approach

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[2]For jump diffusions the resulting partial differential equations become partial-integro-differential equations.

[3]Sporadic application of Fourier and Laplace transforms in financial contexts has been done by McKean (1965), Black and Scholes (1973) and Buser (1986).

is, that Fourier transforms usually exist in horizontal strips of the complex plane whereas Mellin transforms are defined in vertical strips.

The purpose of this thesis is the study of the applicability of Mellin transforms in the field of option pricing. Motivated by the articles of Panini and Srivastav (2004) and Panini and Srivastav (2005) we have developed several extensions. The thesis is based on four papers. Paper 1 (Frontczak and Schöbel (2008)) and Paper 2 (Frontczak and Schöbel (2010)) extend the articles of Panini and Srivastav in a straightforward manner to an analytical pricing of European power options and American put and call options written on stocks with a continuous dividend yield. Paper 3 (Frontczak (2010b)) applies the new transform to a closed-form valuation of European options within the stochastic volatility model of Heston (1993). We propose an equivalent alternative solution and test its accuracy numerically. Finally, in Paper 4 (Frontczak (2010a)), simple analytical approximations for the free boundary associated with the pricing of American options are derived and compared to other approaches found in the literature.

Contents of the thesis were presented in concise form at the 11th Symposium on Finance, Banking, and Insurance in Karlsruhe (2008), the International Conference on Price, Liquidity, and Credit Risks in Konstanz (2008), and the 16th Annual Meeting of the German Finance Association (DGF) in Frankfurt (2009).

The remainder of the thesis is organized as follows. Chapter 2 introduces fundamental concepts from stochastic calculus that are frequently used in mathematical finance. We give a short introduction into the Black/Scholes and Merton model and the risk neutral valuation paradigm. In Chapter 3 we introduce the Mellin transform in one and two dimensions and review some of its functional properties. Chapters 4 and 5 are basically devoted to the pricing of American options. After presenting a detailed literature survey, we review the different formulations of the pricing problem. Thereafter we present the analytic solutions using Mellin transforms. We start with

American put options. After characterizing the price using the new integral approach, we prove the equivalence of three different types of integral equations for the American put. We also show how the Mellin-type equations may be used to derive the pricing formula for the perpetual American put. In a next step we propose a modification of the framework that is applicable for pricing any call option-like derivative. We use the modification for an analytic valuation of American call options. The analysis results in a new integral characterization of American call prices and the corresponding free boundaries. The formulas allow us to recover all the important theoretical properties of the pricing functions. For the modified transform, we show in addition how Gauss-Laguerre quadrature may be applied for numerical evaluation. Finally, we use the Mellin transform approach to derive simple analytical approximations for the free boundary associated with the pricing of American options and compare them to other approaches found in the literature. The stochastic volatility model of Heston (1993) is discussed in Chapter 6. We derive an alternative solution for the pricing problem and provide numerical tests concerning its accuracy and flexibility. Chapter 7 concludes the thesis.

Chapter 2

Foundations

This chapter deals with fundamental concepts from stochastic calculus used in continuous-time mathematical finance. We also review the no-arbitrage and the risk-neutral valuation principles in the Black/Scholes/Merton framework. Both principals are essential in the theory of pricing financial securities and are closely linked mathematically with profound financial implications. A first introduction in the one-dimensional case is given. The theoretical concepts presented in this chapter are well known and are described on a more rigorous level in several textbooks: Karatzas and Shreve (1991), Bingham and Kiesel (1998), Karatzas and Shreve (1998), Duffie (2001), Kijima (2002), Øksendal (2003), Shreve (2004), Musiela and Rutkowski (2004) or Elliott and Kopp (2005).

2.1 Stochastic Calculus

First, we introduce some notations and definitions. The stochastic setting that will be used is a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and a filtration of sub σ-algebras $\mathbb{F} = \{\mathfrak{F}_t : 0 \leq t \leq T\}$ with $\mathfrak{F}_s \subseteq \mathfrak{F}_t \subseteq \mathfrak{F}$ for $s \leq t$. Furthermore we assume that the filtration $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness, and that $\mathfrak{F}_0$ is trivial, i.e. for every $A \in \mathfrak{F}_0$ either $\mathbb{P}(A) = 0$

or $\mathbb{P}(A) = 1$. The filtration represents a family of information sets that become continuously available to any market participant as time passes[4]. A (one-dimensional) real valued stochastic process $\{X_t\} = \{X_t,\, t \in [0, T]\}$ is a family of real valued random variables. If $\{X_t\}$ is $\mathfrak{F}_t - \mathfrak{B}$-measurable for each t, where $\mathfrak{B}$ is the Borel σ-algebra, the stochastic process is said to be adapted to $\mathbb{F}$. The only source of uncertainty in the market is captured by the process $\{W_t\} = \{W_t,\, t \in [0, T]\}$, called a (one-dimensional) standard Brownian motion.

Definition 2.1.1 *A continuous-time stochastic process $\{W_t\} = \{W_t,\, t \geq 0\}$ is called a Brownian motion with drift μ, diffusion coefficient σ, and start in $x \in \mathbb{R}$, if*

1. *$W_0 = x$ a.s.*

2. *$\{W_t\} = \{W_t,\, t \geq 0\}$ has independent increments, i.e. if $0 \leq r < s \leq t < u < \infty$, then $W_u - W_t$ and $W_s - W_r$ are independent.*

3. *The increment $W_{t+s} - W_t$ is normally distributed with mean μs and variance $\sigma^2 s$, i.e.*

$$\forall t \geq 0, \forall s > 0: \quad W_{t+s} - W_t \sim N(\mu s, \sigma^2 s).$$

4. *W_t has continuous sample paths, i.e.*

$$P[\{\omega \in \Omega \mid [0, \infty) \ni t \mapsto W_t(\omega) \text{ is continuous}\}] = 1 \ \text{a.s.}$$

The process $\{W_t\} = \{W_t,\, t \geq 0\}$ is called standard, if $x = 0, \mu = 0$ and $\sigma^2 = 1$.

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[4]$\mathfrak{F}_t$ may represent different type of information. In our case, $\mathfrak{F}_t$ will represent the information one can obtain from the observed prices in financial markets up to time t.

Stochastic processes under consideration will be defined in terms of their stochastic differential equations (SDEs):

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_0 = x, \tag{2.1}$$

where μ and σ are measurable functions from $\mathbb{R} \times [0, T]$ to $\mathbb{R}$. The functions μ and σ are called drift and diffusion of the process, respectively. Sufficient conditions for a unique (path-by-path) solution are called the growth condition and the Lipschitz condition.

C1 Growth condition

There exists a constant $K > 0$ such that

$$\mu^2(x, t) + \sigma^2(x, t) \leq K(1 + x^2), \qquad (x, t) \in \mathbb{R} \times [0, T].$$

C2 Lipschitz condition

There exists a constant $L > 0$ such that

$$|\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq L|x - y|, \quad x, y \in \mathbb{R}, t \in [0, T].$$

For a proof, see Karatzas and Shreve (1991) or Øksendal (2003). A prominent process for which the Lipschitz condition is not satisfied is the CIR or Feller process

$$dX_t = \kappa(\theta - X_t)dt + \sigma X_t^\alpha dW_t, \quad X_0 = x,$$

with constant κ, θ, σ and α where $0.5 \leq \alpha < 1$. The process, however, has a well known solution. If the SDE (2.1) has a unique t-continuous and adapted solution X_t, it can be expressed in an equivalent integral form as

$$X_t = x + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s. \tag{2.2}$$

Definition 2.1.2 *The process X_t defined in (2.1) is called an Itô process, if the functions μ and σ satisfy the following conditions:*

$$\mathbb{P}\Big[\int_0^t |\mu(X_s, s)| \, ds < \infty, \ \forall t \geq 0\Big] = 1,$$

$$\mathbb{P}\Big[\int_0^t \sigma(X_s, s)^2 ds < \infty, \ \forall t \geq 0\Big] = 1.$$

The following theorem is known as Itô's formula or Itô's lemma. It represents a fundamental result in stochastic calculus and is frequently used in financial mathematics:

Theorem 2.1.3 *Let X_t be an Itô process and let $g(x,t) \in C^{2,1}(\mathbb{R} \times [0, \infty))$. Then $Y_t = g(X_t, t)$ is again an Itô process whose dynamics are given by*

$$dY_t = \frac{\partial g}{\partial t}(X_t, t)dt + \frac{\partial g}{\partial x}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2, \qquad (2.3)$$

where $(dX_t)^2 = dX_t \cdot dX_t$ is computed according to the rules $dW_t \cdot dW_t = dt$, and $dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0$.

The above theorem may be used to prove the famous Feynman-Kac-Theorem. The theorem establishes a link between (parabolic) partial differential equations and stochastic processes. Loosely speaking, the theorem allows solving PDEs by computing expectations, and vice versa. The formula is fundamental in pricing derivative securities. The idea was invented by Feynman (1948) and developed by Kac (1949).

Theorem 2.1.4 *Let $F(x,t) \in C^{2,1}(\mathbb{R}^+ \times [0,T))$ be a solution of the PDE*

$$\frac{\partial F(x,t)}{\partial t} + L_t\, F(x,t) - r(x,t)F(x,t) = 0, \qquad (2.4)$$

with

$$L_t F(x,t) = \mu(x,t)\frac{\partial F(x,t)}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 F(x,t)}{\partial x^2},$$

and the boundary condition $F(x,T) = g(x)$. Then the solution has the form

$$F(x,t) = E_t\left[g(X_T) \cdot e^{-\int_t^T r(X_u, u)du}\right], \qquad (2.5)$$

where the expectation is taken with respect to the process X_t defined in (2.1)[5].

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[5] The operator L_t is called the generator of $(X_t)_{t \in [0,T]}$. Furthermore, it is worth mentioning that a solution to the PDE may not be unique. Also, when $r(x,t) < 0$, which is not realistic from an economic point of view when r denotes the interest rate, solutions may not exist or may exist only for $T - t < \tau$, $\tau \in [0,T]$. For a more sophisticated discussion of technical conditions, see Durrett (1996).

A sketch of the proof is presented below.

Applying Itô's formula to $F(X_t, t)$ gives

$$dF(X_t, t) = \left(\frac{\partial F(X_t, t)}{\partial t} + L_t F(X_t, t)\right) dt + dM_t$$

where

$$dM_t = \frac{\partial F(X_t, t)}{\partial x} \sigma(X_t, t) \, dW_t.$$

Then, by assumption, it follows after integration that

$$F(X_T, T) = F(X_t, t) + \int_t^T r(X_s, s) F(X_s, s) \, ds + \int_t^T \frac{\partial F(X_s, s)}{\partial x} \sigma(X_s, s) \, dW_s.$$

Since this is a linear equation, it is solved as

$$F(X_T, T) = e^{\int_t^T r(X_u, u) du} \left[F(X_t, t) + \int_t^T e^{-\int_t^s r(X_u, u) du} dM_s \right].$$

The last integral is an Itô integral, thus by taking expectations and using the terminal condition we finally get

$$\begin{aligned}
F(X_t, t) &= E\left[g(X_T) \cdot e^{-\int_t^T r(X_u, u) du} \mid X_t = x \right] \\
&= E_t\left[g(X_T) \cdot e^{-\int_t^T r(X_u, u) du} \right],
\end{aligned} \tag{2.6}$$

where we have used a short-hand notation for the conditional expectation. In the case of a constant r we obtain

$$F(x, t) = e^{-r(T-t)} E_t[g(X_T)]. \tag{2.7}$$

If X_t denotes the time t price of an asset, i.e. $X_t = S_t$, then $F(S, t)$ can be viewed as the current price of a derivative, which is given as the discounted expected payoff at maturity under the probability measure $\mathbb{P}$.

2.2 The Black/Scholes/Merton Framework

This section presents the ideas underlying the no-arbitrage and the risk-neutral valuation principles in the Black/Scholes/Merton model (sometimes

also called the log-normal model). The no-arbitrage principle denotes the assumption, that there should be no possibility for a risk-free gain without initial capital. Thus, all investments with certain payoffs should have the same yield. The log-normal model assumes that the asset price $S_t, t \in [0, T]$, evolves according to the SDE:

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t, \tag{2.8}$$

with initial value $S_0 \in (0, \infty)$, and where μ is the subjective expected return on the asset, q is the dividend yield, and $\sigma > 0$ is the volatility. The dividend yield is also assumed to be positive. However, if $q < 0$ it can be regarded as a cost of carry factor. The asset price process is called a (one dimensional) geometric Brownian motion. All three parameters are assumed to be constant. The process $\{W_t\}$ is a standard Brownian motion, under the physical (or statistical) probability measure $\mathbb{P}$. The market is frictionless and continuous, i.e. there are no transaction costs, no taxes, trading takes place continuously, assets are infinite divisible, and unlimited investing is allowed at a constant risk-free rate r[6]. A straightforward application of Itô's formula leads to the following solution of (2.8):

$$S_T = S_t \exp \left(\left(\mu - q - \frac{1}{2}\sigma^2 \right)(T - t) + \sigma \left(W_T - W_t \right) \right), \tag{2.9}$$

for all $t \in [0, T]$. We are interested in the pricing of contingent claims on the underlying asset S. For a given payoff function $g : \mathbb{R}^+ \to \mathbb{R}$, the European contingent claim written on S can be regarded as a financial instrument that pays the holder $g(S_T)$ at expiry T. For the European put option we have $g(S_T) = (X - S_T)^+$, for all $S \in \mathbb{R}^+$. The corresponding European call option is treated analogously. Assuming that the current price of the security $F = F(S, t)$ is suitably differentiable, i.e. $F(S, t) \in C^{2,1}(\mathbb{R}^+ \times [0, T))$, it

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[6]Trading strategies are subject to a weak condition. To ensure a well-defined value process, the quadratic variation of a portfolio must be finite $\mathbb{P}$-a.s.

follows from a dynamic replication strategy that

$$\frac{\partial F}{\partial t} + (r - q) S_t \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S^2} - rF = 0 \qquad (2.10)$$

on $\mathbb{R}^+ \times [0, T)$. This PDE, along with the boundary conditions

$$F(S, T) = g(S_T) \qquad \text{on} \quad \mathbb{R}^+$$

$$F(0, t) = g(0)e^{-r(T-t)} \qquad \text{on} \quad [0, T) \qquad (2.11)$$

$$\lim_{S \to \infty} F(S, t) = g(\infty)e^{-r(T-t)} \qquad \text{on} \quad [0, T),$$

characterizes the price of a traded derivative on the underlying S. It is sometimes called "the fundamental valuation equation" because it applies to any (European) contingent claim, independent of its payoff structure. What changes across securities are the relevant boundary conditions (2.11). For a more comprehensive introduction to modeling derivative securities as partial differential equations, see Wilmott et al. (1993).

Since μ does not enter (2.10), one implication of the fundamental valuation equation is that two investors can agree on the fair price of a derivative without taking into consideration their individual views about the expected performance of the underlying asset. Furthermore, the equation also applies to index options, foreign currency options, and options on futures where the dividend yield q can be replaced by the average dividend yield on stocks composing the index, the foreign risk-free interest rate, and the domestic risk-free interest rate, respectively.

From a mathematical point of view, the fundamental valuation equation is a backward-in-time parabolic partial differential equation, and the terms $(r - q)S_t\frac{\partial F}{\partial S}, \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 F}{\partial S^2}$, and $-rF$ are called convection term, diffusion term and reaction term, respectively. In this sense (2.10) is a convection-diffusion PDE. In finance, the partial derivatives $\frac{\partial F}{\partial S}, \frac{\partial^2 F}{\partial S^2}$ and $\frac{\partial F}{\partial t}$ denote the delta, gamma, and theta of a derivative, respectively. They give information about different dimensions of risk in the derivative (see Hull (2006)).

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In some cases, dependent on the specific payoff structure, (2.10) with the conditions (2.11) can be solved analytically. When the payoff is specialized to $g(S_T) = (X - S_T)^+$, we can use a change of variables technique to reduce it to the heat equation, and apply standard methods to get the solution

$$P^E(S,t) = Xe^{-r(T-t)}N(-d_2(S,X,T)) - S_te^{-q(T-t)}N(-d_1(S,X,T)), \quad (2.12)$$

where $P^E(S,t)$ denotes the value of a European put option, $N(\cdot)$ denotes the cumulative standard normal distribution, and

$$d_1(S,X,T) = \frac{\ln \frac{S_t}{X} + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad (2.13)$$

$$d_2(S,X,T) = d_1(S,X,T) - \sigma\sqrt{T-t}. \quad (2.14)$$

The expression is the celebrated Black-Scholes-Merton formula for a European put option. Similar arguments lead to the European call option

$$C^E(S,t) = S_te^{-q(T-t)}N(d_1(S,X,T)) - Xe^{-r(T-t)}N(d_2(S,X,T)), \quad (2.15)$$

with $d_{1/2}(S,X,T)$ as defined in (2.13) and (2.14). The second formula can also be derived directly using the put-call-parity relationship for European options. This no-arbitrage condition relates call and put options with identical maturities and strikes, and states that

$$C^E(S,t) - P^E(S,t) = S_te^{-q(T-t)} - Xe^{-r(T-t)} \quad (2.16)$$

for all $t \in [0,T]$. The condition reflects the fact that the portfolios on both sides of the equation (2.16) have the same payoff $(S_T - X)^+ - (X - S_T)^+ = S_T - X$ at expiry. Because the terminal payoffs are identical, in order to preclude arbitrage opportunities, their prices up to maturity must be equal. The fundamental valuation equation (2.10) has the key property that it does not involve any variables that are affected by the risk preferences of market participants, i.e. it does not depend on μ. The only variables that are relevant for pricing derivatives are the risk-free interest rate r, the volatility

of the underlying asset σ, and the dividend yield q. This gives rise to the risk-neutral valuation approach. The main idea is to use a change of measure and transform the random stock price process into a martingale. The risk-neutral valuation approach is therefore also referred to as the martingale approach, and was first elaborated by Cox and Ross (1976), although the idea of risk-neutral probabilities goes back to Arrow (Arrow (1964) and Arrow (1970)). Its formalization is essentially due to Harrison and Kreps (1979) and Harrison and Pliska (1981). An excellent summary of the martingale approach and its applications in finance can be found in Bingham and Kiesel (1998), Karatzas and Shreve (1998), Duffie (2001), Musiela and Rutkowski (2004), and Elliott and Kopp (2005). For extensions and generalizations, the reader is referred to the articles of Delbaen and Schachermayer (1994), Delbaen and Schachermayer (1995), Pham and Touzi (1999), Kabanov and Stricker (2001), and Broadie and Detemple (2004).

The fundamental step in risk-neutral valuation is the conversion of asset prices into martingales. If $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space with a filtration $\mathbb{F} = \{\mathfrak{F}_t : 0 \leq t \leq T\}$ a real valued stochastic process $\{X_t\} = \{X_t, t \in [0, T]\}$ is called a $\mathbb{P}$-martingale with respect to the filtration $\mathbb{F}$, if for all $t \in [0, T]$,

- X_t is adapted to $\mathfrak{F}_t$,

- X_t is integrable, i.e. $E[|X_t|] < \infty$,

- $E[X_t|\mathfrak{F}_s] = E_s[X_t] = X_s$ $\mathbb{P}$-a.s. for all $0 \leq s \leq t$.

Similarly, if a stochastic process follows a trend and increases or decreases on average, the third condition is $E_s[X_t] \geq X_s$, or $E_s[X_t] \leq X_s$, and the process is called a sub- or supermartingale, respectively.

Converting random processes such as stock prices into martingales is accomplished by changing the drift of such processes and leaving the volatility unchanged. This can be achieved by changing the underlying probability measure without changing the outcomes themselves, thus changing the

probability measure $\mathbb{P}$ to a equivalent probability measure $\mathbb{Q}$[7]. The basis of changing form $\mathbb{P}$ to $\mathbb{Q}$ is provided by the Girsanov theorem[8]. For the one-dimensional case, the statement is the following:

Let $(\lambda_t)_{t\in[0,T]}$ be a measurable, adapted process with $\int_0^T \lambda_t^2 dt < \infty$ a.s., and define the new exponential process $(\xi_t)_{t\in[0,T]}$ by

$$\xi_t := \exp\left\{ -\int_0^t \lambda_s \, dW_s - \frac{1}{2} \int_0^t \lambda_s^2 \, ds \right\}. \tag{2.17}$$

Then ξ_t is measurable, continuous and positive. Also, since $d\xi_t = -\xi_t \lambda_t dW_t$ it follows that ξ_t is a (local) martingale. Further if λ_t suffices Novikov's condition

$$E\left(\exp\left\{ \frac{1}{2} \int_0^T \lambda_s^2 \, ds \right\} \right) < \infty,$$

then ξ_t is a continuous martingale, and we have

Theorem 2.2.1 *Let λ be as above and satisfy the Novikov condition. Let ξ be the corresponding martingale. Define the process $(W_t^*)_{t\in[0,T]}$ by*

$$W_t^* := W_t + \int_0^t \lambda_s \, ds. \tag{2.18}$$

Then under the equivalent probability measure $\mathbb{Q}$[9] defined on $(\Omega, \mathfrak{F})$ with the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T$, the new process W_t^* is a Brownian motion.

For proofs, see Dothan (1990), Karatzas and Shreve (1991), Revuz and Yor (1991), Protter (1992), and Øksendal (2003).

As a special case, consider λ_t to be constant, $\lambda_t = \lambda$ for all $t \in [0, T]$. Then

$$\xi_t = \exp\left(-\frac{1}{2}\lambda^2 t - \lambda W_t \right), \tag{2.19}$$

---

[7]The measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathfrak{F})$ are (mutually) equivalent if they have the same null sets, i.e. if, for any $A \in \mathfrak{F}$, we have $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$.

[8]Sometimes referred to as the Cameron-Martin-Girsanov theorem, depending on the degree of generality.

[9]Henceforth referred to as the equivalent martingale measure.

with initial value $\xi_0 = 1$. An application of Itô's formula shows that ξ_t is indeed a martingale, and for any $A \in \mathfrak{F}$, the equivalent martingale measure $\mathbb{Q}$ is given by $\mathbb{Q}(A) = E^{\mathbb{P}}[1_A \xi_T] = \int_A \xi_T d\mathbb{P}$.

Now turning back to our initial stock price process given by the SDE

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t,$$

with initial value $S_0 \in (0, \infty)$, and with W_t as a $\mathbb{P}$-Brownian motion, it is apparent that we can write the SDE in the following form:

$$dS_t = (r - q)S_t dt + \sigma S_t(dW_t + \lambda dt), \qquad (2.20)$$

where

$$\lambda = \frac{\mu - r}{\sigma}. \qquad (2.21)$$

λ is known as the market price of risk or the Sharpe ratio. It measures the excess return over the risk-free interest rate per unit of standard deviation, and therefore can be seen as a reward or risk premium per unit of risk.

Under the unique martingale measure $\mathbb{Q}$ on $(\Omega, \mathfrak{F})$ given by means of the Radon-Nikodym derivative

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}T - \frac{\mu - r}{\sigma}W_T \right), \qquad (2.22)$$

W_t^*, defined by $dW_t^* = dW_t + \lambda dt$, is a Brownian motion. Thus, we can write

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t^*, \qquad (2.23)$$

under $\mathbb{Q}$. The solution is given by equation (2.9) where the expected rate of return μ is replaced by the risk free interest rate r and W_t is replaced by W_t^*. Thus, under the martingale measure $\mathbb{Q}$, the expected total return on the asset equals the risk free rate, i.e.

$$E_t^{\mathbb{Q}}\left[\frac{dS_t}{S_t} + q\, dt\right] = r\, dt,$$

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where $E_t^{\mathbb{Q}}[\cdot]$ is the conditional expectation with respect to W_t^*. Thus, risk premiums can be ignored, and this, in turn, is exactly what would be required by risk neutral market participants.

We continue the section by giving a description of the stock price behavior under the new measure $\mathbb{Q}$. Doing so, it is natural to consider the discounted asset price process $S_t^* = D_t S_t$ where D_t is the discount process. This process can be interpreted as (the inverse of) the money market account[10], and is defined by

$$D_t = e^{-rt}. \tag{2.24}$$

From Itô's formula we get

$$d(D_t S_t) = D_t S_t (\mu - q - r)dt + D_t S_t \sigma dW_t.$$

The $\mathbb{Q}$-dynamics of the discounted asset price process $D_t S_t$ are:

$$d(D_t S_t) = -D_t S_t \, q \, dt + D_t S_t \sigma dW_t^*.$$

Integrating both sides yields

$$e^{-rT} S_T = D_t S_t + \int_t^T D_u S_u \sigma \, dW_u^* - \int_t^T D_u S_u q \, du \,,$$

or equivalently

$$S_t + \int_t^T e^{-r(u-t)} S_u \sigma \, dW_u^* = e^{-r(T-t)} S_T + \int_t^T e^{-r(u-t)} S_u q \, du \tag{2.25}$$

for all $t \in [0, T]$. Taking conditional expectations with respect to $\mathbb{Q}$, we see that

$$S_t = E_t^{\mathbb{Q}} \Big[e^{-r(T-t)} S_T + \int_t^T e^{-r(u-t)} S_u q \, du \Big]. \tag{2.26}$$

Under the equivalent martingale measure $\mathbb{Q}$, the asset price equals the expected value of the discounted dividends, augmented by the expected value

---

[10]In other terms, we use the money market account as a numéraire. "Historically", this was done by Harrison and Pliska (1981). Although this choice seems natural, other numéraires are possible for which the equivalent martingale measure exists.

of the discounted terminal price. Notice that the discount factor depends on the risk-free rate. Again, this implies that, under $\mathbb{Q}$, the asset is priced as if the market were risk neutral; hence the terminology risk neutral measure. Furthermore, we have

$$E_t^{\mathbb{Q}}[S_T] = S_t e^{(r-q)(T-t)} \quad \forall t \in [0, T]. \tag{2.27}$$

The quantity on the right-hand side is known as the forward price of a dividend paying stock.

The results concerning the martingale property of discounted asset prices are not limited to stocks but can be transferred to other risky claims (see Broadie and Detemple (2004) or Shreve (2004)). This essential result is known as the Fundamental Theorem of Asset Pricing. Accordingly, in case of non-constant interest rates, the price of a contingent claim $F = F(S, t)$ with the payoff function $g(S_T)$ at maturity T is given by

$$F(S, t) = E^{\mathbb{Q}}\left[\frac{D_T}{D_t} F(S, T) \,\Big|\, \mathfrak{F}_t \right] = E^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} g(S_T) \,\Big|\, \mathfrak{F}_t \right] \tag{2.28}$$

with

$$D_t = e^{-\int_0^t r_s ds}.$$

The martingale property of the underlying asset, as well as derivative prices and their representation as conditional expectations under $\mathbb{Q}$, still hold in higher-dimensional settings where there are several underlying risky assets.

Chapter 3

The Mellin Transform

In this chapter we shall give a brief introduction to Mellin transforms. Our presentation is based on Panini (2004), Chapters 2.2 and 2.3. In a first step, the introduction is restricted to one dimension. We present and prove some basic operational properties of the integral transform. Finally we extend the definition to higher dimensions. A detailed presentation of the topic including proofs and examples can be found in Titchmarsh (1986), Sneddon (1972), and Debnath and Bhatta (2007) for the one dimensional case and in Brychkov et al. (1992), Hai and Yakubovich (1992), and Reed (1944) for the two dimensional case.

3.1 Definition and Basic Properties

Robert Hjalmar Mellin (1854-1933) gave his name to the Mellin transform, although Riemann had already worked with this integral transform in his seminal paper on prime numbers in 1876. It associates to a locally Lebesgue integrable function $f(x)$ defined over positive real numbers the complex function $M(f(x), \omega)$ defined by

$$M(f(x), \omega) := \widetilde{f}(\omega) = \int_0^\infty f(x)\, x^{\omega-1}\, dx. \tag{3.1}$$

The Mellin transform variable ω is a complex number, $\omega = Re(\omega) + iIm(\omega)$, where i is the imaginary unit, and $Re(\cdot)$ and $Im(\cdot)$ denote the real and imaginary parts, respectively. The integral transform itself is defined on a vertical strip in the ω-plane, whose boundaries are determined by the asymptotic behavior of $f(x)$ as $x \to 0^+$ and $x \to \infty$. The largest strip (a, b) in which the integral converges is called the fundamental strip. The conditions $f(x) = O(x^u)$ for $x \to 0^+$ and $f(x) = O(x^v)$ for $x \to \infty$, when $u > v$, guarantee the existence of $M(f(x), \omega)$ in the strip $(-u, -v)$. Thus, the existence is granted for locally integrable functions, whose exponent in the order at 0 is strictly larger than the exponent of the order at infinity.

Conversely, if $f(x)$ is an integrable function with fundamental strip (a, b), then if c is such that $a < c < b$ and $f(c + it)$ is integrable, the equality

$$M^{-1}(\widetilde{f}(\omega)) = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(\omega)\, x^{-\omega}\, d\omega \tag{3.2}$$

holds almost everywhere. Moreover, if $f(x)$ is continuous, then the equality holds everywhere on $(0, \infty)$. Obviously, M and M^{-1} are linear integral operators.

Two important examples of Mellin transforms are presented:

If $f(x) = e^{-x}$, then

$$M(e^{-x}, \omega) = \widetilde{f}(\omega) = \int_0^\infty e^{-x} x^{\omega-1} dx = \Gamma(\omega), \quad Re(\omega) > 0,$$

so the Gamma function is defined as the Mellin transform of e^{-x}.

If $f(x) = (e^x - 1)^{-1}$, then

$$
\begin{aligned}
M\left(\frac{1}{e^x - 1}, \omega\right) = \widetilde{f}(\omega) &= \int_0^\infty \frac{1}{e^x - 1} x^{\omega-1} dx \\
&= \sum_{n=1}^\infty \int_0^\infty e^{-nx} x^{\omega-1} dx \\
&= \sum_{n=1}^\infty \frac{\Gamma(\omega)}{n^\omega} = \Gamma(\omega)\zeta(\omega), \quad Re(\omega) > 1,
\end{aligned}
$$

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where we have used that

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1 - e^{-x}},$$

and hence

$$\sum_{n=1}^{\infty} e^{-nx} = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1}.$$

The function $\zeta(\omega) = \sum_{n=1}^{\infty} \frac{1}{n^\omega}, Re(\omega) > 1$, is the famous Riemann zeta function.

Simple changes of variables in the definition of the Mellin transforms yield a whole set of transformation rules and facilitate the computations. In particular, if $f(x)$ admits the Mellin transform on the strip (a, b) and α, β are positive reals, then

$$M(x^\alpha f(x), \omega) = \widetilde{f}(\omega + \alpha) \quad \text{on} \quad (a, b),$$

follows directly from the definition. The change of variable $t = \alpha x$ gives immediately

$$M(f(\alpha x), \omega) = \alpha^{-\omega} \widetilde{f}(\omega) \quad \text{on} \quad (a, b).$$

Further relations of this kind are

$$M(f(x^\alpha), \omega) = \frac{1}{\alpha} \widetilde{f}\left(\frac{\omega}{\alpha}\right) \quad \text{on} \quad (a\alpha, b\alpha),$$

$$M(f\left(\frac{1}{x}\right), \omega) = -\widetilde{f}(-\omega) \quad \text{on} \quad (-b, -a),$$

$$M(x^\beta f(x^\alpha), \omega) = \frac{1}{\alpha} \widetilde{f}\left(\frac{\omega + \beta}{\alpha}\right) \quad \text{on} \quad (a\alpha, b\alpha),$$

$$M(f(x) \ln(x), \omega) = \frac{d}{d\omega} \widetilde{f}(\omega) \quad \text{on} \quad (a, b).$$

The Mellin transform of a derivative equals

$$M(\frac{d}{dx} f(x), \omega) = -(\omega - 1) \widetilde{f}(\omega - 1),$$

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provided $x^{\omega-1}f(x)$ vanishes as $x \to 0^+$ and as $x \to \infty$. The relation can be rewritten as

$$M\Big(x\frac{d}{dx}f(x),\omega\Big) = -\omega\widetilde{f}(\omega)$$

provided $x^{\omega}f(x)$ vanishes as $x \to 0^+$ and as $x \to \infty$. The statement is proved straightforwardly using integration by parts. Finally, the property can be extended to

$$M\Big(\frac{d^n}{dx^n}f(x),\omega\Big) = (-1)^n \frac{\Gamma(\omega)}{\Gamma(\omega-n)}\widetilde{f}(\omega-n),$$

for n a positive integer, provided that for $k = 0, 1, ..., n-1$

$$\lim_{x \to 0^+} x^{\omega-k-1}f^{(n-k-1)}(x) = \lim_{x \to \infty} x^{\omega-k-1}f^{(n-k-1)}(x) = 0.$$

The Mellin transform of an integral of $f(x)$ is given by

$$M\Big(\int_0^x f(t)dt,\omega\Big) = -\frac{1}{\omega}\widetilde{f}(\omega+1).$$

For proofs, examples and more relations of this kind, we refer to Titchmarsh (1986), Sneddon (1972) or Debnath and Bhatta (2007).

The change of variables $x = e^t$ in definition shows that the Mellin transform is closely related to the (two-sided) Laplace and Fourier transform defined by, respectively,

$$L(f(t),\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-t\omega}\,dt\,,$$

$$F(f(t),\omega) = \int_{-\infty}^{\infty} f(t)\,e^{it\omega}\,dt\,.$$

Now, making the transformation stated above shows that

$$M(f(x),\omega) = L(f(e^{-x}),\omega) = F(f(e^{-x}),-i\,\omega). \tag{3.3}$$

3.2 Convolution Theorems

Important results for Mellin transforms are the celebrated convolution theorems. Assume that $\widetilde{f}(\omega)$ and $\widetilde{g}(\omega)$ are two Mellin transforms of the functions $f(x)$ and $g(x)$, respectively. Then

$$M(f(x)g(x), \omega) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(z)\widetilde{g}(\omega - z)\, dz, \qquad (3.4)$$

$$M\left[\int_0^\infty f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}, \omega\right] = \widetilde{f}(\omega)\widetilde{g}(\omega), \qquad (3.5)$$

and

$$M\left[\int_0^\infty f(x\xi)g(\xi)d\xi, \omega\right] = \widetilde{f}(\omega)\widetilde{g}(1 - \omega). \qquad (3.6)$$

PROOF: Assume that $\widetilde{f}(\omega)$ and $\widetilde{g}(\omega - z)$ have a common strip of analyticity. Take the vertical line $Re(\omega) = c$ to lie within the common strip. Then

$$
\begin{aligned}
\int_0^\infty f(x)g(x)x^{\omega-1}\, dx &= \int_0^\infty g(x)x^{\omega-1}\, dx \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(z)\, x^{-z}\, dz \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(z)\, dz \int_0^\infty g(x)x^{\omega-z-1}\, dx \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(z)\widetilde{g}(\omega - z)\, dz.
\end{aligned}
$$

This proves the first statement. The second follows directly from the definition. We have

$$
\begin{aligned}
M\left[\int_0^\infty f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}, \omega\right] &= \int_0^\infty x^{\omega-1}dx \int_0^\infty f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi} \\
&= \int_0^\infty f(\xi)\frac{d\xi}{\xi} \int_0^\infty x^{\omega-1}g\left(\frac{x}{\xi}\right)dx \\
&= \int_0^\infty f(\xi)\frac{d\xi}{\xi} \int_0^\infty (\xi\eta)^{\omega-1}g(\eta)\xi d\eta \\
&= \int_0^\infty f(\xi)\xi^{\omega-1}d\xi \int_0^\infty \eta^{\omega-1}g(\eta)d\eta \\
&= \widetilde{f}(\omega)\widetilde{g}(\omega),
\end{aligned}
$$

where we have used the transformation $x = \eta\xi$. The last equality follows from

$$
\begin{aligned}
M\left[\int_0^\infty f(x\xi)g(\xi)d\xi, \omega\right] &= \int_0^\infty x^{\omega-1}dx \int_0^\infty f(x\xi)g(\xi)d\xi \\
&= \int_0^\infty g(\xi)d\xi \int_0^\infty \eta^{\omega-1}\xi^{-\omega+1}f(\eta)\frac{d\eta}{\xi} \\
&= \int_0^\infty \xi^{1-\omega-1}g(\xi)d\xi \int_0^\infty \eta^{\omega-1}f(\eta)d\eta \\
&= \widetilde{g}(1-\omega)\widetilde{f}(\omega),
\end{aligned}
$$

with $\eta = x\xi$. This completes the proof. $\qquad\square$

The first identity is known as Parseval's identity. Note also the special case

$$
\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{f}(z)\widetilde{g}(1-z)dz. \tag{3.7}
$$

3.3 Mellin Transforms in Higher Dimensions

For higher dimensional problems one can extend the concept of Mellin transforms to functions of several variables. For instance, the double Mellin transform of a function $f(x_1, x_2)$ is defined by

$$
M(f(x_1, x_2), \omega_1, \omega_2) := \widetilde{f}(\omega_1, \omega_2) = \int_0^\infty \int_0^\infty f(x_1, x_2)\, x_1^{\omega_1-1}x_2^{\omega_2-1}\, dx_1dx_2\,, \tag{3.8}
$$

for all functions f so that the double integral converges (see Brychkov et al. (1992), p. 194). Conversely, the inverse double Mellin transform is given by

$$
f(x_1, x_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_1-i\infty}^{c_1+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} \widetilde{f}(\omega_1, \omega_2)\, x^{-\omega_1}x^{-\omega_2}\, d\omega_1 d\omega_2\,, \tag{3.9}
$$

provided that the integral exists. Reed (1944) proves conditions for the existence. A convolution-type theorem similar to the one dimensional case

is given by

$$M^{-1}(\widetilde{f}(\omega_1, \omega_2)\widetilde{g}(\omega_1, \omega_2), x_1, x_2) = \int_0^\infty \int_0^\infty f(\xi, \eta) g\left(\frac{x_1}{\xi}, \frac{x_2}{\eta}\right) \frac{1}{\xi\eta} \, d\xi d\eta \,.$$

(3.10)

More about the double Mellin transform can be found in Brychkov et al. (1992), Hai and Yakubovich (1992), and Reed (1944).

Chapter 4

American Options

This chapter discusses the pricing of standard American options on a single underlying asset in the log-normal model. After reviewing the literature, we give brief formulations of the problem. American options are analyzed in detail in Wilmott et al. (1993), Kwok (1998), Elliott and Kopp (2005), and Detemple (2006).

The main difference between European and American options is that American options can be exercised at any time before and including expiry. Since the early exercise privilege should have a non-negative value, it is natural to expect an American option to be worth more than the corresponding European option. The additional cost for early exercise is called the "early exercise premium". However, it turns out that, under special circumstances, the early exercise premium becomes zero, indicating that it is never optimal/rational to exercise the American option prior to expiry. Merton (1973) shows that the American call option on a non-dividend-paying stock should never be exercised early. Thus prices of European and American calls on non-dividend paying stocks must be equal. This does not apply if the underlying asset pays dividends. The American put option always offers an optimal early exercise policy. The early exercise feature constitutes a "free boundary problem" (McKean (1965)) and makes the pricing and hedging of

American-style derivatives mathematically challenging. The free boundary or early exercise boundary specifies the conditions under which the American option should be exercised optimally prior to maturity. In the case of an American put option, it is the set of all critical stock prices $S^*(t)$ such that, when the stock price at time t, $S_t = S(t)$, falls below $S^*(t)$, it becomes optimal to exercise the American put at time t before maturity. Similarly, the American call should be exercised prematurely if the stock price at time t rises above some critical value $S^*(t)$. However, the optimal early exercise policy is not known ex ante and must be determined simultaneously as a part of the valuation problem.

4.1 Literature Survey

4.1.1 Standard American Options

The pricing of American-style options and more exotic products has become a challenging issue and demands sophisticated mathematical tools. Due to the complexity, different analytical and numerical treatments have been developed. Consequently, a great field of research has been created throughout the last three decades.

The extensive literature on numerical methods for American option pricing comprises finite difference and element methods, penalty methods, binomial trees and simulation techniques. Brennan and Schwartz (1978) initially proposed a finite difference scheme for the purpose of pricing American options. A proof of the convergence of the algorithm is given in Jaillet et al. (1990). The approach has been refined and extended in various ways and is still in the focus of current interest (Zhao et al. (2007), Tangman et al. (2008), Khaliq et al. (2008) and Hu et al. (2009) among others). Cox et al. (1979) used a binomial tree lattice for an accurate valuation which still enjoys great popularity. Extensions of the initial work can be found in Leisen

and Reimer (1996), Leisen (1998), Rogers and Stapleton (1998), Chang and Palmer (2007), Liang et al. (2007), and Jourdain and Zanette (2008), among others. Moreover, Monte Carlo methods, firstly introduced by Boyle (1977), were modified to solve the forward-simulation-backward-induction valuation problem and to provide accurate American option prices. Good references are the articles of Boyle et al. (1997), Broadie and Glasserman (1997), Longstaff and Schwartz (2001), Rogers (2002), Glasserman and Yu (2004a), Glasserman and Yu (2004b) or Milstein et al. (2004). The least-square Monte Carlo method of Longstaff and Schwartz (2001) turned out to have desirable properties and is widely used. Detailed analyses of the regression algorithm are given in Clement et al. (2002), Moreno and Navas (2003), Stentoft (2004a), and Stentoft (2004b). Recently, Belomestny and Milstein (2006) developed a simulation-based framework for American options that is based on consumption processes.

Besides numerical methods, one can distinguish two main categories of analytical pricing approaches: the PDE-based approach and the probabilistic approach. The different mathematical aspects lead to different but equivalent formulations of the problem. The most prominent are

- Free boundary formulation

- Integral equation formulation

- Optimal stopping formulation

- Linear complementarity formulation

- Primal-dual formulation

- Viscosity solution formulation.

Firth (2005) offers a comprehensive survey. The first method, similar to the solution of Stefan's problem from physics, expresses the price of the American option as the solution of a non-homogeneous PDE. The PDE formulation

goes back to Merton (1973), who first gave it an economic interpretation, although McKean (1965) presented a first solution of the free boundary problem in form of an integral expression. Many alternative methods based on the PDE approach were proposed for the purpose of pricing the American option and the free boundary by approximation. These methods include the works of Geske and Johnson (1984), MacMillan (1986), Barone-Adesi and Whaley (1987), Barone-Adesi and Elliott (1991), Bunch and Johnson (1992), Allegretto et al. (1995), or Ju and Zhong (1999). Of special interest is the paper of Geske and Johnson (1984) in which the authors characterize American options as compound European options. The mathematical complexity is illustrated by the American put price, which is expressed as an infinite series of multivariate normal terms. An alternative approximation of American option prices that is based on an extension of the compound option approach can be found in Lee and Paxson (2003).

Mallier and Alobaidi (2000) use Laplace transforms to value American options on dividend paying stocks. They derive an integral equation of the Fredholm-type specifying the optimal exercise boundary depending on the relationship between the risk-free interest rate and the dividend yield. Based on the same integral transform, Zhu (2006a) presents another analytical approximation of the free boundary and the price of an American put option. In a different publication (Zhu (2006b)), he derives a closed-form analytical solution for the non-homogeneous PDE of an American put option on a non-dividend paying stock. The solution has the form of a Taylor's series expansion, which contains an infinite number of terms. Another approach for solving the free boundary problem explicitly was recently proposed by Muthuraman (2008).

The second set of methods comes from probability theory. It focuses on expressing the current price of an American option as a discounted expectation of the specific option's payoff under the risk-neutral measure. This optimal stopping characterization is perhaps the most intuitive description

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of the problem. A complete formulation goes back to Bensoussan (1984) and Karatzas (1988). The paper of Jaillet et al. (1990) links the probabilistic approach to variational inequalities. See also Parkinson (1977) and Myneni (1992) for further references. The approach is used by Bjerksund and Stensland (1993) for a closed-form approximation that is based on a flat boundary restriction. Another probabilistic approximation scheme is explored in Jourdain and Martini (2001) and Jourdain and Martini (2002).

At the beginning of the 1990's, a breakthrough was achieved by characterizing the price of an American option as the sum of the corresponding European option plus an early exercise premium. These integral representations, also known as the early exercise representations, have been obtained by Kim (1990), Jacka (1991), and Carr et al. (1992). An analysis can also be found in Jamshidian (1992). These representations are exact solutions and implicitly characterize the free boundary in terms of a recursive integral equation. They were the starting point for new approximations for the American option price and/or the free boundary. Huang et al. (1996) use Richardson extrapolation to solve the integral expression. Ju (1998) approximates the early exercise boundary by a piece-wise exponential function. Bunch and Johnson (2000) derive expressions for the early exercise boundary using a new characterization of the option's price in terms of its time derivative. Sullivan (2000) introduces a Gaussian quadrature method to approximate the price of an American put using Chebyshev polynomials. Kallast and Kivinukk (2003) apply Newton's method for the approximation of the integrals arising in the early exercise representation. The approximation error of these numerical procedures is studied in Heider (2007). Two new analytical approximations for the critical stock price and a detailed numerical comparison of some existing approximations are given in Li (2010b).

Other popular methods are those of Broadie and Detemple (1996), Carr (1998) and Ingersoll (1998). Broadie and Detemple (1996) provide a pricing method based on a lower and upper bound. Carr (1998) determines

accurate prices using a randomization approach whereas Ingersoll (1998) approximates American options using barrier derivatives. See also Chiarella et al. (1999) for an application of path integrals for the pricing of American options. Broadie and Detemple (2004) give an excellent review of existing tools and methods.

The key to determining the value of the American option is finding the critical stock price $S^*(t)$ for all $t \leqq T$. Unfortunately, finding the critical stock price of a finite living American option in closed form seems to be impossible[11]. Even approximative solutions tend to be mathematically complex. The investigation of local and global properties[12] of the critical stock price attracted the interest of many researchers from the field of mathematical finance including the works of Barles et al. (1995), Kuske and Keller (1998), Evans et al. (2002), Bunch and Johnson (2000), Knessl (2001), Lamberton and Villeneuve (2003), Chen and Chadam (2003), Mallier and Alobaidi (2004), Chen and Chadam (2006), and Zhang and Li (2010), among others. Little et al. (2000) derive a one-dimensional integral equation for the free boundary by reducing the dimension of the underlying problem. Ševčovič (2001) analyzes the free boundary of American call options using a Fourier integral transformation and derives a nonlinear singular integral equation determining its shape. Peskir (2005) applies a change-of-variable formula to prove that the critical boundary of an American put option can be characterized as a unique solution of an integral equation arising in the early exercise representation. A numerical analysis of the early exercise boundary can be found in Basso et al. (2002), and Basso et al. (2004).

Ekström (2004a) and Chen et al. (2008) give two different proofs for the convexity of the free boundary of an American put option on a non-dividend paying stock. The general case seems to be more subtle and is by the cur-

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[11]For perpetual American options, $S^*(t)$ is independent of time and can be expressed in closed-form. Zhu's (2006b) paper also contains an infinite series expression for $S^*(t)$.

[12]By local properties we mean the short-time behavior of $S^*(t)$, i.e. the behavior of $\lim_{t \to T} S^*(t)$. The term global accounts for monotonicity, convexity etc.

rent date still an open problem. Chen et al. (2008), p. 187, point out that private communications with Detemple suggest that, for a particular choice of parameters, the early exercise boundary may not be convex.

Price relations between call and put options, commonly known as put call parity relations, are easily established for European options. For American options the situation is not so obvious. Parity and duality relations for American options and free boundaries are studied in Chesney and Gibson (1993), Carr and Chesney (1994), and McDonald and Schroder (1998). Several extensions can be found in Detemple (2001), Fajardo and Mordecki (2003), Fajardo and Mordecki (2006), and Carr and Lee (2009). Duality relations for perpetual American options are derived in Alfonsi and Jourdain (2008).

In the last decade there has also been some interest in generalizing the pricing framework of standard American options to a broader class of payoff functions. Natural extensions are American straddles and strangles. Alobaidi and Mallier (2002) apply a partial Laplace transform to locate the free boundary of an American straddle using an integral equation. In Alobaidi and Mallier (2006) the framework is extended to the valuation of American straddles close to expiry. Using Fourier transform techniques, Kim's (1990) method has also been generalized by Chiarella and Ziogas (2005a) to the case of American strangles.

4.1.2 Multidimensional and Exotic American Options

The extension to an analytical pricing of American options written on several assets is not straightforward, since the characterization of the early exercise region becomes highly nontrivial. Most numerical techniques use either lattice-based approaches or focus on combining advanced simulation methods with stochastic dynamic programming. Generalizations of the binomial tree approach of Cox et al. (1979) to multidimensional valuation problems were proposed by Boyle (1988), Boyle et al. (1989), Madan et al. (1989) or He (1990), among others. First algorithms for the valuation of multidimen-

sional American options based on simulation are those of Barraquand and Martineau (1995) and Raymar and Zwecher (1997). Broadie and Glasserman (2004) introduce a stochastic mesh method for pricing high-dimensional American options. Other popular algorithms based on simulation can be found in the articles of Andersen and Broadie (2004) and Ibáñez and Zapatero (2004). See also Ibáñez (2004) for a theoretical work based on Monte Carlo methodology concerning prices and early exercise boundaries of securities with multiple early exercise opportunities. Theoretical results of Glasserman and Yu (2004b) are extended in Firth (2005) by employing regression basis functions with a martingale property. The result is an improved Monte Carlo simulation, and computationally cheap lower and upper bounds to the American option price. Villeneuve and Zanette (2002) propose two numerical methods for pricing American options on two stocks based on the ADI algorithm. Berridge and Schumacher (2008) propose an irregular grid method for the valuation. Li (2010a) presents a quasi-analytical method for pricing multidimensional American options based on interpolating two arbitrage bounds.

Although much progress has been made in the understanding of early exercise boundaries of American options written on a single asset, the situation complicates significantly in the multidimensional case. When the optimal exercise policy is specified by several assets, the shape of the early exercise region cannot be determined by simple arguments. Also, intuition may be false. Gerber and Shiu (1996) analyze perpetual American options written on two underlying assets. While the understanding of finite-living contracts was befogged and mostly based on conjectures, it was Broadie and Detemple (1997) who first give a profound mathematical and economic clarification. Broadie and Detemple (1997) provide characterizations of the shape of early exercise regions of American options on multiple assets and valuation formulas for a class of convex and non-convex payoff functions. These include options on the maximum of two assets, dual strike options, spread options,

exchange options, options on the product and powers of the product, options on the arithmetic average of two assets and American capped exchange options. This work has been completed and extended by Villeneuve (1999), who provides additional results concerning the asymptotic behavior of the early exercise boundary. He also presents further results concerning the valuation of various types of multidimensional American options, such as the finite-living and perpetual American put on the minimum of two assets. See also Detemple et al. (2003) for the valuation of American call options on the minimum of two dividend-paying assets.

Analytical valuation of exotic options with the early exercise feature is similarly demanding. Most research results are relatively new. Early theoretical work based on a general PDE framework is Barraquand and Pudet (1996). American Asian options are considered in Wu et al. (1999), Hansen and Jorgensen (2000), Peskir and Uys (2005), Dai and Kwok (2006) or Lo et al. (2009). A lattice based framework for pricing American Asian options is explored in Dai and Lyuu (2009). American-style lookback and barrier options are studied both theoretically and numerically in Karatzas and Wang (2000), Gao et al. (2000), Dai (2000), Haug (2001), Ait-Sahlia et al. (2003), Ait-Sahlia et al. (2004), Dai and Kwok (2004), Dai et al. (2004), Lai and Lim (2004), Dai and Kwok (2005), Dai and Kwok (2006), and Chang et al. (2009). Gaudenzi and Zanette (2009) present a method for pricing American barrier options using binomial trees. An extension of the Monte Carlo approach to the valuation of path-dependent American-style derivatives can be found in Fujiwara and Kijima (2007). Analytical pricing of American chooser options is examined in Detemple and Emmerling (2009).

4.1.3 Other Price Processes

Kim and Yu (1996) study the American option pricing problem for general underlying asset price processes. Similarly, Detemple and Tian (2002) consider American options for a class of diffusion processes. Based on the integral

equation approach, they consider American options in stochastic volatility models, with stochastic interest rates, American bond options, and American options in the constant elasticity of variance (CEV) model. Ekström (2003) derives the closed-form solution for a perpetual American put option in the CEV model. Properties of American option prices in the CEV model are investigated in Ekström (2004b). A general numerical approach to pricing American-style derivatives that is applicable to any Markovian setting is presented in Laprise et al. (2006).

Pham (1997) and Zhang (1997) have done first theoretical work on American options in a jump diffusion environment. Perpetual American options in a jump-diffusion model are priced analytically in Mordecki (1999), and Aase (2005), and in a general Levy model setting in Mordecki (2002). Pricing formulas for finite-living American options are provided in Gukhal (2001), and Gukhal (2004), where the compound option approach is extended to jump-diffusion processes. Chiarella and Ziogas (2009) value American call options in a jump-diffusion setting applying Fourier transforms. Solutions for perpetual and non-perpetual American options can also be found in Chesney and Jeanblanc (2004). Bounds for perpetual American option prices in terms of standard American options are derived in Ekström (2006). Liang et al. (2010) study the optimal convergence of a binomial tree scheme for American options in a jump diffusion model. Bayraktar and Xing (2009) develop an approximation scheme for American options on jump diffusions by computing a sequence of functions iteratively. American and European options in multi-factor jump-diffusion models are considered in Levendorskiĭ (2008).

Touzi (1999) establishes fundamental properties of American options on stocks with stochastic volatility. Chevalier (2005) considers the asymptotic behavior of the free boundary of American options in a local volatility model. Numerical approaches to American options on stocks with stochastic volatility are those of Zvan et al. (1998), Clarke and Parrott (1999), Ikonen and

Toivanen (2007), and Zhylyevskyy (2010). The articles of Tzavalis and Wang (2003) and Chiarella and Ziogas (2005b) include frameworks based on an analytical approximation.

In the last few years, much progress has been achieved in the research of American options under general Levy processes. A good reference for the application of Levy processes in finance is Schoutens (2003). Through the years, a number of different Levy processes has been proposed for modeling financial markets which fit empirical observations. Prominent candidates are the symmetric Variance Gamma process (Madan and Seneta (1990)), asymmetric Variance Gamma process (Madan et al. (1998)), the Normal Inverse Gaussian process (Barndorff-Nielsen (1998)), and the CGMY process (Carr et al. (2003) and Carr et al. (2007)). American options under the Variance Gamma process are considered in Hirsa and Madan (2004). Stentoft (2008) proposes an econometric framework for pricing American options in a model with time-varying volatility and conditional skewness and leptokurtosis, using GARCH processes and the Normal Inverse Gaussian distribution. Further theoretical works on pricing American options under Levy processes include the papers of Boyarchenko and Levendorskiĭ (2002), Boyarchenko and Levendorskiĭ (2004), Boyarchenko and Levendorskiĭ (2005), Boyarchenko and Levendorskiĭ (2008), Levendorskiĭ (2004), Levendorskiĭ (2006) and Eberlein et al. (2008). Finite and infinite living American options for a large class of Levy processes are also studied in Ivanov (2007). Lamberton and Mikou (2008) consider the asymptotic behavior of the critical stock price of an American put in an exponential Levy model. Numerical methods are developed in Almendral (2005), Almendral and Oosterlee (2007), Maller et al. (2006), and Fang and Oosterlee (2009).

4.2 Problem Formulation

4.2.1 Preliminaries

We consider American put and call options on dividend paying stocks. Let $P^A(S, t)$ and $C^A(S, t)$ be the prices at the point (S, t), respectively. It is obvious that the early exercise feature imposes the conditions

$$P^A(S, t) \geq (X - S(t))^+, \qquad t \in [0, T], \tag{4.1}$$

for the American put, and

$$C^A(S, t) \geq (S(t) - X)^+, \qquad t \in [0, T], \tag{4.2}$$

for the American call, respectively. Non-fulfillment of these conditions implies arbitrage possibilities. For an informal proof, consider the contrary and assume that at a certain time t the American put $P^A(S, t)$ is worth less than its intrinsic payoff. Then riskless arbitrage profits are realized by buying the option for $P^A(S, t)$, buying the asset for $S(t)$ and immediately exercising the option. The strategy results in a profit of $X - S(t) - P^A(S, t) > 0$. Similar arguments applied to the American call establish (4.2). The critical stock price at which one should exercise the American option prematurely, is called the optimal exercise boundary, the early exercise boundary, or simply the free boundary. Due to its time dependency, it will be denoted by $S^*(t)$. It is clear that the early exercise boundaries will differ for American puts and calls, and therefore the more proper notation of $S^*(t) = S_P^*(t)$ for puts and $S^*(t) = S_C^*(t)$ for calls should be used[13]. Also, since early exercise is only optimal for in-the-money options, it follows that $0 < S_P^*(t) \leq X$ and $X \leq S_C^*(t) < \infty$ for all $t \in [0, T]$. If confusion is excluded we will write $S^*(t)$ for the early exercise boundary for notational simplicity.

The American pricing problem must be solved in the domain

$$\mathcal{D} = \{(S, t) \in \mathbb{R}^+ \times [0, T]\}.$$

---

[13]Analytic relations between $S_P^*(t)$ and $S_C^*(t)$ will be studied later.

The solution domain $\mathcal{D}$ is divided by $S^*(t)$ into two sets, the continuation region $\mathcal{C}$ and the stopping region $\mathcal{S}$. The continuation region is the set of all S and t combinations where the American option is alive, whereas in the stopping region the American option is exercised or dead. More formally, we have for the American put

$$\mathcal{C}^P = \{(S,t) \in \mathbb{R}^+ \times [0,T] : P^A(S,t) > (X-S)^+\},$$

$$\mathcal{S}^P = \{(S,t) \in \mathbb{R}^+ \times [0,T] : P^A(S,t) = (X-S)^+\}.$$

Similarly,

$$\mathcal{C}^C = \{(S,t) \in \mathbb{R}^+ \times [0,T] : C^A(S,t) > (S-X)^+\},$$

$$\mathcal{S}^C = \{(S,t) \in \mathbb{R}^+ \times [0,T] : C^A(S,t) = (S-X)^+\}.$$

The next figure summarizes the results graphically. It also indicates the high contact or smooth pasting conditions

$$P^A(S^*,t) = X - S^*(t) \qquad \text{and} \qquad \left.\frac{\partial P^A(S,t)}{\partial S}\right|_{S(t)=S^*(t)} = -1, \qquad (4.3)$$

for puts and

$$C^A(S^*,t) = S^*(t) - X \qquad \text{and} \qquad \left.\frac{\partial C^A(S,t)}{\partial S}\right|_{S(t)=S^*(t)} = 1, \qquad (4.4)$$

for calls, respectively.

Proposition 4.2.1 *Let $P^A(S,t)$ be the value of the American put option. Then*

- *$P^A(S,t)$ is continuous on $\mathbb{R}^+ \times [0,T]$.*

- *$P^A(\cdot,t)$ is convex and non-increasing on $\mathbb{R}^+$ for every $t \in [0,T]$.*

- *$P^A(S,\cdot)$ is non-decreasing on $[0,T]$ for every $S \in \mathbb{R}^+$.*

- *$-1 \leq \frac{\partial P^A(S,t)}{\partial S} \leq 0$ in $\mathcal{D}$ and $\frac{\partial P^A(S,t)}{\partial S} = -1$ in the interior of $\mathcal{S}^P$.*
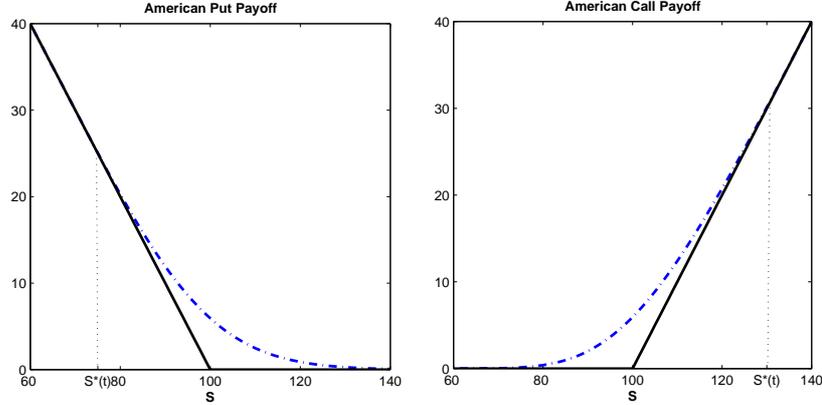
Figure 4.1: Price functions of American options with strike price $X = 100$. For the American put, the price function touches the intrinsic payoff tangentially at $(S^*(t), X - S^*(t))$. For $S < S^*(t)$, the American put price becomes $X - S$. Similarly, the price function of an American call touches the intrinsic payoff tangentially at $(S^*(t), S^*(t) - X)$. For $S > S^*(t)$, the American call price becomes $S - X$.

Analogously, we have for the American call

- *$C^A(S, t)$ is continuous on $\mathbb{R}^+ \times [0, T]$.*

- *$C^A(\cdot, t)$ is convex and non-decreasing on $\mathbb{R}^+$ for every $t \in [0, T]$.*

- *$C^A(S, \cdot)$ is non-increasing on $[0, T]$ for every $S \in \mathbb{R}^+$.*

- *$0 \leq \frac{\partial C^A(S,t)}{\partial S} \leq 1$ in $\mathcal{D}$ and $\frac{\partial C^A(S,t)}{\partial S} = 1$ in the interior of $\mathcal{S}^C$.*

PROOF: The first part is proved in Elliott and Kopp (2005), the second in Detemple (2006). □

As pointed out by Detemple (2006), the continuity of the price functions

imply the stopping regions to be closed sets. Therefore, one may conclude that the sets $\{S_P^*(t) : t \in [0, T]\}$, and $\{S_C^*(t) : t \in [0, T]\}$ belong to $\mathcal{S}^P$ and $\mathcal{S}^C$, respectively. Finally, one deduces that

$$\mathcal{S}^P = \{(S, t) \in \mathbb{R}^+ \times [0, T) : S \leq S_P^*(t)\},$$

and

$$\mathcal{S}^C = \{(S, t) \in \mathbb{R}^+ \times [0, T) : S \geq S_C^*(t)\}.$$

The next proposition summarizes the results on the structure of the free boundaries.

Proposition 4.2.2 *Let $S_P^*(t)$ and $S_C^*(t)$ be the early exercise boundaries of an American put and call options, respectively. Then*

- *$S_P^*(t)$ is continuous on $[0, T)$.*

- *$S_P^*(t)$ does not depend on the current price of the asset, $S(0)$.*

- *$S_P^*(t)$ is linearly homogeneous in X.*

- *$S_P^*(t)$ is non-decreasing with respect to calendar time t, i.e. non-increasing with respect to time to maturity $\tau = T - t$, and has the limiting values*

$$\lim_{t \to T} S_P^*(t) = \lim_{\tau \to 0} S_P^*(\tau) = \min(X, \frac{r}{q}X)$$

and $\lim_{\tau \to \infty} S_P^*(\tau) = \frac{\gamma}{\gamma - 1}X$, where
$\gamma = (-(r - q - \frac{1}{2}\sigma^2) - \sqrt{(r - q - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2})/\sigma^2.$

Similarly,

- *$S_C^*(t)$ is continuous on $[0, T)$.*

- *$S_C^*(t)$ does not depend on the current price of the asset, $S(0)$.*

- *$S_C^*(t)$ is linearly homogeneous in X.*

- $S_C^*(t)$ *is non-increasing with respect to calendar time* t, *i.e. non-decreasing with respect to time to maturity* $\tau = T - t$, *and has the limiting values*

$$\lim_{t \to T} S_C^*(t) = \lim_{\tau \to 0} S_C^*(\tau) = \max(X, \frac{r}{q}X)$$

and $\lim_{\tau \to \infty} S_C^*(\tau) = \frac{\beta}{\beta-1}X$, *where*
$\beta = (-(r - q - \frac{1}{2}\sigma^2) + \sqrt{(r - q - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2})/\sigma^2$.

PROOF: Proofs for most of these statements can be found in the textbooks of Elliott and Kopp (2005) or Detemple (2006). See also Myneni (1992). Especially, the limiting values are due to Kim (1990). A proof of the linear homogeneity property with respect to the strike price is given in Basso et al. (2004), although it has also been obtained by Gao et al. (2000) concerning American barrier options. □

The next figure displays the functional form of the free boundaries for a special set of parameters. From the last proposition, it follows that the limiting values for $\tau = 0$ and $\tau = \infty$ give lower and upper bounds for the free boundaries. Also, when $r = 0$ it follows that $S_P^*(\tau) \to 0$ as $\tau \to 0$. Furthermore, since $S_P^*(\tau)$ decreases monotonically in τ, we have $S_P^*(\tau) \to 0$ for all values of τ. This shows that, for a zero interest rate, it is never optimal to exercise an American put option prior to maturity. In a similar manner, when $q = 0$ we have that $S_C^*(\tau) \to \infty$ as $\tau \to 0$. Again, since $S_C^*(t)$ increases monotonically in τ, it follows that $S_C^*(\tau) \to \infty$ for all values of τ. This result confirms Merton's (1973) result that it is never optimal to exercise an American call on a non-dividend paying stock prematurely. Finally, it is worth mentioning that, if there is no uncertainty in the underlying price process, i.e. $\sigma = 0$, the early exercise boundaries become constants, $\min(X, \frac{r}{q}X)$ for puts and $\max(X, \frac{r}{q}X)$ for calls, respectively. The intuition underlying these results is, that if there is no uncertainty, it is profitable to exercise early if the payoffs and local gains are non-negative: $S \leq X$ and $rX - qS \geq 0$ for puts and $S \geq X$ and $qS - rX \geq 0$ for calls, respectively. For $\tau \to \infty$, the optimal
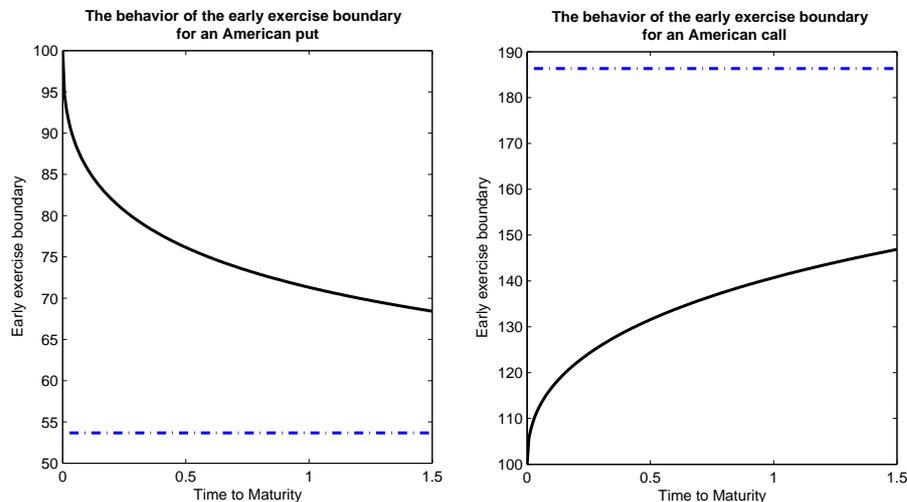
Figure 4.2: Early exercise boundaries of American options as a function of time to maturity with fixed parameters $X = 100, r = q = 0.05, \tau = 1.5$ and $\sigma = 0.2$. The dotted lines represent the constant free boundaries of perpetual American options. The exact numerical values are 53.67 for the put and 186.33 for the call, respectively.

exercise policy becomes a constant and it becomes optimal to exercise at the first hitting time of this constant barrier.

4.2.2 Optimal Stopping Formulation

Possibly the most intuitive formulation of the American pricing problem comes from probability theory and is called the Optimal Stopping Formulation. Since the early exercise feature imposes an optimal decision when the contract has to be exercised, the valuation equation (2.28) must be general-

ized to

$$F(S,t) = \sup_{\tau \in \mathfrak{T}_{t,T}} E_t^{\mathbb{Q}}\left[e^{-r(\tau-t)}g(S_\tau)\right], \tag{4.5}$$

where $\mathfrak{T}_{t,T}$ is the set of all stopping times between t and T. If $F(S,t)$ is an American put, then

$$P^A(S,t) = \sup_{\tau \in \mathfrak{T}_{t,T}} E_t^{\mathbb{Q}}\left[e^{-r(\tau-t)}(X - S(\tau))^+\right]. \tag{4.6}$$

From the theory of optimal stopping, it follows that the first optimal stopping time after t is

$$\varrho_t = \inf\{u \in [t,T] : P^A(S,u) = (X - S(u))^+\}. \tag{4.7}$$

Analogously, for $C^A(S,t)$. Similar to a European put, the American put is expressed in terms of the expected discounted payoff under the risk neutral probability measure $\mathbb{Q}$. However, since the holder has the additional privilege of exercising the option at any time, the supremum is taken over all possible exercise times. One can use the formulation to prove all the properties of the price function and the free boundary. However, good skills in stochastic calculus are a prerequisite. Details can be found in the articles of Bensoussan (1984), Karatzas (1988), Jacka (1991), and Myneni (1992).

4.2.3 Free Boundary Formulation

The free boundary formulation of American (call) options goes back to McKean (1965). A good description is also given in Wilmott et al. (1993). A rigorous justification for the equivalence of the Optimal Stopping Formulation and the Free Boundary Formulation is given in Carr et al. (1992) or Peskir (2005).

The American put option problem can be formulated as the solution of

$$\frac{\partial P^A}{\partial t} + (r-q)\,S_t\,\frac{\partial P^A}{\partial S} + \frac{1}{2}\sigma^2\,S_t^2\,\frac{\partial^2 P^A}{\partial S^2} - rP^A = 0\,, \tag{4.8}$$

on $\mathbb{R}^+ \times [0, T)$, along with the boundary conditions

$$P^A(S, T) = (X - S(T))^+, \tag{4.9}$$

$$P^A(S^*, t) = X - S^*(t), \tag{4.10}$$

$$\left. \frac{\partial P^A(S, t)}{\partial S} \right|_{S(t) = S^*(t)} = -1. \tag{4.11}$$

In the continuation region $\mathcal{C}^P$ the price function is governed by the BSM-PDE. The equation must be solved along the smooth pasting conditions. Advanced theory of free boundary value problems shows that, using the principle of Duhamel[14], the solution can be written formally as

$$
\begin{aligned}
P^A(S, \tau) \;=\; & e^{-r\tau} \int_0^X (X - S(T)) \psi(S(T), S) dS(T) \\
& + \int_0^\tau e^{-r\xi} \int_0^{S^*(\tau - \xi)} (rX - qS(\xi)) \psi(S(\xi), S) dS(\xi) d\xi,
\end{aligned}
\tag{4.12}
$$

where $\tau = T - t$, ξ is a dummy variable for the time elapsed after time t, and $\psi(S(\xi), S)$ is the transition density function given by

$$\psi(S(\xi), S) = \frac{1}{S(\xi)\sqrt{2\pi\xi\sigma^2}} \exp\left(-\frac{\left(\ln S(\xi) - \ln S - (r - q - \frac{1}{2}\sigma^2)\xi \right)^2}{2\xi\sigma^2} \right). \tag{4.13}$$

The first part of (4.12) is the European put and the integral represents the early exercise premium. A financial interpretation of the necessity of the smooth pasting conditions in terms of a dynamic trading strategy is given in Carr et al. (1992). Jamshidian (1992) gives an interpretation of the early exercise premium as a delay exercise compensation. Further evaluations of the integrals in (4.12) give the following characterization of the American

---

[14]See Kwok (1998).

put price:

$$
\begin{aligned}
P^A(S,\tau) \;=\;& P^E(S,\tau) \\
& + \int_0^\tau rXe^{-r\xi} N(-d_2(S, S^*(\tau-\xi), \xi)) d\xi \\
& - \int_0^\tau qSe^{-q\xi} N(-d_1(S, S^*(\tau-\xi), \xi)) d\xi \qquad (4.14)
\end{aligned}
$$

where $\tau = T - t$, $P^E(S,\tau)$ is the European put price from (2.12), and

$$
d_1(x,y,t) = \frac{\ln \frac{x}{y} + (r - q + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},
$$

$$
d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.
$$

Similar arguments give the American call price as

$$
\begin{aligned}
C^A(S,\tau) \;=\;& C^E(S,\tau) \\
& + \int_0^\tau qSe^{-q\xi} N(d_1(S, S^*(\tau-\xi), \xi)) d\xi \\
& - \int_0^\tau rXe^{-r\xi} N(d_2(S, S^*(\tau-\xi), \xi)) d\xi, \qquad (4.15)
\end{aligned}
$$

where $\tau = T - t$, $C^E(S,\tau)$ is the European call price from (2.15). These are the early exercise representations due to Kim (1990), Jacka (1991), and Carr et al. (1992).

4.2.4 Integral Equation Formulation

The decomposition formulas (4.14) and (4.15) express American option prices in terms of the unknown early exercise boundary. However, since they hold in the entire domain $\mathcal{D}$, they particularly hold on the boundary of the exercise region before maturity, where $S(t) = S^*(t), t < T$, or equivalently, $S(\tau) = S^*(\tau), \tau > 0$. Applying the conditions $P^A(S^*, \tau) = X - S_P^*(\tau)$ for puts, and $C^A(S^*, \tau) = S_C^*(\tau) - X$ for calls respectively, each early exercise boundary is

uniquely characterized implicitly by a recursive non-linear integral equation

$$
\begin{aligned}
X - S^*(\tau) \;=\;& P^E(S^*, \tau) \\
&+ \int_0^\tau r X e^{-r\xi} N(-d_2(S^*(\tau), S^*(\tau - \xi), \xi)) d\xi \\
&- \int_0^\tau q S^*(\tau) e^{-q\xi} N(-d_1(S^*(\tau), S^*(\tau - \xi), \xi)) d\xi \quad (4.16)
\end{aligned}
$$

where $S^*(\cdot) = S_P^*(\cdot)$ and

$$
\begin{aligned}
S^*(\tau) - X \;=\;& C^E(S^*, \tau) \\
&+ \int_0^\tau q S^*(\tau) e^{-q\xi} N(d_1(S^*(\tau), S^*(\tau - \xi), \xi)) d\xi \\
&- \int_0^\tau r X e^{-r\xi} N(d_2(S^*(\tau), S^*(\tau - \xi), \xi)) d\xi, \quad (4.17)
\end{aligned}
$$

for $S^*(\cdot) = S_C^*(\cdot)$. In each case the solution procedure starts at maturity $\tau = 0$ with $S^*(0)$. Also, solving for the early exercise boundary $S^*(\tau)$ for some $\tau > 0$ requires the knowledge of $S^*(\xi), 0 < \xi \leq \tau$. A numerical implementation of the early exercise representations consists of first solving the integral equations for the free boundary, and computing the prices from (4.14) and (4.15) taking the free boundary curve as an input.

Since the above formulae use the cumulative normal distribution function, which itself is an integral expression, they may be regarded as two-dimensional characterizations of the free boundary. Another two-dimensional integral equation is presented in Kwok (1998). In the exercise region, however, the price functions have nice properties which can be exploited to reduce the dimensionality. Little et al. (2000) achieve a reduction to a one-dimensional integral equation by exploiting the fact that, in the exercise region, the second derivative of an American option with respect to the underlying must be zero when evaluated at the boundary. In the case of an American put, their

final result characterizing the free boundary is

$$S^*(\tau)\left(\sigma e^{-q\tau - \frac{1}{2}d_1^2(S^*(\tau),X,\tau)} + q\sqrt{2\pi\tau}\right) = rX\sqrt{2\pi\tau}$$

$$+ qS^*(\tau)\sqrt{\tau}\int_0^\tau e^{-q\xi - \frac{1}{2}d_1^2(S^*(\tau),S^*(\tau-\xi),\xi)}\left(\frac{d_2(S^*(\tau),S^*(\tau-\xi),\xi)}{\xi}\right)d\xi$$

$$- rX\sqrt{\tau}\int_0^\tau e^{-r\xi - \frac{1}{2}d_2^2(S^*(\tau),S^*(\tau-\xi),\xi)}\left(\frac{d_1(S^*(\tau),S^*(\tau-\xi),\xi)}{\xi}\right)d\xi. \quad (4.18)$$

The formula holds for any $\tau > 0$ and does not involve the cumulative normal distribution function. The corresponding expression for the free boundary of an American call can be found in Detemple (2006). Additionally, Detemple (2006) points out that such a reduction is not unique implying a whole set of one-dimensional integral equations characterizing the free boundary which may be used for a numerical computation. A numerical comparison of the performance of a large family of analytical approximations for the early exercise boundary is given in Li (2010b).

4.2.5 Other Formulations

The free boundary problem associated with the pricing of American options can be formulated as a linear complementarity problem. The link between the Optimal Stopping Formulation and variational inequalities is provided in Jaillet et al. (1990). A concise derivation of the linear complementarity formulation can be found in Kwok (1998) or Firth (2005).

The American option problem has also been formulated as a viscosity solution (Benth et al. (2003)). By applying the theory of dynamic programming, the authors derive a semilinear BSM-type partial differential equation whose unique solution equals the American option price. The formulation allows a solution on a fixed domain, with no explicit free boundary. The "price" for this is a discontinuous non-linearity. Benth et al. (2003) note that the partial differential equation resulting from the viscosity solution formulation allows

an interpretation as the infinitesimal version of the early exercise premium representation.

A further method coming from probability theory, which was recently developed to characterize American options, is called the Primal Dual Formulation. Rogers (2002) introduced the idea of a dual pricing procedure by simulating the paths of the payoff and a suitable Lagrangian martingale. This gives sharp upper bounds for the American option price. Similar ideas for constructing tight bounds for the price were developed in Haugh and Kogan (2004) and Andersen and Broadie (2004).

4.2.6 Symmetry Relations

The early exercise feature inherent in the pricing of American options implies the non-validity of the put-call parity for European options. The parity can be replaced by the so-called put-call symmetry. The symmetry relation for American options is a theoretical statement that makes it possible to infer the American call value on a dividend paying stock by a reparametrization of the American put price function, and vise versa. Once the functional relation is established, it additionally allows to infer the behavior of the early exercise boundary of one contract when knowing the other. The proof is essentially due to McDonald and Schroder (1998). Several extensions can be found in Detemple (2001), Fajardo and Mordecki (2003), and Fajardo and Mordecki (2006).

Let $P^A(S, \tau) = P^A(S, \tau, X, r, q)$ be the price function of an American put option. Then, prior to exercise, the corresponding call price $C^A(S, \tau, X, r, q)$ equals

$$C^A(S, \tau, X, r, q) = P^A(X, \tau, S, q, r), \qquad (4.19)$$

i.e. an American call on a dividend paying asset S with strike price X and maturity date τ exactly equals an American put on a dividend paying asset X with strike price S and the same maturity date τ. Also, the roles of r and

q are interchanged. The intuition behind the symmetry comes from foreign exchange options. If $\tau^P = \tau^P(S, X, r, q)$ denotes the optimal exercise time before maturity of an American put, then

$$\tau^C(S, X, r, q) = \tau^P(X, S, q, r), \tag{4.20}$$

where $\tau^C(S, X, r, q)$ denotes the optimal exercise time for the corresponding symmetric call. The relationship between the respective early exercise boundaries is

$$S_C^*(\tau, S, r, q) = \frac{SX}{S_P^*(\tau, X, q, r)}, \tag{4.21}$$

where $S_P^*(\cdot)$ and $S_C^*(\cdot)$ denote the critical stock prices of puts and calls, respectively. Especially,

$$S_C^*(\tau, X, r, q) = \frac{X^2}{S_P^*(\tau, X, q, r)}. \tag{4.22}$$

The symmetry relation states that an American call price can be inferred from the corresponding American put price by interchanging S and X and r and q. The same is true for the free boundaries. A rigorous proof is given in Detemple (2006). Another one can be found in Kwok (1998), where a homogeneity property of the price functions is used to establish the relation. An application of (4.22) is for instance

$$\lim_{\tau \to 0} S_C^*(\tau, X, r, q) = \frac{X^2}{\lim_{\tau \to 0} S_P^*(\tau, X, q, r)} = \frac{X^2}{\min(X, \frac{q}{r}X)} = \max(X, \frac{r}{q}X).$$

The other extreme, $\lim_{\tau \to \infty} S_C^*(\tau, X, r, q)$, can be determined similarly using (4.22) and the limiting value of $S_P^*(\tau, X, r, q)$. The validity of the symmetry relations is not restricted to the log-normal model and holds for a wide class of diffusions.

Chapter 5

American Options and Mellin Transforms

This chapter is concerned with the analytical pricing of American options applying Mellin transforms[15]. The American put is considered first. Thereafter, we propose a modification of the transform to be applicable for a valuation of European and American calls. After proving all relevant theoretical and economically meaningful properties of the derived formulae, we propose the Gauss-Laguerre quadrature for an efficient pricing. Finally, we use the pricing functions based on Mellin transforms to derive simple and accurate approximations of the free boundary.

5.1   The American Put Option

Before dealing with American options it is convenient to start with their European counterparts. Instead of focusing on a standard European put, we consider a broader class of payoff functions called power options. Power options offer flexibility to investors and are of practical interest since many

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[15]The Sections 5.1, 5.2, and 5.3 are based on the papers Frontczak and Schöbel (2008), Frontczak and Schöbel (2010), and Frontczak (2010a), respectively.

OTC-traded options exhibit such a payoff structure. After characterizing the payoff of a power put option, we show how the Mellin transform can be applied to derive a BSM-like formula for this derivative. The results are used afterwards to price the American put analytically.

5.1.1 The European Power Put Option

A European power put option is an option with a non-linear payoff given by the difference between the strike price X and the underlying asset price at maturity raised to a strictly positive power

$$P_n^E(S, T) = \max(X - S_T^n, 0), \qquad n > 0. \tag{5.1}$$

For $n = 1$ we have the plain vanilla put as a special case. For references to power options see for example Esser (2003) and Macovschi and Quittard-Pinon (2006)[16]. Our goal is to derive a valuation formula for European power put options using Mellin transform techniques. Assuming the log-normal risk neutral dynamics of Chapter 2, we may apply Ito's Lemma to $\mathcal{S}_t = S_t^n$ to get

$$d\mathcal{S}_t = \left(n(r - q) + \frac{1}{2}n(n-1)\sigma^2\right)\mathcal{S}_t\, dt + n\sigma\, \mathcal{S}_t\, dW_t\,. \tag{5.2}$$

The new process is identified as a new Geometric Brownian motion. Now it is straightforward to derive the PDE for any derivative F written on $\mathcal{S}$:

$$\frac{\partial F}{\partial t} + n(\frac{1}{2}\sigma^2(n-1) + (r-q))\mathcal{S}\,\frac{\partial F}{\partial \mathcal{S}} + \frac{1}{2}\sigma^2 n^2\, \mathcal{S}^2\, \frac{\partial^2 F}{\partial \mathcal{S}^2} - rF = 0. \tag{5.3}$$

---

[16]Macovschi and Quittard-Pinon (2006) also study polynomial options with a payoff of the form

$$P_n^E(S, T) = \max(X - S_T, 0)^n, \qquad \text{for} \quad n > 0\,.$$

Since by the binomial theorem we have that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

an extension of the framework to polynomial options may also be possible.

Especially, for European power put options P_n^E we have

$$\frac{\partial P_n^E}{\partial t} + n(\frac{1}{2}\sigma^2(n-1) + (r-q))\mathcal{S}\frac{\partial P_n^E}{\partial \mathcal{S}} + \frac{1}{2}\sigma^2 n^2 \mathcal{S}^2 \frac{\partial^2 P_n^E}{\partial \mathcal{S}^2} - rP_n^E = 0 \quad (5.4)$$

with boundary conditions

$$\lim_{\mathcal{S}\to\infty} P_n^E(\mathcal{S},t) = 0 \qquad \text{on} \quad [0,T], \qquad (5.5)$$

$$P_n^E(\mathcal{S},T) = \theta(\mathcal{S}) = (X - \mathcal{S})^+ \qquad \text{on} \quad [0,\infty), \qquad (5.6)$$

and

$$P_n^E(0,t) = Xe^{-r(T-t)} \qquad \text{on} \quad [0,T]. \qquad (5.7)$$

For $n = 1$ the solution to the above PDE is the celebrated BSM formula given in (2.12).

Let $\widetilde{P}_n^E(\omega,t)$ denote the Mellin transform of $P_n^E(\mathcal{S},t)$. Then a straightforward application shows that the Mellin transform of the PDE equals

$$\frac{\partial \widetilde{P}_n^E(\omega,t)}{\partial t} + \frac{1}{2}n^2\sigma^2\big[\omega^2 + \omega(1-\kappa_2) - \kappa_1\big]\widetilde{P}_n^E(\omega,t) = 0 \qquad (5.8)$$

where

$$\kappa_2 = \frac{n-1}{n} + \frac{2(r-q)}{n\sigma^2}$$

and

$$\kappa_1 = \frac{2r}{n^2\sigma^2}.$$

This is a ordinary differential equation (ODE) which general solution is given by

$$\widetilde{P}^E(\omega,t) = c(\omega) \cdot e^{-\frac{1}{2}n^2\sigma^2 \cdot Q(\omega) \cdot t} \qquad (5.9)$$

where we have set

$$Q(\omega) = \omega^2 + \omega(1 - \kappa_2) - \kappa_1, \qquad (5.10)$$

and $c(\omega)$ a constant depending on the boundary conditions. Now, the terminal condition gives

$$c(\omega) = \widetilde{\theta}(\omega,t) \cdot e^{\frac{1}{2}n^2\sigma^2 \cdot Q(\omega) \cdot T} \qquad (5.11)$$

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where
$$\widetilde{\theta}(\omega,t) = \widetilde{\theta}(\omega) = X^{\omega+1}\Big(\frac{1}{\omega} - \frac{1}{\omega+1}\Big) \tag{5.12}$$

is the Mellin transform of the terminal condition and is independent of n. Using the inverse Mellin transform we see that the price of a European power put option is given by

$$
\begin{aligned}
P_n^E(\mathcal{S},t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{P}_n^E(\omega,t)\,\mathcal{S}^{-\omega}\,d\omega \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega,t)\cdot e^{\frac{1}{2}n^2\sigma^2\cdot Q(\omega)\cdot(T-t)}\,\mathcal{S}^{-\omega}\,d\omega \tag{5.13}
\end{aligned}
$$

with $(\mathcal{S},t)\in(0,\infty)\times[0,T)$, $c\in(0,\infty)$ a constant, $\{\omega\in\mathbb{C}\,|\,0<Re(\omega)<\infty\}$, and $\widetilde{\theta}(\omega,t)$ and $Q(\omega)$ as defined in equations (5.12) and (5.10), respectively. To derive a "BSM-like" formula, we follow Panini and Srivastav (2004) and use the convolution property of Mellin transforms from Chapter 3

$$P_n^E(\mathcal{S},t) = \int_0^\infty \theta(u)\cdot\phi\Big(\frac{\mathcal{S}}{u}\Big)\cdot\frac{1}{u}\,du \tag{5.14}$$

where $\phi(u)$ is to be determined. First, observe that for $\beta_1 = \frac{1}{2}n^2\sigma^2(T-t)$ we have

$$
\begin{aligned}
\frac{1}{2}n^2\sigma^2(T-t)Q(\omega) &= \beta_1\Big[\Big(\omega+\frac{1-\kappa_2}{2}\Big)^2 - \Big(\frac{1-\kappa_2}{2}\Big)^2 - \kappa_1\Big] \\
&= \beta_1\big[(\omega+\beta_2)^2 - \beta_2^2 - \kappa_1\big] \tag{5.15}
\end{aligned}
$$

where we have set $\beta_2 = \frac{1-\kappa_2}{2}$. Thus, we can write for the put price

$$P_n^E(\mathcal{S},t) = e^{-\beta_1(\beta_2^2+\kappa_1)}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega,t)\cdot e^{\beta_1(\omega+\beta_2)^2}\,\mathcal{S}^{-\omega}\,d\omega. \tag{5.16}$$

Now, $\widetilde{\phi}(\omega)$ is the Mellin transform of

$$e^{\beta_1(\omega+\beta_2)^2} = \int_0^\infty \phi(\mathcal{S})\,\mathcal{S}^{\omega-1}\,d\mathcal{S}. \tag{5.17}$$

Using the transformation (see Erdelyi et al. (1954))

$$e^{\theta\omega^2} = \int_0^\infty \frac{1}{2\sqrt{\pi\theta}}e^{-\frac{(\ln\mathcal{S})^2}{4\theta}}\,\mathcal{S}^{\omega-1}\,d\mathcal{S} \quad,\ Re(\theta)\geq 0$$

we get

$$\phi(\mathcal{S}) = \phi(\mathcal{S}, t) = \frac{\mathcal{S}^{\beta_2}}{n\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{\ln\mathcal{S}}{n\sigma\sqrt{T-t}}\right)^2}. \qquad (5.18)$$

The European power put price can therefore be expressed as

$$
\begin{aligned}
P_n^E(\mathcal{S}, t) &= \frac{e^{-\beta_1(\beta_2^2+\kappa_1)}}{n\sigma\sqrt{2\pi(T-t)}} \int_0^X (X-u)\left(\frac{\mathcal{S}}{u}\right)^{\beta_2} e^{-\frac{1}{2}\left(\frac{\ln\frac{\mathcal{S}}{u}}{n\sigma\sqrt{T-t}}\right)^2} \cdot \frac{1}{u}\, du \\
&= \frac{e^{-\beta_1(\beta_2^2+\kappa_1)}}{n\sigma\sqrt{2\pi(T-t)}} \cdot X \cdot \mathcal{S}^{\beta_2} \int_0^X \frac{1}{u^{\beta_2+1}} e^{-\frac{1}{2}\left(\frac{\ln\frac{\mathcal{S}}{u}}{n\sigma\sqrt{T-t}}\right)^2}\, du \\
&\quad - \frac{e^{-\beta_1(\beta_2^2+\kappa_1)}}{n\sigma\sqrt{2\pi(T-t)}} \cdot \mathcal{S}^{\beta_2} \int_0^X \frac{1}{u^{\beta_2}} e^{-\frac{1}{2}\left(\frac{\ln\frac{\mathcal{S}}{u}}{n\sigma\sqrt{T-t}}\right)^2}\, du \qquad (5.19)
\end{aligned}
$$

with

$$\beta_1 = \frac{1}{2}n^2\sigma^2(T-t), \qquad \beta_2 = \frac{1-\kappa_2}{2},$$

and

$$\kappa_2 = \frac{n-1}{n} + \frac{2(r-q)}{n\sigma^2}.$$

To evaluate the first integral use the new variable

$$\gamma = \frac{1}{n\sigma\sqrt{T-t}}\left(\ln\left(\frac{\mathcal{S}}{u}\right) - \beta_2 n^2\sigma^2(T-t)\right).$$

For the second integral use the slightly different transformation

$$\gamma = \frac{1}{n\sigma\sqrt{T-t}}\left(\ln\left(\frac{\mathcal{S}}{u}\right) - (\beta_2-1)n^2\sigma^2(T-t)\right).$$

Finally, the first part of (5.19) is determined as

$$X\, e^{-r(T-t)}\, N(-d_{2,n}(\mathcal{S}, X, T))$$

where

$$d_{2,n}(\mathcal{S}, X, T) = \frac{\ln\frac{\mathcal{S}}{X} + n\left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{n\,\sigma\,(T-t)}. \qquad (5.20)$$

The second integral is evaluated using the transformation suggested above and the result is

$$-e^{((n-1)r-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t)}\,\mathcal{S}\,N(-d_{1,n}(\mathcal{S},X,T))$$

where

$$d_{1,n}(\mathcal{S},X,T) = \frac{\ln\frac{\mathcal{S}}{X} + n\left(r-q+(n-\frac{1}{2})\sigma^2\right)(T-t)}{n\,\sigma\,(T-t)}\,. \qquad (5.21)$$

The price of a power put option is therefore given by

$$P_n^E(\mathcal{S},t) = X\,e^{-r(T-t)}\,N(-d_{2,n}) - e^{((n-1)r-nq+\frac{1}{2}n(n-1)\sigma^2)(T-t)}\,\mathcal{S}\,N(-d_{1,n}) \qquad (5.22)$$

with $\mathcal{S} = S^n$, and $d_{1,n}$ and $d_{2,n}$ given in (5.21) and (5.20), respectively.

5.1.2   The American Put Option

Although possible, we don't pursue the valuation of American power options. Instead we focus our attention to the classical problem of valuing plain vanilla options on dividend paying stocks. Therefore, $n = 1$ is fixed.

Following Kwok (1998) we extend the domain of the BSM PDE by setting $P^A(S,t) = X - S(t)$ for $S(t) < S^*(t)$. Then $P^A = P^A(S,t)$ satisfies the non-homogeneous PDE:

$$\frac{\partial P^A}{\partial t} + (r-q)\,S\,\frac{\partial P^A}{\partial S} + \frac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 P^A}{\partial S^2} - rP^A = f \qquad (5.23)$$

with

$$f = f(S,t) = \begin{cases} -rX + qS\,, & \text{for } 0 < S \le S^*(t) \\ 0\,, & \text{for } S > S^*(t) \end{cases} \qquad (5.24)$$

on $(0,\infty) \times [0,T)$. Furthermore, we have the boundary conditions

$$\lim_{S\to\infty} P^A(S,t) = 0 \qquad \text{on } [0,T)\,, \qquad (5.25)$$

$$P^A(S,T) = \theta(S) = (X - S_T)^+ \qquad \text{on } [0,\infty) \qquad (5.26)$$

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and

$$P^A(0,t) = X \qquad \text{on} \quad [0,T]. \tag{5.27}$$

The "smooth pasting conditions" at $S^*(t)$ are:

$$P^A(S^*,t) = X - S^*(t) \qquad \text{and} \qquad \frac{\partial P^A}{\partial S}\bigg|_{S(t)=S^*(t)} = -1. \tag{5.28}$$

The Mellin transform of (5.23) is given by

$$\frac{\partial \widetilde{P}^A(\omega,t)}{\partial t} + \frac{1}{2}\sigma^2\big[\omega^2 + \omega(1-\kappa_2) - \kappa_1\big]\widetilde{P}^A(\omega,t) = \widetilde{f}(\omega,t) \tag{5.29}$$

where $\kappa_2 = \frac{2(r-q)}{\sigma^2}$ and $\kappa_1 = \frac{2r}{\sigma^2}$, and

$$\widetilde{f}(\omega,t) = \int_0^\infty f(S,t)\, S^{\omega-1}dS = -\frac{rX}{\omega}(S^*(t))^\omega + \frac{q}{\omega+1}(S^*(t))^{\omega+1}. \tag{5.30}$$

The general solution to this non-homogeneous ODE is given by

$$
\begin{aligned}
\widetilde{P}^A(\omega,t) \;=\; & c(\omega)e^{-\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot t} \\
& + \int_t^T \frac{rX}{\omega}(S^*(x))^\omega e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(x-t)}\,dx \\
& - \int_t^T \frac{q}{\omega+1}(S^*(x))^{\omega+1} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(x-t)}\,dx \\
\;=\; & \widetilde{\theta}(\omega)e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(T-t)} \\
& + \int_t^T \frac{rX}{\omega}(S^*(x))^\omega e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(x-t)}\,dx \\
& - \int_t^T \frac{q}{\omega+1}(S^*(x))^{\omega+1} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(x-t)}\,dx
\end{aligned} \tag{5.31}
$$

where $Q(\omega)$ is defined in equation (5.10) and $\widetilde{\theta}(\omega)$ is the terminal condition given in equation (5.12). Again, Mellin inversion of (5.31) yields

$$
\begin{aligned}
P^A(S,t) = \; & \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega) \cdot e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(T-t)}\, S^{-\omega}\,d\omega \\
& + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(x-t)}\,dx\,d\omega \\
& - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega+1}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega)\cdot(x-t)}\,dx\,d\omega.
\end{aligned} \tag{5.32}
$$

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Now, notice that the first term in equation (5.32) is the plain vanilla European put price from (5.13) and the last two terms capture the early exercise premium. Therefore, we finally arrive at

Theorem 5.1.1 *The American put option $P^A(S,t)$ satisfies the decomposition*

$$
\begin{aligned}
P^A(S,t) = P^E(S,t) \\
+ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{rX}{\omega} \left(\frac{S}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (x-t)} \, dx d\omega \\
- \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{qS^*(x)}{\omega+1} \left(\frac{S}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (x-t)} \, dx d\omega \, ,
\end{aligned} \quad (5.33)
$$

where $(S,t) \in (0,\infty) \times [0,T)$, $c \in (0,\infty)$, $\{\omega \in \mathbb{C} \,|\, 0 < Re(\omega) < \infty\}$, and

$$
Q(\omega) = \omega^2 + \omega(1 - \kappa_2) - \kappa_1 \, ,
$$

with $\kappa_2 = \frac{2(r-q)}{\sigma^2}$, and $\kappa_1 = \frac{2r}{\sigma^2}$. The implicit equation for the free boundary is given by

$$
\begin{aligned}
X - S^*(t) = P^E(S^*(t), t) \\
+ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{rX}{\omega} \left(\frac{S^*(t)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (x-t)} \, dx d\omega \\
- \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{qS^*(x)}{\omega+1} \left(\frac{S^*(t)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (x-t)} \, dx d\omega.
\end{aligned} \quad (5.34)
$$

We point out that equation (5.34) can be used to recover the asymptotics regarding the optimal exercise price of American put options at expiry. The first part of the proof below partially follows Chiarella et al. (2004).

Proposition 5.1.2 *If $t \to T$ the free boundary of the American put satisfies*

$$
\lim_{t \to T} S^*(t) = \min\left(X, \frac{r}{q}X\right). \quad (5.35)
$$

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PROOF: Change the time variable, $t \mapsto \tau = T - t$, to obtain

$$X - S^*(\tau) = P^E(S^*(\tau), \tau)$$

$$+ \frac{1}{2\pi i} \int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{rX}{\omega} \left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)} \, d\omega dx$$

$$- \frac{1}{2\pi i} \int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{qS^*(x)}{\omega+1} \left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)} \, d\omega dx . \quad (5.36)$$

A simple factorization and rearrangement produces the following implicit equation for $S^*(\tau)$:

$$\frac{S^*(\tau)}{X} = \frac{1 - e^{-r\tau} + e^{-r\tau} N(d_2(S^*(\tau), X, \tau)) - r \cdot I_1(\tau)}{1 - e^{-q\tau} + e^{-q\tau} N(d_1(S^*(\tau), X, \tau)) - q \cdot I_2(\tau)} \quad (5.37)$$

where

$$I_1(\tau) = \frac{1}{2\pi i} \int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega} \left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)} \, d\omega dx \quad (5.38)$$

and

$$I_2(\tau) = \frac{1}{2\pi i} \int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega+1} \left(\frac{S^*(\tau)}{S^*(x)}\right)^{-(\omega+1)} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)} \, d\omega dx . \quad (5.39)$$

Notice first that the critical stock price is bounded from above, i.e. $S^*(\tau) \leq X, \forall \tau > 0$ (see for example Jacka (1991), Prop. 2.2.2). To find the value $S^*(0^+) = \lim_{\tau \to 0^+} S^*(\tau)$, in a first step we evaluate the limits involving d_1 and d_2. We have

$$\lim_{\tau \to 0^+} d_1(S^*(\tau), X, \tau) = \begin{cases} 0, & \text{for } S^*(0^+) = X \\ -\infty, & \text{for } S^*(0^+) < X . \end{cases}$$

Similarly,

$$\lim_{\tau \to 0^+} d_2(S^*(\tau), X, \tau) = \begin{cases} 0, & \text{for } S^*(0^+) = X \\ -\infty, & \text{for } S^*(0^+) < X . \end{cases}$$

If $\lim_{\tau \to 0^+} S^*(\tau) = X$ we have

$$\lim_{\tau \to 0^+} N(d_1(S^*(\tau), X, \tau)) = \lim_{\tau \to 0^+} N(d_2(S^*(\tau), X, \tau)) = \frac{1}{2}$$

and
$$\lim_{\tau \to 0^+} \frac{S^*(\tau)}{X} = \frac{\frac{1}{2} - \lim_{\tau \to 0^+} r \cdot I_1(\tau)}{\frac{1}{2} - \lim_{\tau \to 0^+} q \cdot I_2(\tau)}. \tag{5.40}$$

It is easily verified that both expressions $I_1(\tau)$ and $I_2(\tau)$ tend to zero as $\tau \to 0^+$. As a result we have $\lim_{\tau \to 0^+} S^*(\tau) = X$ being a possible solution. In the second case where $\lim_{\tau \to 0^+} S^*(\tau) < X$, the implicit equation for $S^*(\tau)$ reads

$$\lim_{\tau \to 0^+} \frac{S^*(\tau)}{X} = \frac{r}{q} \cdot \lim_{\tau \to 0^+} \frac{I_1(\tau)}{I_2(\tau)}. \tag{5.41}$$

But

$$I_1(\tau) = \int_0^\tau \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega} \left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)} \, d\omega dx$$

and a simple application of the residue theorem (see for example Freitag and Busam (2000)) shows that the inner integral equals

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega} \left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)} \, d\omega = e^{-r(\tau - x)} \tag{5.42}$$

and thus

$$I_1(\tau) = \frac{1}{r}\left(1 - e^{-r\tau}\right). \tag{5.43}$$

In the same manner we apply the residue theorem to the second integral to get

$$I_2(\tau) = \frac{1}{q}\left(1 - e^{-q\tau}\right). \tag{5.44}$$

Obviously, the above calculations can be used to prove the limits in the first case, i.e. for $\lim_{\tau \to 0^+} S^*(\tau) = X$, as well. Putting the results together we arrive at

$$\lim_{\tau \to 0^+} \frac{S^*(\tau)}{X} = \frac{r}{q} \cdot \lim_{\tau \to 0^+} \frac{\frac{1}{r}\left(1 - e^{-r\tau}\right)}{\frac{1}{q}\left(1 - e^{-q\tau}\right)} = \lim_{\tau \to 0^+} \frac{1 - e^{-r\tau}}{1 - e^{-q\tau}}. \tag{5.45}$$

Now, use the rule of l'Hospital to establish the second assertion. Recalling that the result holds only when $S^*(0^+) < X$, it follows that $r < q$. Combining both results confirms Kim's formula. $\qquad\square$

5.1.3 The Equivalence of Integral Representations

In this section we prove explicitly the equivalence of three types of integral representations for American put options[17]. We show the equivalence of the integral representation derived herein, the representation obtained by Kim (1990), Jacka (1991), and Carr et al. (1992).

Proposition 5.1.3 *The following three representations for American put options are equivalent:*

- *Representation derived herein using Mellin transforms*

$$P^A(S,t) = P^E(S,t)$$
$$+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\left(\frac{S}{S^*(x)}\right)^{-\omega}e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}dxd\omega$$
$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega+1}\left(\frac{S}{S^*(x)}\right)^{-\omega}e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}dxd\omega \quad (5.46)$$

  with $Q(\omega)$ given in (5.10).

- *Representation obtained by Kim (1990) and Jacka (1991)*

$$P^A(S,\tau) = P^E(S,\tau)$$
$$+\int_0^\tau rXe^{-r(\tau-\xi)}\cdot N(-d_2(S,S^*(\xi),\tau-\xi))d\xi$$
$$-\int_0^\tau qSe^{-q(\tau-\xi)}\cdot N(-d_1(S,S^*(\xi),\tau-\xi))d\xi \quad (5.47)$$

  where $\tau = T - t, S = S(\tau), S \geq S^*(\tau)$, and

$$d_1(x,y,t) = \frac{\ln\frac{x}{y} + (r-q-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$

$$d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.$$

---

[17]Chiarella et al. (2004) use the incomplete Fourier transform to survey the integral representations of American call options.

- *Decomposition derived by Carr et al. (1992))*

$$P^A(S, \tau) = max(X - S, 0) + \frac{1}{2}\sigma^2 S \int_0^\tau \frac{e^{-q(\tau - \xi)}}{\sigma\sqrt{\tau - \xi}} \cdot N'(-d_1(S, X, \tau - \xi))d\xi$$

$$+ \int_0^\tau rXe^{-r(\tau - \xi)}\Big[N(-d_2(S, S^*(\xi), \tau - \xi)) - N(-d_2(S, X, \tau - \xi))\Big]d\xi$$

$$- \int_0^\tau qSe^{-q(\tau - \xi)}\Big[N(-d_1(S, S^*(\xi), \tau - \xi)) - N(-d_1(S, X, \tau - \xi))\Big]d\xi \quad (5.48)$$

where $\tau = T - t, S = S(\tau), S \geq S^*(\tau)$, and d_1 and d_2 as above.

PROOF: A change of the time variable in the "Mellin representation" $t \mapsto \tau = T - t$ leads to

$$P^A(S, \tau) = P^E(S, \tau)$$
$$+ \frac{1}{2\pi i} \int_0^\tau \int_{c-i\infty}^{c+i\infty} rX\frac{1}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)}d\omega dx$$
$$- \frac{1}{2\pi i} \int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{qS^*(x)}{\omega + 1}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)}d\omega dx$$

or using a more compact form

$$P^A(S, \tau) = P^E(S, \tau) - \int_0^\tau \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(\omega, x) \cdot \widetilde{\phi}(\omega, x) \cdot S^{-\omega} d\omega dx$$

with

$$\widetilde{f}(\omega, x) = -\frac{rX}{\omega}(S^*(x))^\omega + \frac{q}{\omega + 1}(S^*(x))^{\omega + 1}$$

and

$$\widetilde{\phi}(\omega, x) = e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)}$$

the Mellin transforms of $f(S, x)$ and $\phi(S, x)$, respectively. Using the convolution theorem we can write

$$P^A(S, \tau) = P^E(S, \tau) - \int_0^\tau \int_0^\infty f(u, x) \cdot \phi\Big(\frac{S}{u}, x\Big) \cdot \frac{1}{u} du\, dx.$$

Now, we have

$$P^A(S, \tau) = P^E(S, \tau) - \int_0^\tau h(S, x)dx \quad (5.49)$$

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where

$$h(S,x) = -rXe^{-\beta_1(\beta_2^2+\kappa_1)}\frac{S^{\beta_2}}{\sigma\sqrt{2\pi(\tau-x)}}\int_0^{S^*(x)}\frac{1}{u^{\beta_2+1}}e^{-\frac{1}{2}\left(\frac{\ln\frac{S}{u}}{\sigma\sqrt{\tau-x}}\right)^2}du$$

$$+qe^{-\beta_1(\beta_2^2+\kappa_1)}\frac{S^{\beta_2}}{\sigma\sqrt{2\pi(\tau-x)}}\int_0^{S^*(x)}\frac{1}{u^{\beta_2}}e^{-\frac{1}{2}\left(\frac{\ln\frac{S}{u}}{\sigma\sqrt{\tau-x}}\right)^2}du, \quad (5.50)$$

and $\beta_1 = \frac{1}{2}\sigma^2(\tau-x)$, $\beta_2 = \frac{1-\kappa_2}{2}$, $\kappa_1 = \frac{2r}{\sigma^2}$, and $\kappa_2 = \frac{2(r-q)}{\sigma^2}$. Transforming variables

$$\gamma = \frac{1}{\sigma\sqrt{\tau-x}}\left(\ln\left(\frac{S}{u}\right)-\beta\sigma^2(\tau-x)\right) \quad (5.51)$$

for the first integral in (5.50), and

$$\gamma = \frac{1}{\sigma\sqrt{\tau-x}}\left(\ln\left(\frac{S}{u}\right)-(\beta-1)\sigma^2(\tau-x)\right) \quad (5.52)$$

for the second yields to

$$h(S,x) = -rX\,e^{-r(\tau-x)}\cdot N(-d_2(S,S^*(x),\tau-x))$$

$$+qS\,e^{-q(\tau-x)}\cdot N(-d_1(S,S^*(x),\tau-x)). \quad (5.53)$$

Finally, change the variables from x to ξ and the equivalence of (5.46) and (5.47) follows.

For the second equivalence, observe that we can write the European put as

$$P^E(S,\tau) = X\cdot H(X-S) - X\cdot H(X-S)$$

$$+Xe^{-r\tau}N(-d_2(S,X,\tau)) - Se^{-q\tau}N(-d_1(S,X,\tau)),(5.54)$$

where $H(x)$ is the Heaviside step function given by

$$H(x) = \begin{cases} 1, & \text{for } x > 0 \\ \dfrac{1}{2}, & \text{for } x = 0 \\ 0, & \text{for } x < 0. \end{cases}$$

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The reason for the appearance of the factor $1/2$ at the point of discontinuity will become obvious below. Given the limit result that

$$\lim_{\tau \to 0} d_1(S, X, \tau) = \lim_{\tau \to 0} d_2(S, X, \tau) = \begin{cases} \infty & \text{for } S > X \\ 0 & \text{for } S = X \\ -\infty & \text{for } S < X \end{cases}$$

we can express $P^E(S, \tau)$ as

$$
\begin{aligned}
P^E(S, \tau) &= X \cdot H(X - S) - S\,e^{-q\tau} N(-d_1(S, X, \tau)) + \left[X\,e^{-r\xi} N(-d_2(S, X, \xi)) \right]\Big|_0^\tau \\
&= X \cdot H(X - S) - S\,e^{-q\tau} N(-d_1(S, X, \tau)) \\
&\quad + X \int_0^\tau \left[e^{-r\xi} N'(-d_2(S, X, \xi)) \cdot \frac{\partial}{\partial \xi}(-d_2(S, X, \xi)) \right. \\
&\qquad\qquad \left. -r\,e^{-r\xi} N(-d_2(S, X, \xi)) \right] d\xi \\
&= X \cdot H(X - S) - S\,e^{-q\tau} N(-d_1(S, X, \tau)) \\
&\quad -r X \int_0^\tau e^{-r\xi} N(-d_2(S, X, \xi))\, d\xi \\
&\quad + X \int_0^\tau e^{-r\xi} N'(-d_2(S, X, \xi)) \frac{\partial}{\partial \xi}\left[-(d_1(S, X, \xi) - \sigma\sqrt{\xi}) \right] d\xi \\
&= X \cdot H(X - S) - S\,e^{-q\tau} N(-d_1(S, X, \tau)) \\
&\quad -r X \int_0^\tau e^{-r\xi} N(-d_2(S, X, \xi))\, d\xi \\
&\quad + X \int_0^\tau e^{-r\xi} N'(-d_2(S, X, \xi)) \frac{\partial}{\partial \xi}(-d_1(S, X, \xi))\, d\xi \\
&\quad + X \int_0^\tau e^{-r\xi} N'(-d_2(S, X, \xi)) \frac{\sigma}{2\sqrt{\xi}}\, d\xi, \qquad\qquad (5.55)
\end{aligned}
$$

where $N'(x) = n(x)$ is the density function of a standard normal distributed random variable x. Now, we have

$$N'(-d_2(S, X, \xi)) = N'(d_2(S, X, \xi))$$

$$N'(-d_1(S, X, \xi)) = N'(d_1(S, X, \xi))$$

and

$$S e^{-q\xi} N'(d_1(S, X, \xi)) = X e^{-r\xi} N'(d_2(S, X, \xi)).$$

Thus,

$$
\begin{aligned}
P^E(S,\tau) &= (X-S)\cdot H(X-S) + S\cdot H(X-S) - S\,e^{-q\tau}N(-d_1(S,X,\tau)) \\
&\quad -r\,X\int_0^\tau e^{-r\xi}N(-d_2(S,X,\xi))\,d\xi \\
&\quad +S\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\partial}{\partial\xi}(-d_1(S,X,\xi))\,d\xi \\
&\quad +S\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\sigma}{2\sqrt{\xi}}\,d\xi \\
&= max(X-S,0) \\
&\quad +\frac{1}{2}\sigma^2 S\int_0^\tau e^{-q\xi}\,N'(-d_1(S,X,\xi))\frac{1}{\sigma\sqrt{\xi}}\,d\xi \\
&\quad -r\,X\int_0^\tau e^{-r\xi}N(-d_2(S,X,\xi))\,d\xi \\
&\quad -S\Big[e^{-q\tau}N(-d_1(S,X,\tau)) - H(X-S) \\
&\quad -\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\partial}{\partial\xi}(-d_1(S,X,\xi))\Big]\,d\xi. \qquad (5.56)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
P^E(S,\tau) &= max(X-S,0) + \frac{1}{2}\sigma^2 S\int_0^\tau e^{-q\xi}\cdot N'(-d_1(S,X,\xi))\frac{1}{\sigma\sqrt{\xi}}\,d\xi \\
&\quad -r\,X\int_0^\tau e^{-r\xi}N(-d_2(S,X,\xi))\,d\xi \\
&\quad -S\Big[\big[e^{-q\xi}N(-d_1(S,X,\xi))\big]\Big|_0^\tau \\
&\quad -\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\partial}{\partial\xi}(-d_1(S,X,\xi))\,d\xi\Big]. \qquad (5.57)
\end{aligned}
$$

Changing the integration variable from ξ to $\tau-\xi$ gives

$$
\begin{aligned}
P^E(S,\tau) &= max(X-S,0) + \frac{1}{2}\sigma^2 S\int_0^\tau \frac{e^{-q(\tau-\xi)}}{\sigma\sqrt{\tau-\xi}}\cdot N'(-d_1(S,X,\tau-\xi))\,d\xi \\
&\quad -r\,X\int_0^\tau e^{-r(\tau-\xi)}N(-d_2(S,X,\tau-\xi))\,d\xi \\
&\quad +qS\int_0^\tau e^{-q(\tau-\xi)}N(-d_1(S,X,\tau-\xi))\,d\xi. \qquad (5.58)
\end{aligned}
$$

Now, substitute this expression into Kim's representation and rearrange terms. This completes the proof. $\qquad\square$

Remark 5.1.4 *It is worth mentioning that a second proof for the first equivalence was found by the author. This proof makes no explicit use of the convolution theorem.*

SECOND PROOF: The starting point is equation (5.1.16) in Panini and Srivastav (2004). Including dividends it is straightforward to extend the result and show that equation (5.46) is equivalent to

$$P^A(S,\tau) = P^E(S,\tau) + \int_0^\tau I_1(\xi)d\xi - \int_0^\tau I_2(\xi)d\xi, \qquad (5.59)$$

where

$$I_1(\xi) = \frac{rX}{2\sqrt{\pi\bar{\zeta}}}\, e^{-r\xi}\, e^{-\bar{\zeta}c^2+\beta c} \int_0^\infty e^{-cx}\, e^{-\frac{(\beta-x)^2}{4\bar{\zeta}}}\, dx \qquad (5.60)$$

and

$$I_2(\xi) = \frac{qS^*(\tau-\xi)}{2\sqrt{\pi\bar{\zeta}}}\, e^{-r\xi}\, e^{-\bar{\zeta}c^2+\beta c} \int_0^\infty e^{-(c+1)x}\, e^{-\frac{(\beta-x)^2}{4\bar{\zeta}}}\, dx \qquad (5.61)$$

with $\xi = \tau - x$, $\bar{\zeta} = \frac{1}{2}\sigma^2\xi$ and

$$\beta = \bar{\zeta}(2c+1-\kappa_2) - \ln\left(\frac{S(\tau)}{S^*(\tau-\xi)}\right). \qquad (5.62)$$

Now, the integrals can be expressed as

$$I_1(\xi) = \frac{rX}{2\sqrt{\pi\bar{\zeta}}}\, e^{-r\xi}\, e^{-\bar{\zeta}c^2+\beta c}\, e^{-\frac{b}{4\bar{\zeta}}} \int_0^\infty e^{-\frac{x^2}{4\bar{\zeta}}}\, e^{-\frac{a_1 x}{4\bar{\zeta}}}\, dx \qquad (5.63)$$

and

$$I_2(\xi) = \frac{qS^*(\tau-\xi)}{2\sqrt{\pi\bar{\zeta}}}\, e^{-r\xi}\, e^{-\bar{\zeta}c^2+\beta c}\, e^{-\frac{b}{4\bar{\zeta}}} \int_0^\infty e^{-\frac{x^2}{4\bar{\zeta}}}\, e^{-\frac{a_2 x}{4\bar{\zeta}}}\, dx \qquad (5.64)$$

with

$$a_1 = 2\bar{\zeta}(\kappa_2 - 1) + 2\ln\left(\frac{S(\tau)}{S^*(\tau-\xi)}\right),$$

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$$a_2 = 2\bar{\zeta}(\kappa_2 + 1) + 2\ln\left(\frac{S(\tau)}{S^*(\tau - \xi)}\right),$$

and $b = \beta^2$. From Gradshteyn and Ryzhik (2007), p.336, we have

$$\int_0^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right)dx = \sqrt{\pi\beta}\exp\left(\beta\gamma^2\right)\left[1 - \Phi(\gamma\sqrt{\beta})\right] \qquad (5.65)$$

for $Re(\beta) > 0$ and where $\Phi(x)$ denotes the error function at x

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\,.$$

After simplifying the expressions for $I_1(\xi)$ and $I_2(\xi)$ become, respectively:

$$
\begin{aligned}
I_1(\xi) &= \frac{rX}{2}\left(\frac{S^*(\tau - \xi)}{S(\tau)}\right)^c e^{-r\xi}e^{\bar{\zeta}c(c+1-\kappa_2)}e^{\frac{1}{4\zeta}\left(\frac{a_1^2}{4}-b\right)}\left[1 - \Phi\left(\frac{a_1}{4\bar{\zeta}}\right)\right] \\
&= \frac{rX}{2}\left(\frac{S^*(\tau - \xi)}{S(\tau)}\right)^c\left(\frac{S^*(\tau - \xi)}{S(\tau)}\right)^{-c}e^{-r\xi}\left[1 - \Phi\left(\frac{a_1}{4\bar{\zeta}}\right)\right] \\
&= \frac{rX}{2}e^{-r\xi}\left[1 - \Phi\left(\frac{a_1}{4\bar{\zeta}}\right)\right] \qquad (5.66)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(\xi) &= \frac{qS^*(\tau - \xi)}{2}\left(\frac{S^*(\tau - \xi)}{S(\tau)}\right)^c e^{-r\xi}e^{\bar{\zeta}c(c+1-\kappa_2)}e^{\frac{1}{4\zeta}\left(\frac{a_2^2}{4}-b\right)}\left[1 - \Phi\left(\frac{a_2}{4\bar{\zeta}}\right)\right] \\
&= \frac{qS^*(\tau - \xi)}{2}\left(\frac{S^*(\tau - \xi)}{S(\tau)}\right)^c\left(\frac{S(\tau)}{S^*(\tau - \xi)}\right)^{c+1}e^{-r\xi}e^{\frac{1}{4}\bar{\zeta}[(\kappa_2+1)^2-(\kappa_2-1)^2]}\left[1 - \Phi\left(\frac{a_2}{4\bar{\zeta}}\right)\right] \\
&= \frac{qS(\tau)}{2}e^{-q\xi}\left[1 - \Phi\left(\frac{a_2}{4\bar{\zeta}}\right)\right]. \qquad (5.67)
\end{aligned}
$$

Using the connection between the error function and the cumulative standard normal distribution function

$$\Phi(x) = 2\,N(\sqrt{2}\,x) - 1 \qquad (5.68)$$

we have, respectively:

$$I_1(\xi) = rX\,e^{-r\xi}\,N\left(-\frac{a_1}{2}\frac{1}{\sqrt{2\bar{\zeta}}}\right) \qquad (5.69)$$

and

$$I_2(\xi) = qS(\tau)\,e^{-q\xi}\,N\Big(-\frac{a_2}{2}\frac{1}{\sqrt{2\zeta}}\Big). \tag{5.70}$$

Finally, observe that

$$-\frac{a_1}{2}\frac{1}{\sqrt{2\zeta}} = \frac{1}{2}\sigma\,(1-\kappa_2)\sqrt{\xi} - \frac{1}{\sigma\sqrt{\xi}}\ln\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big) \tag{5.71}$$

and

$$-\frac{a_2}{2}\frac{1}{\sqrt{2\zeta}} = -\frac{1}{2}\sigma\,(1+\kappa_2)\sqrt{\xi} - \frac{1}{\sigma\sqrt{\xi}}\ln\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big) \tag{5.72}$$

and Kim's integral representation follows immediately. This completes the second proof. $\qquad\square$

5.1.4 Perpetual American Puts and Mellin Transforms

In this section, we show how to use the Mellin transform approach to derive closed-form solutions for perpetual American put options. An extension of Panini and Srivastav (2005) to dividend-paying stocks is provided.

Proposition 5.1.5 *It $T \to \infty$ the free boundary of the perpetual American put becomes*

$$S_\infty^* = \frac{\omega_2}{\omega_2 + 1}X, \tag{5.73}$$

where

$$\omega_2 = \frac{\kappa_2 - 1}{2} + \frac{\sqrt{(\kappa_2 - 1)^2 + 4\kappa_1}}{2} \tag{5.74}$$

and the perpetual American put equals

$$P_\infty^A(S,t) = \Big(\frac{S}{S_\infty^*}\Big)^{-\omega_2}(X - S_\infty^*), \qquad for\ S > S_\infty^*. \tag{5.75}$$

PROOF: The roots of $Q(\omega)$ defined in (5.10) are given by

$$\omega_1 = \frac{\kappa_2 - 1}{2} - \frac{\sqrt{(\kappa_2 - 1)^2 + 4\kappa_1}}{2}$$

and
$$\omega_2 = \frac{\kappa_2 - 1}{2} + \frac{\sqrt{(\kappa_2 - 1)^2 + 4\kappa_1}}{2}.$$

Thus, we have
$$Q(\omega) = (\omega - \omega_1)(\omega - \omega_2)$$

with $\omega_1 \le -1 < 0 < \omega_2 \le \kappa_1$. The limiting cases $\omega_1 = -1$ and $\omega_2 = \kappa_1$ are special roots for $q = 0$. We will determine the unknown critical stock price $S^*(t)$ using the second smooth pasting condition.

Notice, that for the valuation formula (5.33) to hold as $T \to \infty$, it is necessary that $Re(Q(\omega)) < 0$, i.e. $0 < Re(\omega) < \omega_2$.

Using the second smooth pasting condition we obtain as $T \to \infty$

$$-1 = \left.\frac{\partial P^A}{\partial S}\right|_{S=S^*} = \left.\frac{\partial P^E}{\partial S}\right|_{S=S^*} + \left.\frac{\partial P_1}{\partial S}\right|_{S=S^*} + \left.\frac{\partial P_2}{\partial S}\right|_{S=S^*}$$

where the free boundary $S^* = S^*_\infty$ is now independent of time, and P_1 and P_2 denote the second and third term in the valuation formula (5.33), respectively.

Now, the delta of a European put option on a dividend-paying stock is determined as
$$\frac{\partial P^E}{\partial S} = -e^{-q(T-t)} \cdot N(-d_1(S, X, T))$$

with
$$d_1(S, X, T) = \frac{\ln \frac{S}{X} + (r - q + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

It follows that as $T \to \infty$

$$\left.\frac{\partial P^E}{\partial S}\right|_{S=S^*_\infty} \to 0.$$

Now consider the P_1 term. The limit $T \to \infty$ gives

$$
\begin{aligned}
\frac{\partial P_1}{\partial S} &= -\frac{rX}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{S} \left(\int_t^\infty \left(\frac{S}{S^*_\infty} \right)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (x-t)} \, dx \right) d\omega \\
&= -\frac{rX}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{S} \left(\frac{S}{S^*_\infty} \right)^{-\omega} \left[\frac{1}{\frac{1}{2}\sigma^2 \cdot Q(\omega)} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (x-t)} \Big|_t^\infty \right] d\omega
\end{aligned}
$$

Therefore,

$$\frac{\partial P_1}{\partial S}\Big|_{S=S_\infty^*} = \frac{\kappa_1 X}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{S_\infty^*} \cdot \frac{1}{(\omega - \omega_1)(\omega - \omega_2)}\, d\omega. \qquad (5.76)$$

Similarly, the P_2 term is determined as

$$\begin{aligned}
\frac{\partial P_2}{\partial S} &= \frac{q}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\int_t^\infty \frac{\omega}{\omega+1} \left(\frac{S}{S_\infty^*}\right)^{-(\omega+1)} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (x-t)}\, dx \right) d\omega \\
&= -\frac{2q}{\sigma^2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{\omega+1} \left(\frac{S}{S_\infty^*}\right)^{-(\omega+1)} \cdot \frac{1}{Q(\omega)}\, d\omega.
\end{aligned}$$

Therefore,

$$\frac{\partial P_2}{\partial S}\Big|_{S=S_\infty^*} = (\kappa_2 - \kappa_1) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{(\omega+1)(\omega-\omega_1)(\omega-\omega_2)}\, d\omega. \qquad (5.77)$$

To evaluate both integrals we consider the integration path (or contour path) in the complex plane outlined in the next figure. An application of the residue theorem (see Freitag and Busam (2000)) gives

$$\frac{\partial P_1}{\partial S}\Big|_{S=S_\infty^*} = \kappa_1 X \frac{1}{S_\infty^*(\omega_1 - \omega_2)} \qquad (5.78)$$

and

$$\frac{\partial P_2}{\partial S}\Big|_{S=S_\infty^*} = (\kappa_2 - \kappa_1) \left[\frac{\omega_1}{(\omega_1+1)(\omega_1-\omega_2)} - \frac{1}{(\omega_1+1)(\omega_2+1)} \right]. \qquad (5.79)$$

Finally, we get for the critical stock price[18]

$$\begin{aligned}
S_\infty^* &= \frac{\kappa_1(\omega_1+1)}{\omega_1(\kappa_1-\kappa_2)} X \\
&= \frac{\omega_2}{\omega_2+1} X.
\end{aligned} \qquad (5.80)$$

---

[18]Merton's result (1973)

$$S_\infty^* = \frac{\kappa_1}{\kappa_1+1} X$$

is obtained as a special case for $q = 0$.
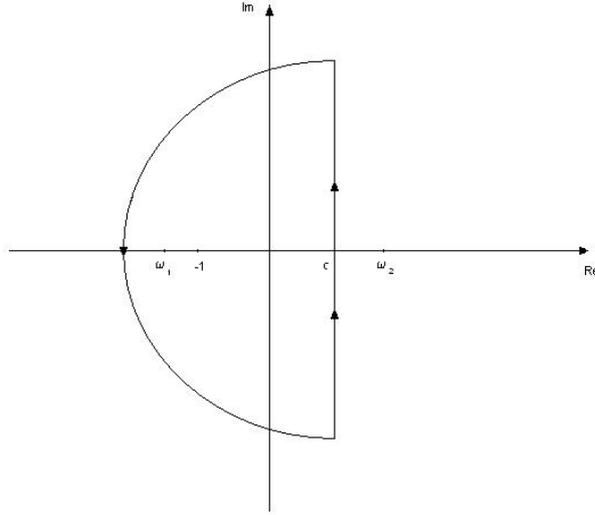
Figure 5.1: Integration path for the critical stock price of the perpetual American put option.

Observe that since $S^*(t)$ is non-decreasing in t (see Kim (1990), p. 560, Jacka (1991), Proposition 2.2.2 for a reference) we have the lower and upper bounds for $S^*(t)$ given by

$$S^*_\infty \leq S^*(t) \leq S^*(T) = \min\left(X, \frac{r}{q}X\right) \qquad \forall t \in [0, T]. \tag{5.81}$$

The price for the perpetual American put is given by

$$P^A_\infty(S, t) = -\frac{\kappa_1 X}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{S}{S^*_\infty}\right)^{-\omega} \frac{1}{\omega(\omega - \omega_1)(\omega - \omega_2)} \, d\omega$$
$$+ \frac{2q}{\sigma^2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} S^*_\infty \left(\frac{S}{S^*_\infty}\right)^{-\omega} \frac{1}{(\omega + 1)(\omega - \omega_1)(\omega - \omega_2)} \, d\omega. \tag{5.82}$$

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Once again, we apply the residue theorem to determine the first integral as

$$\left(\frac{S}{S_\infty^*}\right)^{-\omega_2} \frac{\kappa_1 X}{\omega_2(\omega_2 - \omega_1)}.$$

The second integral is evaluated as

$$-\frac{2q}{\sigma^2}\left(\frac{S}{S_\infty^*}\right)^{-\omega_2} \frac{S_\infty^*}{(\omega_2 + 1)(\omega_2 - \omega_1)}.$$

Thus, we finally get for the perpetual American put price

$$
\begin{aligned}
P_\infty^A(S,t) &= \left(\frac{S}{S_\infty^*}\right)^{-\omega_2} \frac{\kappa_1 X}{\omega_2(\omega_2 - \omega_1)} - \frac{2q}{\sigma^2}\left(\frac{S}{S_\infty^*}\right)^{-\omega_2} \frac{S_\infty^*}{(\omega_2 + 1)(\omega_2 - \omega_1)} \\
&= \left(\frac{S}{S_\infty^*}\right)^{-\omega_2} \frac{X}{\omega_2 + 1} \\
&= \left(\frac{S}{S_\infty^*}\right)^{-\omega_2} (X - S_\infty^*), \qquad \text{for } S > S_\infty^*.
\end{aligned}
\tag{5.83}
$$

This establishes the result. □

5.2 The American Call Option

This section illustrates a modification of Mellin transforms that is applicable to any call option-like derivative, i.e. any derivative with a linearly in S as S tends to infinity increasing payoff function. The modification must be done in such a way that the existence of the integral arising in the definition will be guaranteed. The modification will be used to value call options in the log-normal and square root stochastic volatility model of Heston (Chapter 6). Starting with the log-normal model, we establish some theoretical results of the option's price, and put special interest on the application of Gauss-Laguerre quadrature for an efficient numerical evaluation of American calls.

5.2.1 Modification and First Application

The objective of this section is to propose a modification of the original definition of Mellin transforms to be applicable to any call option-like derivative.

The approach is illustrated considering plain vanilla European call options but the modified definition can be used for any payoff structures with the above property.

Let $C^E(S,t)$ denote the price of a European call option. Since $C^E(S,t) = O(1)$ for $S \to 0^+$ and $C^E(S,t) = O(S)$ for $S \to \infty$ the integral arising in the definition of Mellin transforms will not exist. We therefore propose the modified Mellin transform for call options defined by

$$M(C^E(S,t), -\omega) = \widetilde{C}^E(\omega, t) := \int_0^\infty C^E(S,t)\, S^{-(\omega+1)}\, dS\,, \qquad (5.84)$$

where $1 < Re(\omega) < \infty$. Conversely, the inverse of the modified Mellin transform is given by

$$C^E(S,t) = M^{-1}\big(\widetilde{C}^E(\omega,t)\big) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{C}^E(\omega,t)\, S^\omega\, d\omega\,, \qquad (5.85)$$

with $1 < c < \infty$.

To apply the modified definition to the European call option, recall that $C^E(S,t)$ satisfies (2.10) along with the boundary conditions

$$
\begin{aligned}
C^E(S,T) &= (S(T) - X)^+ & \text{on} \quad \mathbb{R}^+ \\
C^E(0,t) &= 0 & \text{on} \quad [0,T),
\end{aligned}
$$

and

$$\lim_{S \to \infty} C^E(S,t) = \infty \qquad \text{on} \quad [0,T).$$

An application of the modified Mellin transform again converts the PDE into an ODE

$$\frac{\partial \widetilde{C}^E(\omega,t)}{\partial t} + \frac{1}{2}\sigma^2 Q(\omega)\, \widetilde{C}^E(\omega,t) = 0 \qquad (5.86)$$

where $Q(\omega)$ has a very similar structure as for put options and equals

$$Q(\omega) = \omega^2 - \omega(1 - \kappa_2) - \kappa_1\,, \qquad (5.87)$$

with $\kappa_1 = \frac{2r}{\sigma^2}$ and $\kappa_2 = \frac{2(r-q)}{\sigma^2}$. Imposing the boundary conditions the ODE is solved as

$$\widetilde{C}^E(\omega, t) = X^{-\omega+1}\Big(\frac{1}{\omega-1} - \frac{1}{\omega}\Big) \cdot e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)} \tag{5.88}$$

Hence, using (5.85), we see that the price of a European call option equals

$$C^E(S, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X^{-\omega+1}\Big(\frac{1}{\omega-1} - \frac{1}{\omega}\Big) \cdot e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)} S^\omega d\omega \tag{5.89}$$

with $(S, t) \in (0, \infty) \times [0, T]$, $c \in (1, \infty)$ a constant, $\{\omega \in \mathbb{C} \mid 1 < Re(\omega) < \infty\}$. For completeness we give an explicit proof of the equivalence of the above integral expression and the BSM solution (2.15).

Proposition 5.2.1 *Equations (5.89) and (2.15) are equivalent.*

PROOF: First, observe that

$$\begin{aligned}
C^E(S, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} S\Big(\frac{S}{X}\Big)^{\omega-1} \frac{1}{\omega-1} e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)} d\omega \\
&\quad - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X\Big(\frac{S}{X}\Big)^{\omega} \frac{1}{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)} d\omega \,.
\end{aligned}$$

Now write $\omega = c + iy, 1 < c < \infty$ and $\zeta = \frac{1}{2}\sigma^2(T-t)$ to get

$$C^E(S, t) = I_1(S, X, T-t) - I_2(S, X, T-t) \,,$$

with

$$I_1(S, X, T-t)) = Se^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)-\ln(S/X)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c-1-iy}{(c-1)^2+y^2} e^{-\zeta y^2+iy\alpha} \, dy \,,$$

where we have set

$$\alpha = \ln\Big(\frac{S}{X}\Big) + \zeta(2c + \kappa_2 - 1).$$

Similarly,

$$I_2(S, X, T-t) = Xe^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c-iy}{c^2+y^2} e^{-\zeta y^2+iy\alpha} \, dy \,.$$

Using Euler's theorem for the complex valued exponential function $e^{ix} = \cos(x) + i\sin(x)$ we can simplify further and get

$$I_1(S,X,T-t) = Xe^{-r(T-t)+\zeta c^2 + c(\alpha - 2c\zeta)}\frac{1}{\pi}\int_0^\infty e^{-\zeta y^2}\frac{(c-1)\cos(\alpha y) + y\sin(\alpha y)}{(c-1)^2 + y^2}\,dy\,,$$

and

$$I_2(S,X,T-t) = rXe^{-r(T-t)+\zeta c^2 + c(\alpha - 2c\zeta)}\frac{1}{\pi}\int_0^\infty e^{-\zeta y^2}\frac{c\cos(\alpha y) + y\sin(\alpha y)}{c^2 + y^2}\,dy\,,$$

where we have used that $\cos(x)$ and $\sin(x)$ are even and odd functions, respectively. From Gradshteyn and Ryzhik (2007), p. 504, we have: For $a > 0$, $Re(\beta) > 0$, and $Re(\gamma) > 0$:

$$\int_0^\infty e^{-\beta x^2}\sin(ax)\frac{x\,dx}{\gamma^2 + x^2} = -\frac{\pi}{4}e^{\beta\gamma^2}\left[2\sinh a\gamma + e^{-\gamma a}\Phi\left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}}\right)\right.$$
$$\left. - e^{\gamma a}\Phi\left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}}\right)\right] \qquad (5.90)$$

and

$$\int_0^\infty e^{-\beta x^2}\cos(ax)\frac{dx}{\gamma^2 + x^2} = \frac{\pi}{4\gamma}e^{\beta\gamma^2}\left[2\cosh a\gamma - e^{-\gamma a}\Phi\left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}}\right)\right.$$
$$\left. - e^{\gamma a}\Phi\left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}}\right)\right] \qquad (5.91)$$

where $\Phi(x)$ is the error function. Inserting $\beta = \zeta$, $a = \alpha$, $\gamma = c - 1$ and $\gamma = c$, respectively, and simplifying gives

$$\begin{aligned} I_1(S,X,T-t) &= Xe^{-r(T-t)+\zeta c^2 + c(\alpha - 2c\zeta)}\frac{1}{2}e^{\zeta(c-1)^2} \\ &\cdot\left(\cosh((c-1)\alpha) - \sinh((c-1)\alpha) - e^{-(c-1)\alpha}\Phi\left((c-1)\sqrt{\zeta} - \frac{\alpha}{2\sqrt{\zeta}}\right)\right) \\ &= Xe^{-r(T-t)+\zeta c^2 + c(\alpha - 2c\zeta)}e^{\zeta(c-1)^2 - (c-1)\alpha}\frac{1}{2}\left(1 - \Phi\left((c-1)\sqrt{\zeta} - \frac{\alpha}{2\sqrt{\zeta}}\right)\right), \end{aligned}$$

where in the last step we have used the relation $\cosh(x) - \sinh(x) = e^{-x}$. In the same manner we obtain for $I_2(S,X,T-t)$

$$I_2(S,X,T-t) = Xe^{-r(T-t)+\zeta c^2 + c(\alpha - 2c\zeta)}e^{\zeta c^2 - c\alpha}\frac{1}{2}\left(1 - \Phi\left(c\sqrt{\zeta} - \frac{\alpha}{2\sqrt{\zeta}}\right)\right).$$

Now, the exponentials can be simplified further to get

$$I_1(S, X, T-t) = Se^{-q(T-t)}\frac{1}{2}\left(1 - \Phi\left((c-1)\sqrt{\zeta} - \frac{\alpha}{2\sqrt{\zeta}}\right)\right),$$

and

$$I_2(S, X, T-t) = Xe^{-r(T-t)}\frac{1}{2}\left(1 - \Phi\left(c\sqrt{\zeta} - \frac{\alpha}{2\sqrt{\zeta}}\right)\right).$$

The final step in our proof is to use the connection between the error function $\Phi(x)$ and the normal distribution function $N(x)$ from the last section given by the relation $\Phi(x) = 2N(\sqrt{2}x) - 1$, and observing that

$$\frac{\alpha}{\sqrt{2\zeta}} - (c-1)\sqrt{2\zeta} = \frac{\ln\left(\frac{S}{X}\right) + \zeta(\kappa_2 + 1)}{\sigma\sqrt{T-t}} = d_1(S, X, T-t),$$

and

$$\frac{\alpha}{\sqrt{2\zeta}} - c\sqrt{2\zeta} = \frac{\ln\left(\frac{S}{X}\right) + \zeta(\kappa_2 - 1)}{\sigma\sqrt{T-t}} = d_2(S, X, T-t).$$

This completes the proof. $\qquad\qquad\square$

5.2.2   The American Call Option

To derive a decomposition formula for the American call $C^A(S, t)$, similarly to the previous section we will use Mellin transforms to solve the resulting non-homogeneous PDE

$$\frac{\partial C^A}{\partial t} + (r - q)\, S\, \frac{\partial C^A}{\partial S} + \frac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 C^A}{\partial S^2} - rC^A = f \qquad (5.92)$$

with

$$f = f(S, t) = \begin{cases} rX - qS & \text{for} & S^*(t) \le S(t) < \infty \\ 0 & \text{for} & 0 < S(t) < S^*(t) \end{cases} \qquad (5.93)$$

on $(0, \infty) \times [0, T]$ with the boundary conditions

$$\lim_{S \to \infty} C^A(S, t) = \infty \qquad \text{on} \quad [0, T), \qquad (5.94)$$

$$C^A(S,T) = \theta(S) = \max(S(T) - X, 0) \qquad \text{on} \quad [0,\infty) \tag{5.95}$$

and

$$C^A(0,t) = 0 \qquad \text{on} \quad [0,T). \tag{5.96}$$

In a similar manner as for American puts, the "smooth pasting conditions" at $S^*(t)$ are:

$$C^A(S^*,t) = S^*(t) - X \qquad \text{and} \qquad \frac{\partial C^A}{\partial S}\Big|_{S(t)=S^*(t)} = 1. \tag{5.97}$$

The resulting non-homogeneous ODE after transformation equals

$$\frac{\partial \widetilde{C}^A(\omega,t)}{\partial t} + \frac{1}{2}\sigma^2 Q(\omega)\,\widetilde{C}^A(\omega,t) = \widetilde{f}(\omega,t) \tag{5.98}$$

where

$$\widetilde{f}(\omega,t) = \frac{rX}{\omega}(S^*(t))^{-\omega} - \frac{q}{\omega - 1}(S^*(t))^{-\omega+1}, \tag{5.99}$$

and $Q(\omega)$ from above. Solving the ODE similar to the American put option case results in

Theorem 5.2.2 *The American call option $C^A(S,t)$ can be expressed as*

$$
\begin{aligned}
C^A(S,t) &= C^E(S,t) \\
&+ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega - 1}\left(\frac{S}{S^*(x)}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx\,d\omega \\
&- \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\left(\frac{S}{S^*(x)}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx\,d\omega.
\end{aligned} \tag{5.100}
$$

where $(S,t) \in (0,\infty) \times [0,T)$, $c \in (1,\infty)$, $\{\omega \in \mathbb{C} \,|\, 1 < Re(\omega) < \infty\}$, and with $Q(\omega)$ from (5.87). In a similar manner we have for the free boundary

$$
\begin{aligned}
S^*(t) - X &= C^E(S^*(t),t) \\
&+ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega - 1}\left(\frac{S^*(t)}{S^*(x)}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx\,d\omega \\
&- \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\left(\frac{S^*(t)}{S^*(x)}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx\,d\omega.
\end{aligned} \tag{5.101}
$$

For completeness we prove the following proposition:

Proposition 5.2.3 *Equation (5.100) is equivalent to the following integral representation derived by Kim (1990)*

$$
\begin{aligned}
C^A(S, \tau) &= C^E(S, \tau) \\
&+ \int_0^\tau qS\, e^{-q(\tau - \xi)}\, N(d_1(S, S^*(\xi), \tau - \xi))\, d\xi \\
&- \int_0^\tau rX\, e^{-r(\tau - \xi)}\, N(d_2(S, S^*(\xi), \tau - \xi))\, d\xi \quad (5.102)
\end{aligned}
$$

where $\tau = T - t$, $S = S(\tau)$, $S \leq S^*(\tau)$, and the arguments d_1 and d_2 are given in (2.13) and (2.14), respectively.

PROOF: A direct proof of the equivalence is similar to that one presented in the previous subsection so we just give the main idea. Set $\tau = T - t$ and $\xi = \tau - x$ and write for the American call price

$$
C^A(S, \tau) = C^E(S, \tau) + \int_0^\tau I_1(\xi)\, d\xi - \int_0^\tau I_2(\xi)\, d\xi, \quad (5.103)
$$

with

$$
I_1(\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{qS^*(\tau - \xi)}{\omega - 1} \left(\frac{S(\tau)}{S^*(\tau - \xi)} \right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} d\omega \quad (5.104)
$$

and

$$
I_2(\xi) = \frac{rX}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega} \left(\frac{S(\tau)}{S^*(\tau - \xi)} \right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} d\omega. \quad (5.105)
$$

Now, with $\omega = c + iy$, $1 < c < \infty$ and $\zeta = \frac{1}{2}\sigma^2\xi$ we have

$$
I_1(\xi) = qS^*(\tau - \xi)e^{-r\xi + \zeta c^2 + c(\alpha - 2c\zeta)} \frac{1}{2\pi} \int_{-\infty}^\infty \frac{c - 1 - iy}{(c - 1)^2 + y^2} e^{-\zeta y^2 + iy\alpha}\, dy, \quad (5.106)
$$

where we have set

$$
\alpha = \ln \left(\frac{S(\tau)}{S^*(\tau - \xi)} \right) + \zeta(2c + \kappa_2 - 1). \quad (5.107)
$$

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Similarly,

$$I_2(\xi) = rXe^{-r\xi+\zeta c^2+c(\alpha-2c\zeta)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c-iy}{c^2+y^2}e^{-\zeta y^2+iy\alpha}\,dy\,. \tag{5.108}$$

From now on the argumentation goes along the same lines as in the proof for the European call and straightforward calculations establish the result. $\square$

5.2.3 Further Analysis and Applications

The following two theoretical properties of the price functions in Theorem 5.2.2 follow immediately:

Proposition 5.2.4 *If $t \to T$ the free boundary of the American call satisfies*

$$\lim_{t\to T} S^*(t) = \max\left(X, \frac{r}{q}X\right). \tag{5.109}$$

PROOF: The proof uses the same arguments as in the put option case and the statement can be verified straightforwardly. $\square$

As a final application, we show in detail how the new integral representation can be used for a closed-form valuation of perpetual American calls. Although the derivation is similar to the put option case, special interest must be put for a call on a non-dividend paying stock.

Proposition 5.2.5 *If $T \to \infty$ the free boundary of the perpetual American call option is given by*

$$S^*_\infty = X\frac{\omega_1}{\omega_1 - 1}, \tag{5.110}$$

where

$$\omega_1 = \frac{1 - \kappa_2}{2} + \frac{\sqrt{(1 - \kappa_2)^2 + 4\kappa_1}}{2}, \tag{5.111}$$

and the closed-form solution for the perpetual American call option equals

$$C^A_\infty(S,t) = \left(\frac{S}{S^*_\infty}\right)^{\omega_1}(S^*_\infty - X), \qquad for\ S < S^*_\infty. \tag{5.112}$$

PROOF: The roots of $Q(\omega)$ defined in (5.87) are given by

$$\omega_{1/2} = \frac{1 - \kappa_2}{2} \pm \frac{\sqrt{(1 - \kappa_2)^2 + 4\kappa_1}}{2} \ .$$

Thus, we have $Q(\omega) = (\omega - \omega_1)(\omega - \omega_2)$ with $-\kappa_1 \leq \omega_2 \leq 0$ and $1 \leq \omega_1 < \infty$. Again, the limiting cases $\omega_1 = 1$ and $\omega_2 = -\kappa_1$ are special roots for $q = 0$. We will determine the unknown critical stock price $S^*(t)$ using the second smooth pasting condition.

For (5.100) to hold as $T \to \infty$, it is necessary that $Re(Q(\omega)) < 0$, i.e. $1 < Re(\omega) < \omega_1$.

Using the second smooth pasting condition we obtain as $T \to \infty$

$$1 = \left.\frac{\partial C^A}{\partial S}\right|_{S=S^*} = \left.\frac{\partial C^E}{\partial S}\right|_{S=S^*} + \left.\frac{\partial C_1}{\partial S}\right|_{S=S^*} + \left.\frac{\partial C_2}{\partial S}\right|_{S=S^*}, \qquad (5.113)$$

where the free boundary $S^* = S^*_\infty$ is now independent of time, and C_1 and C_2 denote the second and third term in (5.100), respectively.

The first summand is the delta of a European call option on a dividend-paying stock and equals

$$\frac{\partial C^E}{\partial S} = e^{-q(T-t)} N(d_1(S, X, T - t))$$

with $d_1(S, X, T - t)$ given in (2.13). It follows[19] that as $T \to \infty$

$$\left.\frac{\partial C^E}{\partial S}\right|_{S=S^*_\infty} \to 0.$$

Now consider the C_1 term. The limit $T \to \infty$ gives

$$\frac{\partial C_1}{\partial S} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^\infty \frac{q\omega}{\omega - 1} \left(\frac{S}{S^*_\infty}\right)^{\omega - 1} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)} \, dx \, d\omega.$$

---

[19]Note that this is not true if $q = 0$. In this case we have

$$\left.\frac{\partial C^E}{\partial S}\right|_{S=S^*_\infty} \to 1.$$

Therefore
$$\frac{\partial C_1}{\partial S}\bigg|_{S=S_\infty^*} = \frac{\kappa_2 - \kappa_1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{(\omega - 1)(\omega - \omega_1)(\omega - \omega_2)} \, d\omega. \qquad (5.114)$$

Similarly, the C_2 term is determined as
$$\frac{\partial C_2}{\partial S} = -\frac{rX}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^\infty \frac{1}{S}\left(\frac{S}{S_\infty^*}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)} \, dx \, d\omega,$$

and we have
$$\frac{\partial C_2}{\partial S}\bigg|_{S=S_\infty^*} = \kappa_1 \frac{X}{S_\infty^*} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{(\omega - \omega_1)(\omega - \omega_2)} \, d\omega. \qquad (5.115)$$

An application of the residue theorem gives
$$\frac{\partial C_1}{\partial S}\bigg|_{S=S_\infty^*} = (\kappa_2 - \kappa_1)\left(\frac{1}{(1 - \omega_1)(1 - \omega_2)} + \frac{\omega_2}{(\omega_2 - 1)(\omega_2 - \omega_1)}\right)$$

and
$$\frac{\partial C_2}{\partial S}\bigg|_{S=S_\infty^*} = \kappa_1 \frac{X}{S_\infty^*} \frac{1}{(\omega_2 - \omega_1)}.$$

Finally, we get for the critical stock price
$$S_\infty^* = X\frac{\kappa_1}{\omega_2 + \kappa_1} = X\frac{\omega_1}{\omega_1 - 1}. \qquad (5.116)$$

Now, the perpetual American call can be expressed as
$$\begin{aligned}
C_\infty^A(S,t) &= \frac{\kappa_2 - \kappa_1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{S}{S_\infty^*}\right)^\omega \frac{S_\infty^*}{(\omega - 1)(\omega - \omega_1)(\omega - \omega_2)} \, d\omega \\
&\quad + \kappa_1 \frac{X}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{S}{S_\infty^*}\right)^\omega \frac{1}{\omega(\omega - \omega_1)(\omega - \omega_2)} \, d\omega.
\end{aligned}$$

Another application of the residue theorem gives us the closed-form solution for the perpetual American call option:
$$\begin{aligned}
C_\infty^A(S,t) &= \left(\frac{S}{S_\infty^*}\right)^{\omega_1} \frac{X}{\omega_1 - 1} \\
&= \left(\frac{S}{S_\infty^*}\right)^{\omega_1} (S_\infty^* - X).
\end{aligned}$$

This completes the proof. $\qquad \square$

Remark 5.2.6 *Note that for $q = 0$ the critical stock price of the perpetual American call option becomes infinite and $C_\infty^A(S,t) = S(t)$.*

5.2.4 Numerical Experiments

It is shown how to use Gauss-Laguerre quadrature for an efficient and accurate pricing of American call options.

From the previous analysis we have

$$C^A(S, \tau) = C^E(S, \tau) + \int_0^\tau I_1(\xi) \, d\xi - \int_0^\tau I_2(\xi) \, d\xi, \qquad (5.117)$$

with $\tau = T - t$,

$$I_1(\xi) = qS^*(\tau - \xi)e^{-r\xi - \zeta c^2 + c\alpha} \frac{1}{\pi} \int_0^\infty e^{-\zeta y^2} \frac{(c-1)\cos(\alpha y) + y\sin(\alpha y)}{(c-1)^2 + y^2} \, dy, \qquad (5.118)$$

and

$$I_2(\xi) = rX e^{-r\xi - \zeta c^2 + c\alpha} \frac{1}{\pi} \int_0^\infty e^{-\zeta y^2} \frac{c\cos(\alpha y) + y\sin(\alpha y)}{c^2 + y^2} \, dy, \qquad (5.119)$$

where again we have set

$$\alpha = \ln\left(\frac{S(\tau)}{S^*(\tau - \xi)}\right) + \zeta(2c + \kappa_2 - 1). \qquad (5.120)$$

From Gradshteyn and Ryzhik (2007), p. 228 and p. 229, we have:

$$\int e^{ax} \sin(bx) \, dx = \frac{e^{ax}\big(a\sin(bx) - b\cos(bx)\big)}{a^2 + b^2} \qquad (5.121)$$

and

$$\int e^{ax} \cos(bx) \, dx = \frac{e^{ax}\big(a\cos(bx) + b\sin(bx)\big)}{a^2 + b^2} \qquad (5.122)$$

so the equations for $I_1(\xi)$ and $I_2(\xi)$ become, respectively:

$$\begin{aligned}
I_1(\xi) &= qS^*(\tau - \xi)e^{-r\xi - \zeta c^2 + c\alpha} \frac{1}{\pi}\bigg(\int_0^\infty \int_0^\infty e^{-\zeta y^2} e^{-(c-1)x} \cos(\alpha y)\cos(xy) \, dx dy \\
&\quad + \int_0^\infty \int_0^\infty e^{-\zeta y^2} e^{-(c-1)x} \sin(\alpha y)\sin(xy) \, dx dy\bigg),
\end{aligned}$$

and

$$I_2(\xi) = rXe^{-r\xi-\zeta c^2+c\alpha}\frac{1}{\pi}\left(\int_0^\infty\int_0^\infty e^{-\zeta y^2}e^{-cx}\cos(\alpha y)\cos(xy)\,dxdy\right.$$
$$\left. +\int_0^\infty\int_0^\infty e^{-\zeta y^2}e^{-cx}\sin(\alpha y)\sin(xy)\,dxdy\right).$$

Now, we use product rules for the sine and cosine function, respectively,

$$\sin(x)\sin(y) = \frac{1}{2}\big(\cos(x-y)-\cos(x+y)\big)$$

$$\cos(x)\cos(y) = \frac{1}{2}\big(\cos(x-y)+\cos(x+y)\big)$$

to obtain

$$I_1(\xi) = A_1\frac{1}{\pi}\left(\int_0^\infty\frac{1}{2}e^{-(c-1)x}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))+\cos(y(\alpha+x))\big)\,dydx\right.$$
$$\left. +\int_0^\infty\frac{1}{2}e^{-(c-1)x}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))-\cos(y(\alpha+x))\big)\,dydx\right),$$

and

$$I_2(\xi) = A_2\frac{1}{\pi}\left(\int_0^\infty\frac{1}{2}e^{-cx}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))+\cos(y(\alpha+x))\big)\,dydx\right.$$
$$\left. +\int_0^\infty\frac{1}{2}e^{-cx}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))-\cos(y(\alpha+x))\big)\,dydx\right),$$

where we have set

$$A_1 = qS^*(\tau-\xi)e^{-r\xi-\zeta c^2+c\alpha}$$

and

$$A_2 = rXe^{-r\xi-\zeta c^2+c\alpha}.$$

Again, from Gradshteyn and Ryzhik (2007), p. 488, we have for $Re(\beta) > 0$:

$$\int_0^\infty e^{-\beta x^2}\cos(bx)dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}e^{-b^2/4\beta}, \tag{5.123}$$

and the last equations for I_1 and I_2 can be simplified to

$$I_1(\xi) = A_1\frac{1}{2\sqrt{\pi\zeta}}\int_0^\infty e^{-(c-1)x}e^{-\frac{(\alpha-x)^2}{4\zeta}}\,dx \tag{5.124}$$

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and

$$I_2(\xi) = A_2 \frac{1}{2\sqrt{\pi\zeta}} \int_0^\infty e^{-cx} e^{-\frac{(\alpha-x)^2}{4\zeta}} \, dx. \tag{5.125}$$

Finally, observe that the integrals can be approximated accurately using Gauss-Laguerre quadrature

$$\int_0^\infty e^{-(c-1)x} e^{-\frac{(\alpha-x)^2}{4\zeta}} \, dx = \frac{1}{c-1} \int_0^\infty e^{-x} f\left(\frac{x}{c-1}\right) dx \tag{5.126}$$

$$\approx \frac{1}{c-1} \sum_{i=1}^n \omega_i f\left(\frac{x_i}{c-1}\right),$$

and

$$\int_0^\infty e^{-cx} e^{-\frac{(\alpha-x)^2}{4\zeta}} \, dx = \frac{1}{c} \int_0^\infty e^{-x} f\left(\frac{x}{c}\right) dx \tag{5.127}$$

$$\approx \frac{1}{c} \sum_{i=1}^n \omega_i f\left(\frac{x_i}{c}\right),$$

where f equals

$$f(x) = e^{-\frac{(\alpha-x)^2}{4\zeta}} \tag{5.128}$$

and ω_i and $x_i, i = 1, 2, ...n$, correspond to the weights and abscissa of the Gauss-Laguerre quadrature. As a final result we have the following approximation for the American call option:

$$C^A(S, \tau) = C^E(S, \tau) + \int_0^\tau I_1(\xi) \, d\xi - \int_0^\tau I_2(\xi) \, d\xi, \tag{5.129}$$

with

$$I_1(\xi) = qS^*(\tau - \xi)e^{-r\xi - \zeta c^2 + c\alpha} \frac{1}{2(c-1)\sqrt{\pi\zeta}} \sum_{i=1}^n \omega_i f\left(\frac{x_i}{c-1}\right) \tag{5.130}$$

and

$$I_2(\xi) = rX e^{-r\xi - \zeta c^2 + c\alpha} \frac{1}{2c\sqrt{\pi\zeta}} \sum_{i=1}^n \omega_i f\left(\frac{x_i}{c}\right), \tag{5.131}$$

with $1 < c < \infty$, $\zeta = 1/2\sigma^2\xi$, and α and f given in equations (5.120) and (5.128), respectively. The weights $\omega_i, i = 1, ..., n$, are determined by

$$
\begin{aligned}
\omega_i &= \frac{1}{x_i(L'_n(x_i))^2} \\
&= \frac{x_i}{(n+1)^2(L_{n+1}(x_i))^2} \,,
\end{aligned}
$$

with $L_n(x)$ the n-th Laguerre polynomial defined by

$$
L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}\left(e^{-x}x^n\right) .
$$

The integrals in equation (5.129) are determined using the trapezoidal rule. Additionally, in equation (5.129) we assume that the critical stock price $S^*(\tau)$ is known for all τ. The calculation is performed using equation (5.101) where the complex integrals are approximated recursively using a n-point Gauss-Laguerre scheme and the time integral is evaluated using the trapezoidal rule. As a specific numerical example, we value a six months American call option with strike price $X = 100$. The parameters (r, q, σ) are varied from $(0.03, 0.07, 0.2)$ (top) to $(0.03, 0.07, 0.4)$ (center) to $(0.07, 0.03, 0.3)$ (bottom). For the valuation we use a 16-point Gauss-Laguerre scheme combined with a 300 time step approximation of the time integral. Furthermore we fix the parameter $c = 4$. The results are shown in the next table. We compare our results to nine other numerical and analytical approaches known in the literature. The "True" value is based on a binomial tree method with $N = 10000$ time steps. The following approaches represent the method proposed by Barone-Adesi and Whaley (1987) (BAW), the four-point method of Geske and Johnson (1984) (GJ4), the modified two-point Geske-Johnson approach of Bunch and Johnson (1992) (BJ2), the four-point schemes of Huang et al. (1996) (HSY4), the lower and upper bound approximation of Broadie and Detemple (1996) (LUBA), the four-point randomization method of Carr (1998) (RAN4), the three-point multi-piece exponential boundary approximation of Ju (1998) (EXP3), an approximation of Ju and Zhong (1999) (JZ),

and the procedure based on Gauss-Laguerre quadrature of this article (GL), respectively.

S	True	BAW	GJ4	BJ2	HSY4	LUBA	RAN4	EXP3	JZ	GL
80	0.2194	0.2300	0.2191	0.2186	0.2199	0.2195	0.2188	0.2196	0.2216	0.2185
90	1.3864	1.4050	1.3849	1.3818	1.3898	1.3862	1.3802	1.3872	1.3857	1.3851
100	4.7825	4.7821	4.7851	4.7862	4.8044	4.7821	4.7728	4.7837	4.7682	4.7835
110	11.0978	11.0409	11.0889	11.2553	11.0686	11.0976	11.0893	11.0993	11.0794	11.1120
120	20.0004	20.0000	20.0073	20.0000	20.0531	20.0000	20.0000	20.0005	20.0000	20.0000
80	2.6889	2.7108	2.6864	2.6827	2.6897	2.6893	2.6787	2.6899	2.6871	2.6788
90	5.7223	5.7416	5.7212	5.7163	5.7361	5.7231	5.7113	5.7237	5.7110	5.7195
100	10.2385	10.2417	10.2451	10.2351	10.2752	10.2402	10.2205	10.2404	10.2143	10.2265
110	16.1812	16.1520	16.1831	16.2107	16.2012	16.1817	16.1629	16.1831	16.1456	16.1756
120	23.3598	23.2883	23.3419	23.4771	23.3288	23.3574	23.3389	23.3622	23.3211	23.3828
80	1.6644	1.6645	1.6644	1.6644	1.6644	1.6644	1.6604	1.6644	1.6644	1.6644
90	4.4947	4.4950	4.4946	4.4947	4.4947	4.4947	4.4959	4.4947	4.4947	4.4947
100	9.2504	9.2513	9.2509	9.2506	9.2506	9.2506	9.2513	9.2506	9.2507	9.2506
110	15.7977	15.7988	15.7973	15.7975	15.7975	15.7975	15.7994	15.7975	15.7977	15.7980
120	23.7061	23.7086	23.7082	23.7062	23.7062	23.7062	23.7027	23.7062	23.7066	23.7060

Table 5.1: Comparison of American call option prices computed ten different ways: a binomial tree method with $N = 10000$ time steps (True), the method proposed by Barone-Adesi and Whaley (1987) (BAW), the four-point method of Geske and Johnson (1984) (GJ4), the modified two-point Geske-Johnson approach of Bunch and Johnson (1992) (BJ2), the four-point schemes of Huang et al. (1996) (HSY4), the lower and upper bound approximation of Broadie and Detemple (1996) (LUBA), the four-point randomization method of Carr (1998) (RAN4), the three-point multi-piece exponential boundary approximation of Ju (1998) (EXP3), an approximation of Ju and Zhong (1999) (JZ), and the procedure based on Gauss-Laguerre quadrature (GL), respectively.

The calculations show that the new method provides comparable results. The accuracy is convincing and the absolute deviations from the "true" value are negligible. Moreover, since the numerical approximation of our integral so-

lution is easy to implement, we suggest the new framework as a capable alternative to existing methods.

5.3   New Approximations for the Free Boundary

This section focuses on deriving new simple analytical approximations for the critical stock price of American options. We use integral representations from the previous two sections to establish the results. Our formulae are consistent with the short- and long-time asymptotics of the early exercise boundary. They also satisfy other financially important limits. Since the log-normal model is still a benchmark model for American options and is widely used in practice, the derived approximations may be of interest for practitioners and traders.

5.3.1   Main Results

The key to determine the value of the American option is finding the critical stock price, which specifies the conditions under which the option should be exercised prior to maturity. Although the integral representations of Kim (1990), Jacka (1991), Carr et al. (1992), and those derived in this thesis are exact from a theoretical point of view, one must solve the problem numerically in two steps. The first step is to determine the critical stock price recursively. Here, one is faced with the backward solution of a system of nonlinear integral equations. This is sometimes regarded as burdensome and computational errors may be inherent. Having determined the critical stock price, American option prices are calculated in a second step by taking the stock price curve as an input for the integration.

Since closed form solutions for the critical stock price of a finite living American option seem to be impossible, one seeks for good approximations to

circumvent the problems resulting from a numerical procedure. From a mathematical point of view this is a challenging issue. Carr (1998), p. 598, describes the situation as "... it is difficult to analytically approximate American option values using boundary approximations that are consistent with the known short- and long-time behavior of the exercise boundary".

Through the years many methods have been developed for an accurate approximation of American option prices and the corresponding early exercise boundaries. Each of the methods has its own advantages and weaknesses, and an assessment must crucially depend on specific theoretical and/or numerical aspects. Additionally, as pointed out by Li (2010b) it is important to realize that approximating the American option price and approximating the critical stock price are two different issues. Although the problems are closely related, it is possible that a method provides good results for the first problem but not for the second, and vice versa. In Li (2010b) he reviews the most prominent analytical approximations of the critical stock price and gives a detailed numerical comparison of their performance.

We start the derivation by recalling from the previous two sections a unique characterization of the free boundary $S^*(t)$ for American puts and calls, respectively:

$$
\begin{aligned}
X - S^*(t) &= P^E(S^*(t), t) \\
&+ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{rX}{\omega} \Big(\frac{S^*(t)}{S^*(x)} \Big)^{-\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)} \, dx d\omega \\
&- \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{qS^*(x)}{\omega+1} \Big(\frac{S^*(t)}{S^*(x)} \Big)^{-\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)} \, dx d\omega \,, \quad (5.132)
\end{aligned}
$$

and

$$
\begin{aligned}
S^*(t) - X &= C^E(S^*(t), t) \\
&+ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{qS^*(x)}{\omega-1} \Big(\frac{S^*(t)}{S^*(x)} \Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)} \, dx d\omega \\
&- \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^T \frac{rX}{\omega} \Big(\frac{S^*(t)}{S^*(x)} \Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)} \, dx d\omega. \quad (5.133)
\end{aligned}
$$

To value the American option one must solve numerically the last integral equations in order to determine the early exercise premium. This can be done using Gauss-Laguerre quadrature as outlined in the last section. This approach, however, does not yield an analytical expression. It is therefore desirable to find accurate approximations for $S^*(t)$.

To derive the approximate solutions, we will apply the smooth pasting conditions for American options. Recall that along the free boundary $S^*(t)$ American option prices as well as the corresponding deltas are continuous functions and must satisfy $\frac{\partial P^A}{\partial S}|_{S=S^*(t)} = -1$ for puts and $\frac{\partial C^A}{\partial S}|_{S=S^*(t)} = 1$ for calls, respectively. Since from now on the analysis is equivalent for both contracts, we will restrict the attention to American put options only, i.e. $S^*(t) = S_P^*(t)$.

Setting $\tau = T - t$, determining the delta of P^A, and letting $S \to S^*$ from the left it is straightforward to get the following equation for $S^*(\tau)$:

$$-1 + e^{-q\tau}N(-d_1(S^*, X, \tau)) = \int_0^\tau I_1(\xi)\,d\xi + \int_0^\tau I_2(\xi)\,d\xi, \qquad (5.134)$$

where d_1 is given by

$$d_1(S, X, \tau) = \frac{\ln\frac{S}{X} + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad (5.135)$$

$\xi = \tau - x$, $N(\cdot)$ denotes the standard normal distribution function, and

$$I_1(\xi) = \frac{-rX}{S^*(\tau)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 Q(\omega)\xi}\,d\omega \qquad (5.136)$$

and

$$I_2(\xi) = q \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{\omega+1} e^{\frac{1}{2}\sigma^2 Q(\omega)\xi}\,d\omega. \qquad (5.137)$$

For a moment we will assume that the stock does not pay any dividends, i.e. $q = 0$. In this case the final equation determining the critical stock price is:

$$-1 + N(-d_1(S^*, X, \tau)) = \int_0^\tau I_1(\xi)\,d\xi \qquad (5.138)$$

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with $I_1(\xi)$ from above, $Q(\omega)$ given by

$$Q(\omega) = \omega^2 + \omega(1 - \kappa_1) - \kappa_1, \tag{5.139}$$

and

$$d_1(S, X, \tau) = \frac{\ln\frac{S}{X} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{5.140}$$

The complex integral $I_1(\xi)$ can be evaluated in closed form. As a result we have the following simple approximation for $S^*(\tau)$:

Proposition 5.3.1 *If the stock pays no dividends, the critical stock price* $S^*(\tau)$ *of an American put option can be determined approximately by solving the following implicit equation:*

$$S^*(\tau) = X \cdot \frac{\kappa_1}{\kappa_1 + 1} \cdot \frac{2\,N(\sqrt{\delta\tau}) - 1}{N(d_1(S^*, X, \tau))}, \tag{5.141}$$

where δ is given by

$$\delta = \frac{1}{4}\sigma^2(1 - \kappa_1)^2 + 2r = \left(\frac{1}{2}\,\sigma\,(1 + \kappa_1)\right)^2, \tag{5.142}$$

$\kappa_1 = \frac{2r}{\sigma^2}$ *and $d_1(S^*, X, \tau)$ is given in (5.140).*

PROOF: We begin by setting $\omega = c + iy$, i.e. $d\omega = idy$ and $\zeta = \frac{1}{2}\sigma^2\xi$. Then

$$\zeta Q(c + iy) = -\zeta(c^2 + y^2) + \zeta iy(2c + 1 - \kappa_1) + \zeta c(2c + 1 - \kappa_1) - r\xi$$

and

$$e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} = e^{-\zeta(c^2+y^2)+\zeta iy(2c+1-\kappa_1)+\zeta c(2c+1-\kappa_1)-r\xi}.$$

The integral $I_1(\xi)$ can be written as

$$I_1(\xi) = \frac{-rX}{S^*(\tau)} \cdot e^{-\zeta c^2 + \zeta c(2c+1-\kappa_1)-r\xi} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\zeta y^2 + iy\varphi}dy, \tag{5.143}$$

with $\varphi = \zeta(2c + 1 - \kappa_1)$. Now, we use Euler's formula for the complex exponential function $e^{iy} = \cos(y) + i\sin(y)$ to get

$$I_1(\xi) = \alpha\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\zeta y^2} \cos(\varphi y)dy + \alpha\frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-\zeta y^2} \sin(\varphi y)dy,$$

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where we have set

$$\alpha = \frac{-rX}{S^*(\tau)} \cdot e^{-\zeta c^2 + \zeta c(2c+1-\kappa_1) - r\xi}. \tag{5.144}$$

Since $\sin(x)$ and $\cos(x)$ are odd and even functions of x, respectively, we see that the last integral vanishes and we obtain:

$$I_1(\xi) = \frac{\alpha}{\pi} \int_0^\infty e^{-\zeta y^2} \cos(\varphi y) dy. \tag{5.145}$$

Form Gradshteyn and Ryzhik (2007), p. 488, we have

$$\int_0^\infty e^{-ay^2} \cos(xy) dy = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{4a}}, \qquad y \neq 0, Re(a) > 0. \tag{5.146}$$

Therefore

$$\begin{aligned}
I_1(\xi) &= \frac{-rX}{2\pi\, S^*(\tau)} \cdot e^{-\zeta c^2 + \zeta c(2c+1-\kappa_1) - r\xi} \cdot \sqrt{\frac{\pi}{\zeta}} \cdot e^{-\frac{\zeta^2(2c+1-\kappa_1)^2}{4\zeta}} \\
&= \frac{-rX}{\sqrt{2\pi}\, S^*(\tau)\, \sigma} \cdot e^{(-\frac{1}{8}\sigma^2(1-\kappa_1)^2 - r)\xi} \cdot \xi^{-\frac{1}{2}}
\end{aligned} \tag{5.147}$$

is independent of c and

$$\int_0^\tau I_1(\xi)\, d\xi = \frac{-rX}{\sqrt{2\pi}\, S^*(\tau)\, \sigma} \int_0^\tau e^{\gamma\xi}\xi^{-\frac{1}{2}} d\xi, \tag{5.148}$$

where

$$\gamma = -\frac{1}{2}\left(\frac{1}{2}\sigma(1-\kappa_1)\right)^2 - r.$$

Finally, we transform the last integral twice to get

$$\int_0^\tau I_1(\xi)\, d\xi = \frac{-2rX}{\sqrt{2\pi}\, S^*(\tau)\, \sigma} \int_0^{\sqrt{\tau}} e^{\gamma y^2}\, dy = \frac{-2rX}{S^*(\tau)\sigma\sqrt{\delta}}\left(N(\sqrt{\delta\tau}) - \frac{1}{2}\right), \tag{5.149}$$

with

$$\delta = \frac{1}{4}\sigma^2(1-\kappa_1)^2 + 2r.$$

The result follows now by observing that $\frac{r}{\sigma\sqrt{\delta}} = \frac{\kappa_1}{\kappa_1+1}$ where $\kappa_1 = \frac{2r}{\sigma^2}$. $\qquad \square$

Although there is a large literature on pricing American options by approximation and also much attention has been put on the critical stock price, we

could not find the formula in the literature. It has the additional appealing feature that it is consistent with known and financially important limits. For instance, we have the following properties which are an immediate consequence of the previous proposition:

Corollary 5.3.2 *The critical stock price $S^*(\tau)$ from (5.141) satisfies:*

$$\lim_{\tau\to\infty} S^*(\tau) = \frac{\kappa_1}{\kappa_1 + 1} X \tag{5.150}$$

$$\lim_{\tau\to 0^+} S^*(\tau) = X \,, \tag{5.151}$$

$$\lim_{r\to 0^+} S^*(\tau) = 0 \,, \quad \forall \tau \geq 0 \,, \tag{5.152}$$

and

$$\lim_{\sigma\to 0^+} S^*(\tau) = X \,, \quad \forall \tau \geq 0 \,. \tag{5.153}$$

PROOF: For $S > 0$ define the function $g(S) = g(S, r, \sigma, \tau)$ by

$$g(S) := N(d_1(S, X, \tau)) - \frac{X}{S} \frac{\kappa_1}{\kappa_1 + 1} (2 N(\sqrt{\delta\tau}) - 1) \,.$$

Since $\lim_{S\to 0^+} g(S) = -\infty$, $\lim_{S\to\infty} g(S) = 1$, and $\partial g/\partial S > 0$ it follows that $S^*(\tau)$ is the unique root of $g(S)$ for $S > 0$. Now, all properties are easily verified. For example, the critical stock price for $\tau \to \infty$ follows immediately. For the second assertion notice that $S^*(\tau) \leq X, \forall \tau$ and use l'Hospital's rule. The third property is obvious. For the last property observe that

$$\lim_{\sigma\to 0^+} \frac{\kappa_1}{\kappa_1 + 1} = 1,$$

and the equation for g becomes

$$g(S, r, 0, \tau) = \begin{cases} 1 - \dfrac{X}{S} & \text{for } S > Xe^{-r\tau} \\ \dfrac{1}{2} - \dfrac{X}{S} & \text{for } S = Xe^{-r\tau} \\ -\dfrac{X}{S} & \text{for } S < Xe^{-r\tau}. \end{cases}$$

This completes the proof. □

Our approximation satisfies all important asymptotic requirements. The limiting values for the free boundary for $\tau \to 0^+$ and $\tau \to \infty$ are results due to McKean (1965) and Merton (1973), and can also be found in Jacka (1991). They describe the asymptotic behavior of the free boundary for an American put with infinite and zero maturity, respectively. Also, it is well-known that an American put option will never be exercised for an interest rate of zero. Finally, the last limit states that for a volatility of zero, the free boundary will equal X. A financial interpretation of this property is given in Detemple (2006).

If we abandon the restriction on q the approximation formula for the free boundary $S^*(\tau)$, although slightly more complicated, preserves its structure.

Proposition 5.3.3 *The critical stock price $S^*(\tau)$ of an American put option on a dividend paying stock can be determined by the implicit equation:*

$$S^*(\tau) = X \cdot \frac{r}{\sigma\sqrt{\delta}} \cdot \frac{2\,N(\sqrt{\delta\tau}) - 1}{e^{-q\tau}[N(d_1(S^*, X, \tau)) - N(\sqrt{(\delta - 2q)\tau})] + c + \frac{1}{2}}, \quad (5.154)$$

where

$$c = \frac{2q + \sigma\sqrt{\delta - 2q}}{2\sigma\sqrt{\delta}} \cdot [2N(\sqrt{\delta\tau}) - 1], \quad (5.155)$$

$$\delta = \left(\frac{1}{2}\sigma(1 - \kappa_2)\right)^2 + 2r = \left(\frac{1}{2}\sigma(1 + \kappa_2)\right)^2 + 2q, \quad (5.156)$$

$\kappa_2 = \frac{2(r-q)}{\sigma^2}$ *and $d_1(S^*, X, \tau)$ is given in (5.135).*

PROOF: We proceed like in the previous proof and write $\omega = c + iy$ and $\zeta = \frac{1}{2}\sigma^2\xi$. If the restriction on the dividend yield q is abandoned, the equation for $Q(\omega)$ becomes

$$\zeta Q(c + iy) = -\zeta(c^2 + y^2) + \zeta iy(2c + 1 - \kappa_2) + \zeta c(2c + 1 - \kappa_2) - r\xi$$

and

$$e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} = e^{-\zeta(c^2 + y^2) + \zeta iy(2c + 1 - \kappa_2) + \zeta c(2c + 1 - \kappa_2) - r\xi}.$$

Since now the critical stock price is determined by

$$-1 + e^{-q\tau} N(-d_1(S^*, X, \tau)) = \int_0^\tau I_1(\xi) \, d\xi + \int_0^\tau I_2(\xi) \, d\xi, \qquad (5.157)$$

where

$$I_1(\xi) = \frac{-rX}{S^*(\tau)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} \, d\omega \qquad (5.158)$$

and

$$I_2(\xi) = q \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{\omega+1} e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} \, d\omega \qquad (5.159)$$

the problem is to evaluate the second integral. We have $\frac{\omega}{\omega+1} = 1 - \frac{1}{\omega+1}$ and therefore

$$I_2(\xi) = q \cdot \alpha \big(I_{2,1}(\xi) - I_{2,2}(\xi) \big) \qquad (5.160)$$

where we have set

$$\alpha = e^{-\zeta c^2 + \zeta c(2c+1-\kappa_2) - r\xi}, \quad \text{and} \quad \varphi = \zeta(2c+1-\kappa_2), \qquad (5.161)$$

and

$$I_{2,1}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\zeta y^2 + iy\varphi} \, dy, \qquad (5.162)$$

$$I_{2,2}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c+1-iy}{(c+1)^2 + y^2} e^{-\zeta y^2 + iy\varphi} \, dy. \qquad (5.163)$$

We identify the integral $I_{2,1}(\xi)$ as identical to that in equation (5.143) (with an adjusted value for φ). Therefore

$$q \cdot \alpha \cdot I_{2,1}(\xi) = \frac{q}{\sigma \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\delta\xi} \cdot \xi^{-\frac{1}{2}} \qquad (5.164)$$

is independent of c with δ given by

$$\delta = \frac{1}{4}\sigma^2(1 - \kappa_2)^2 + 2r. \qquad (5.165)$$

Once again, Euler's formula applied to $I_{2,2}$, and the use of $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ gives

$$I_{2,2}(\xi) = \frac{1}{\pi} \int_0^\infty \frac{c+1}{(c+1)^2 + y^2} e^{-\zeta y^2} \cos(\varphi y) dy + \frac{1}{\pi} \int_0^\infty \frac{y}{(c+1)^2 + y^2} e^{-\zeta y^2} \sin(\varphi y) dy. \qquad (5.166)$$

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For the evaluation of the integrals we use equations (5.90) and (5.91). Inserting $a = \varphi$, $\beta = \zeta$, $\gamma = c + 1$ and simplifying yields

$$
\begin{aligned}
I_{2,2}(\xi) &= \frac{1}{2}e^{\zeta(c+1)^2}\left[\cosh(\varphi(c+1)) - \sinh(\varphi(c+1))\right] \\
&\quad - \frac{1}{2}e^{\zeta(c+1)^2 - \varphi(c+1)}\,\Phi\left((c+1)\sqrt{\zeta} - \frac{\varphi}{2\sqrt{\zeta}}\right) \\
&= \frac{1}{2}e^{\zeta(c+1)^2 - \varphi(c+1)}\left[1 - \Phi\left((c+1)\sqrt{\zeta} - \frac{\varphi}{2\sqrt{\zeta}}\right)\right] \quad (5.167)
\end{aligned}
$$

where again we have used the relation $\cosh(x) - \sinh(x) = e^{-x}$. Now, we have

$$
(c+1)\sqrt{\zeta} - \frac{\varphi}{2\sqrt{\zeta}} = \frac{\sqrt{2}}{4}(1 + \kappa_2)\sigma\sqrt{\xi} \qquad (5.168)
$$

and we observe that the argument in Φ is independent of c. Using the connection between the error function and the standard normal distribution function $\Phi(x) = 2N(\sqrt{2}\,x) - 1$ we arrive at

$$
q\alpha I_{2,2}(\xi) = qe^{-q\xi}N\left(-\frac{1}{2}\sigma(1 + \kappa_2)\sqrt{\xi}\right) \qquad (5.169)
$$

with $\kappa_2 = \frac{2(r-q)}{\sigma^2}$. In summary, we have

$$
\begin{aligned}
I_2(\xi) &= q \cdot \alpha\left(I_{2,1}(\xi) - I_{2,2}(\xi)\right) \\
&= \frac{q}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\delta\xi} \cdot \xi^{-\frac{1}{2}} - qe^{-q\xi}N\left(-\frac{1}{2}\sigma(1 + \kappa_2)\sqrt{\xi}\right) \quad (5.170)
\end{aligned}
$$

is independent of c. We conclude that

$$
\int_0^\tau I_2(\xi)d\xi = \frac{q}{\sigma\sqrt{2\pi}} \cdot \int_0^\tau e^{-\frac{1}{2}\delta\xi} \cdot \xi^{-\frac{1}{2}}d\xi - q\int_0^\tau e^{-q\xi}N\left(-\frac{1}{2}\sigma(1 + \kappa_2)\sqrt{\xi}\right)d\xi.
$$
$$
\qquad (5.171)
$$

The evaluation of the first integral is straightforward and was performed above. The result is

$$
\frac{q}{\sigma\sqrt{2\pi}} \cdot \int_0^\tau e^{-\frac{1}{2}\delta\xi} \cdot \xi^{-\frac{1}{2}}d\xi = \frac{2q}{\sigma\sqrt{\delta}} \cdot \left[N(\sqrt{\delta\tau}) - \frac{1}{2}\right]. \qquad (5.172)
$$

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For the second integral we use integration by parts to obtain

$$
-q \int_0^\tau e^{-q\xi} N\Big(-\frac{1}{2}\sigma(1+\kappa_2)\sqrt{\xi}\Big)d\xi = e^{-q\tau} N\Big(-\frac{1}{2}\sigma(1+\kappa_2)\sqrt{\tau}\Big) - \frac{1}{2}
$$
$$
+\frac{\sigma(1+\kappa_2)}{2\sqrt{\delta}}\Big[N(\sqrt{\delta\tau}) - \frac{1}{2}\Big] \quad (5.173)
$$

Summing up, we have

$$
\int_0^\tau I_1(\xi)d\xi + \int_0^\tau I_2(\xi)d\xi = \frac{-2rX}{S^*(\tau)\sigma\sqrt{\delta}}\Big[N(\sqrt{\delta\tau}) - \frac{1}{2}\Big] + \frac{2q}{\sigma\sqrt{\delta}}\Big[N(\sqrt{\delta\tau}) - \frac{1}{2}\Big]
$$
$$
+e^{-q\tau} N\Big(-\frac{1}{2}\sigma(1+\kappa_2)\sqrt{\tau}\Big) - \frac{1}{2}
$$
$$
+\frac{\sigma(1+\kappa_2)}{2\sqrt{\delta}}\Big[N(\sqrt{\delta\tau}) - \frac{1}{2}\Big]. \quad (5.174)
$$

Using $N(-x) = 1 - N(x)$ and observing that $\frac{1}{2}\sigma(1+\kappa_2) = \sqrt{\delta - 2q}$ the statement follows from (5.157) after slight simplifications. $\qquad\square$

Similarly, we have the requested properties:

Corollary 5.3.4 *The critical stock price $S^*(\tau)$ determined by (5.154) satisfies:*

$$
\lim_{\tau\to\infty} S^*(\tau) = \frac{\omega}{\omega+1}X, \quad (5.175)
$$

$$
\lim_{\tau\to 0^+} S^*(\tau) = X \cdot \min(1, \frac{r}{q}), \quad (5.176)
$$

$$
\lim_{r\to 0^+} S^*(\tau) = 0, \quad \forall \tau \geq 0, \quad (5.177)
$$

and

$$
\lim_{\sigma\to 0^+} S^*(\tau) = X \cdot \min(1, \frac{r}{q}), \quad \forall \tau \geq 0, \quad (5.178)
$$

where

$$
\omega = \frac{\kappa_2 - 1}{2} + \frac{\sqrt{(\kappa_2-1)^2 + 4\kappa_1}}{2}. \quad (5.179)
$$

PROOF: The verification of all properties can be done in analogy to the proof of the previous Corollary. $\qquad\square$

Now we present the approximation of $S^*(\tau)$ for American call options. Since

it is well known that an American call option on a non-dividend paying stock will never be exercised prematurely (Merton (1973)), we assume a strictly positive dividend yield q. Additionally, it is worth to mention that the critical stock price of an American call option can be determined using the corresponding critical stock price of an American put option via the put-call-symmetry relation for American options and free boundaries as outlined in Chapter 4.

A direct approximation is given by the following equation.

Proposition 5.3.5 *The critical stock price $S^*(\tau)$ of an American call option is uniquely determined by the implicit equation:*

$$S^*(\tau) = X \cdot \frac{r}{\sigma \sqrt{\delta}} \cdot \frac{2\, N(\sqrt{\delta\tau}) - 1}{e^{-q\tau}[N(d_1(S^*, X, \tau)) - N(\sqrt{(\delta - 2q)\tau})] + c - \frac{1}{2}} \; , \quad (5.180)$$

where $\kappa_2 = \frac{2(r-q)}{\sigma^2}$ and c, δ, and $d_1(S^*, X, \tau)$ are given in equations (5.155), (5.156), and (5.135), respectively.

PROOF: The proof uses the same arguments as in the put option case and is straightforward. $\square$

Remark 5.3.6 *The proposed approximations reveal a surprising and striking similarity. The only difference comes from the opposite sign of the factor $1/2$ in the denominator in (5.154) and (5.180). Note further that if the stock pays no dividends, the equation characterizing the free boundary becomes*

$$S^*(\tau) = X \cdot \frac{\kappa_1}{\kappa_1 + 1} \cdot \frac{2\, N(\sqrt{\delta\tau}) - 1}{N(d_1(S^*, X, \tau)) - 1}, \quad (5.181)$$

implying that $S^*(\tau)$ becomes infinite for all τ.

Finally, we have the asymptotics:

Corollary 5.3.7 *The critical stock price $S^*(\tau)$ determined by (5.180) satisfies*

$$\lim_{\tau \to \infty} S^*(\tau) = \frac{\omega^*}{\omega^* - 1} X, \tag{5.182}$$

$$\lim_{\tau \to 0^+} S^*(\tau) = X \cdot \max(1, \frac{r}{q}), \tag{5.183}$$

$$\lim_{q \to 0^+} S^*(\tau) = \infty, \quad \forall \tau \geq 0, \tag{5.184}$$

and

$$\lim_{\sigma \to 0^+} S^*(\tau) = X \cdot \max(1, \frac{r}{q}), \quad \forall \tau \geq 0 \tag{5.185}$$

where

$$\omega^* = \frac{1 - \kappa_2}{2} + \frac{\sqrt{(1 - \kappa_2)^2 + 4\kappa_1}}{2}. \tag{5.186}$$

PROOF: Straightforward. $\qquad\square$

Since our analytical approximations of $S^*(\tau)$ for both American options are consistent with the short- and long-time behavior of the free boundary and also satisfy the other limits for r, q, and σ, they may turn out to give accurate values for all τ in between. In the next section we will test our formulas by doing several numerical experiments and comparing the values to other approaches found in the literature. Also, it is worth to point out that our approximations can still be modified to obtain other expressions for $S^*(\tau)$. As an example we consider the American put option. Using approximations of the standard normal tail probabilities (see Bryc (2002) for an overview) we can modify equation (5.141) further. As a result we can get the following analytical expression determining $S^*(\tau)$:

$$\frac{S^*(\tau)}{X} = e^{-\sqrt{\frac{1}{2}\sigma^2 \tau}\, g - (r + \frac{1}{2}\sigma^2)\tau}, \tag{5.187}$$

with

$$g = \pm\sqrt{-\pi \left(\ln\left[4\theta(1 - \theta)\right]\right)}, \tag{5.188}$$

where

$$\theta = \frac{\kappa_1}{\kappa_1 + 1} \frac{X}{S^*(\tau)} \sqrt{1 - e^{-\frac{2}{\pi}\delta\tau}}. \tag{5.189}$$

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Especially for small τ the expressions can be simplified to

$$\frac{S^*(\tau)}{X} = e^{-\sqrt{\frac{1}{2}\sigma^2 \tau}\,\tilde{g} - (r + \frac{1}{2}\sigma^2)\,\tau}, \tag{5.190}$$

with

$$\tilde{g} = \pm\sqrt{-\pi\left(\ln\left[4\tilde{\theta}(1 - \tilde{\theta})\right]\right)}, \tag{5.191}$$

and

$$\tilde{\theta} = \kappa_1 \frac{X}{S^*(\tau)} \sqrt{\frac{\sigma^2 \tau}{2\pi}}. \tag{5.192}$$

The structure of the implicit approximation is similar to a solution presented by Bunch and Johnson (2000) where $S^*(\tau)$ is approximated as the solution to the equation:

$$\frac{S^*(\tau)}{X} = e^{-\sqrt{\sigma^2 \tau}\,g - (r + \frac{1}{2}\sigma^2)\tau}, \tag{5.193}$$

with

$$g = \pm\sqrt{2\ln\left(\frac{\sqrt{\alpha}\sigma^2 \exp(\alpha(r + \sigma^2/2)^2 \tau/(2\sigma^2))}{2r(X/S^*(\tau))\log(X/S^*(\tau))}\right)}, \tag{5.194}$$

where

$$\alpha = 1 - \frac{A}{1 + \frac{(1+\kappa_1)^2}{4}\sigma^2 \tau}, \quad \text{and} \quad A = \frac{1}{2}\left(\frac{\kappa_1}{1 + \kappa_1}\right)^2. \tag{5.195}$$

Similar expressions may be obtained in the general case.

5.3.2 Numerical Experiments

In the following we use the results from the previous section to compute and compare critical stock prices for a range of different parameter values. Having done this, it is straightforward to use them for the purpose of determining American option prices and hedging parameters. The numerical analysis, however, will be restricted to the case of American put options only. This is due to the duality relationship for the free boundaries. Also, explicit numerical experiments show that the proposed approximation of $S^*(\tau)$ for

American calls is very similar in quality.

First, we consider the non-dividend case. We study the behavior of $S^*(\tau)$ functions derived in this paper by generating plots of our $S^*(\tau)$ approximations (equations (5.141), (5.187), and (5.190)) for some notional parameter values: strike $X = 100$, riskless interest rate $r = 0.03$, dividend yield $q = 0.00$, volatility $\sigma = 0.3$, and time to maturity $0 \leq \tau \leq 1$. The calculation of $S^*(\tau)$ is performed easily and quickly using Newton's method. All three
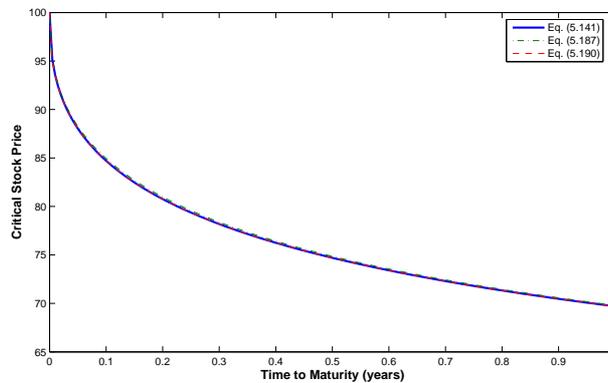
Figure 5.2: The behavior of critical stock price approximations of an American put option as a function of τ. τ varies from zero to one year. Other fixed parameters are: $X = 100, r = 0.03, q = 0$, and $\sigma = 0.3$.

approximations produce curves that are very similar in quality. They illustrate the general shape of the critical stock price as a function of $\tau = T - t$. For American puts $S^*(\tau)$ is monotonically decreasing, bounded by $S^*(\infty)$ for large values of τ and by X for τ approaching zero. Ekström (2004a) and Chen et al. (2008) recently gave two different proofs for the convexity of the early exercise boundary if there are no dividends. For positive values of q the situation is more complex and for specific parameter values the critical stock price may not be convex (see Chen et al. (2008), p.187). For very small τ, $S^*(\tau)$ grows very fast and approaches X with an ever-increasing slope.

The behavior of $S^*(\tau)$ for $\tau \to 0^+$ is not trivial at all and has attracted the interest of many researchers in financial mathematics. Close to expiry, the remaining maturity of the option can be regarded as a small parameter and allows an expansion. The leading-order expansion near expiration was studied by several authors: Barles et al. (1995), Kuske and Keller (1998), Evans et al. (2002), Bunch and Johnson (2000), Knessl (2001), Chen and Chadam (2003), Mallier and Alobaidi (2004), Chen and Chadam (2006), and Zhang and Li (2010) among others. All the studies suggest that the actual behavior of the free boundary crucially depends on the relationship between the interest rate r and the dividend yield q. For a comparison we pick the last named authors who use perturbation methods and derive expressions for $S^*(\tau)$ depending on whether $r > q$, $r = q$, and $r < q$. In order to demonstrate the accuracy of our approximation we will compare our results to their values. The main result of Zhang and Li (2010) is the following theorem regarding the analytical formulas for $S^*(\tau)$:

If $0 \leq q < r$,

$$
\begin{aligned}
S^*(\tau) &= X e^{-\sqrt{2\sigma^2 \tau u(\xi)}}, \\
u(\xi) &= -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{11}{24\xi^3} + O\left(\frac{1}{\xi^4}\right), \\
\xi &= \ln\sqrt{8\pi(r-q)^2\tau/\sigma^2}.
\end{aligned}
$$

If $q = r$,

$$
\begin{aligned}
S^*(\tau) &= X e^{-\sqrt{2\sigma^2 \tau \nu(\eta)}}, \\
\nu(\eta) &= -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta) - \frac{1 - \frac{5}{4\sqrt{2\pi}}}{\eta} + o\left(\frac{1}{\eta}\right), \\
\eta &= \ln(4\sqrt{\pi}r\tau).
\end{aligned}
$$

Finally, if $q > r$,

$$
\begin{aligned}
S^*(\tau) &= \frac{r}{q} X e^{-2\sqrt{\tau^*}w(\sqrt{\tau^*})}, \quad \tau^* = \frac{1}{2}\sigma^2\tau, \quad r^* = \frac{2r}{\sigma^2}, \quad q^* = \frac{2q}{\sigma^2}, \\
w(\sqrt{\tau^*}) &= \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^* + \beta_3\sqrt{\tau^{*3}} + O(\tau^{*2}),
\end{aligned}
$$

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where

$$\beta_0 = 0.451723,$$
$$\beta_1 = 0.144914(r^* - q^*),$$
$$\beta_2 = -0.009801 - 0.041764(r^* + q^*) + 0.014829(r^* - q^*)^2,$$
$$\beta_3 = -0.000618 - 0.002087(r^* - q^*) - 0.015670(r^{*2} - q^{*2}) - 0.001052(r^* - q^*)^3.$$

Zhang and Li (2010) truncate the above expressions gradually and give in each of the above cases four different analytic expressions for the free boundary with slightly different degrees of accuracy (equations termed TA1 to TA4 in the original paper). We consider four (r/q)-combinations: $(0.05; 0)$, $(0.05; 0.05)$, $(0.05; 0.08)$, and $(0.05; 0.1)$. The volatility equals $\sigma = 0.2$. The next figure displays the results for equation (5.154), and the functions TA1 and TA4, respectively. The corresponding limiting values of $S^*(\tau)$ for $\tau \to 0^+$ are 100.0, 100.0, 62.5, and 50.0. As a "small" time to maturity we choose τ to equal one week. The quality of our approximation is convincing. The curves generated by the corresponding formulas are very similar in shape and magnitude. Independent of the (r/q)-combination the differences are small enough to be neglected from a practical point of view.

To get further information about the quality of the approximation derived herein, we continue the numerical analysis by comparing it to $S^*(\tau)$ values found in Zhang and Li (2010). As a benchmark we choose the highly accurate values resulting from the integral-equation approach. The strike price X and the volatility σ are held constant, and equal $X = 100$ and $\sigma = 0.3$, respectively. The (r/q)-combinations under consideration are $(0.05; 0.00)$, $(0.05; 0.05)$, and $(0.05; 0.07)$. Since the authors point out that their expressions are valid only for small maturities, we let τ vary from one week to 3 months. HA is the highly accurate value obtained by solving the integral equations specifying the critical stock price numerically. TA1 to TA4 are the respective truncated solutions. To assess the accuracy we also determine the average absolute error (AAE) and the maximum absolute error (MAE) for

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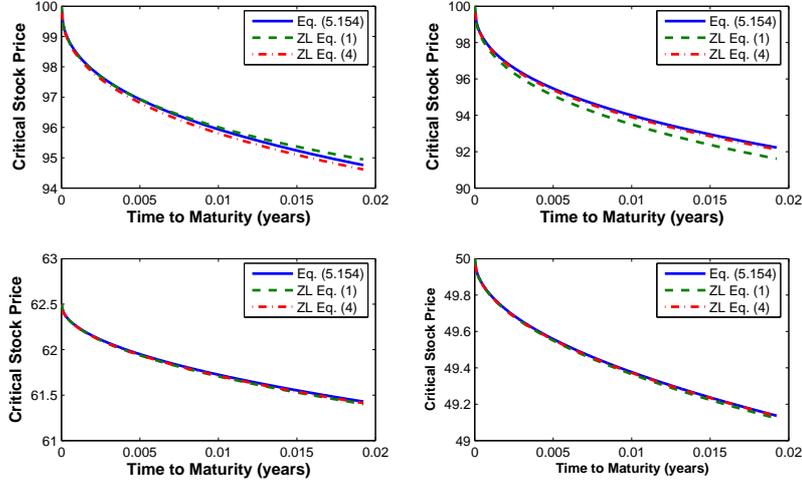
Figure 5.3: The behavior of the critical stock price for small values of τ. We compare our approximation, equation (5.154), to that found in Zhang and Li (2010) where we have picked out two out of four presented formulas (termed TA1 and TA4 in the original paper). Fixed parameter values are $X = 100, r = 0.05, \sigma = 0.2, \tau = 1/52$. In contrast to the previous plot we vary the dividend yield from $q = 0$ (upper left), $q = 0.05$ (upper right), $q = 0.08$ (lower left), and $q = 0.1$ (lower right).

each sample. The results are presented in the next three tables.

We observe that in all three cases the critical stock prices converge for $\tau \to 0^+$, but diverge for large τ. The accuracy of (TA1)-(TA4) decreases with time to maturity, leading to higher AAE- and MAE-values. An exception is the sample with $q > r$ (Table 5.4) where both approaches give very satisfactory results.

In order to be able to study critical stock prices for higher τ-values, we extend the numerics further and compare our results to selected $S^*(\tau)$ values found in the article of Li (2010b), where we adopt the original notation. In Li (2010b) the author compares the performance of a range of promi-

Parameters: $r = 0.05$, $q = 0.00$, $\sigma = 0.3$						
τ	HA	TA1	TA2	TA3	TA4	Eq. (5.141)
1/52	91.33	91.73	91.31	91.26	91.35	91.65
2/52	88.74	89.41	88.68	88.58	88.78	89.14
3/52	86.96	87.88	86.84	86.69	87.04	87.40
4/52	85.57	86.74	85.38	85.16	85.72	86.04
1/12	85.16	86.41	84.95	84.71	85.34	85.64
5/52	84.41	85.82	84.15	83.86	84.66	84.91
6/52	83.42	85.07	83.07	82.71	83.79	83.94
7/52	82.55	84.43	82.10	81.66	83.08	83.09
8/52	81.78	83.88	81.21	80.68	82.51	82.33
1/6	81.30	83.56	80.66	80.06	82.20	81.86
9/52	81.07	83.41	80.39	79.76	82.07	81.63
10/52	80.43	82.99	79.61	78.88	81.75	81.00
11/52	79.84	82.63	78.87	78.03	81.55	80.42
12/52	79.29	82.31	78.17	77.21	81.50	79.87
1/4	78.78	82.02	77.49	76.40	81.59	79.37
AAE		1.84	0.52	1.00	0.82	0.51
MAE		3.24	1.29	2.38	2.81	0.59

Table 5.2: Critical stock prices of an American put option with strike price $X = 100$ in the case $0 \leq q < r$.

Parameters: $r = 0.05$, $q = 0.05$, $\sigma = 0.3$						
τ	HA	TA1	TA2	TA3	TA4	Eq. (5.154)
1/52	88.23	87.69	88.66	88.56	88.43	88.58
2/52	84.76	84.16	85.46	85.30	85.09	85.21
3/52	82.39	81.79	83.31	83.11	82.81	82.90
4/52	80.55	79.99	81.67	81.42	81.05	81.10
1/12	80.01	79.47	81.20	80.94	80.54	80.58
5/52	79.03	78.52	80.34	80.05	79.60	79.62
6/52	77.73	77.29	79.21	78.89	78.36	78.35
7/52	76.59	76.22	78.24	77.89	77.28	77.24
8/52	75.58	75.29	77.38	77.00	76.31	76.24
1/6	74.96	74.73	76.86	76.46	75.72	75.64
9/52	74.67	74.46	76.61	76.21	75.44	75.35
10/52	73.83	73.72	75.92	75.49	74.65	74.53
11/52	73.06	73.05	75.30	74.84	73.91	73.77
12/52	72.35	72.44	74.72	74.24	73.23	73.08
1/4	71.69	71.88	74.19	73.69	72.60	72.43
AAE		0.35	1.58	1.25	0.64	0.61
MAE		0.60	2.50	2.00	0.91	0.74

Table 5.3: Critical stock prices of an American put option with strike price $X = 100$ in the case $q = r$.

Parameters: $r = 0.05$, $q = 0.07$, $\sigma = 0.3$						
τ	HA	TA1	TA2	TA3	TA4	Eq. (5.154)
1/52	69.56	69.56	69.56	69.56	69.56	69.60
2/52	68.81	68.79	68.81	68.81	68.81	68.87
3/52	68.24	68.22	68.24	68.24	68.24	68.31
4/52	67.76	67.73	67.76	67.76	67.76	67.85
1/12	67.62	67.58	67.62	67.62	67.62	67.71
5/52	67.33	67.31	67.34	67.35	67.35	67.43
6/52	66.92	66.93	66.97	66.98	66.98	67.04
7/52	66.51	66.58	66.63	66.64	66.64	66.64
8/52	66.08	66.26	66.32	66.32	66.32	66.23
1/6	65.79	66.05	66.12	66.13	66.13	65.95
9/52	65.64	65.95	66.02	66.03	66.03	65.81
10/52	65.19	65.67	65.74	65.76	65.76	65.39
11/52	64.73	65.40	65.48	65.50	65.50	64.96
12/52	64.28	65.15	65.23	65.25	65.25	64.53
1/4	63.83	64.90	65.00	65.01	65.01	64.10
AAE		0.27	0.30	0.31	0.31	0.14
MAE		1.07	1.17	1.18	1.18	0.27

Table 5.4: Critical stock prices of an American put option with strike price $X = 100$ in the case $q > r$.

nent analytical approximations for the critical stock price: the quadratic approximations of MacMillan (1986), Barone-Adesi and Whaley (1987), and Barone-Adesi and Elliott (1991), the lower bound approximation of Broadie and Detemple (1996), the approximation proposed by Bunch and Johnson (2000), the integral approximation of Zhu (2006a), and the refined quadratic approximations of Li (2010b). The critical stock price computed within the binomial tree framework is chosen as a benchmark. We distinguish between short-term and mid-term options. For each of the two classes we consider two maturity dates: maturities of 0.1 and 0.5 years for short-term options, and maturities of 1 and 3 years for the mid-term options, respectively. The strike price is assumed to be $X = 100$, and the volatility varies from $\sigma = 0.2$ to $\sigma = 0.6$. The riskless interest rate is chosen to equal either 0.02, 0.04, or 0.08. Also, the AAE and the MAE are presented.

The experiments show that the new approximate expressions for the free boundary provide comparable results. The numerical differences between

σ, r	True	IG	QD	BE	IM	QD^+	QD^*	LB	TA	PS	Eq. (5.141)
				Short-term options with $\tau = 0.1$							
0.2, 0.02	87.90	88.98	88.64	87,71	88.38	88.11	87.68	88.31	88.12	85.10	88.31
0.2, 0.04	89.60	89.75	90.22	89.38	90.05	89.76	89.48	89.92	89.75	87.88	89.90
0.2, 0.08	91.44	91.06	91.92	91.19	91.87	91.55	91.40	91.66	91.53	90.56	91.61
0.4, 0.02	74.80	78.46	76.19	74.51	75.54	75.19	74.23	75.59	75.25	67.79	75.64
0.4, 0.04	77.48	79.22	78.74	77.14	78.25	77.82	77.12	78.17	77.74	72.54	78.20
0.4, 0.08	80.47	80.63	81.55	80.08	81.22	80.75	80.29	81.02	80.54	77.37	81.02
0.6, 0.02	63.11	69.17	64.93	62.78	63.91	63.63	62.27	64.17	63.84	52.51	64.27
0.6, 0.04	66.37	69.89	68.07	65.97	67.29	66.84	65.79	67.32	66.87	58.37	67.42
0.6, 0.08	70.06	71.25	71.59	69.58	71.00	70.47	69.72	70.88	70.26	64.55	70.95
				Short-term options with $\tau = 0.5$							
0.2, 0.02	79.04	78.97	80.17	78.60	79.84	79.32	78.87	79.61	79.05	75.95	79.61
0.2, 0.04	82.71	81.82	83.57	82.21	83.50	82.91	82.66	83.10	82.49	81.15	83.02
0.2, 0.08	86.69	85.94	87.25	86.15	87.53	86.77	86.67	86.88	86.15	86.12	86.66
0.4, 0.02	58.10	59.94	59.98	57.54	59.13	58.60	57.68	59.12	58.49	50.50	59.29
0.4, 0.04	63.20	62.74	64.82	62.51	64.23	63.61	63.00	64.03	63.10	58.29	64.15
0.4, 0.08	69.09	67.47	70.34	68.26	70.15	69.37	69.04	69.67	68.40	66.47	69.67
0.6, 0.02	42.14	45.45	44.21	41.58	43.02	42.70	41.61	43.30	42.93	31.89	43.62
0.6, 0.04	47.48	47.90	49.40	46.77	48.43	47.99	47.20	48.51	47.81	40.01	48.81
0.6, 0.08	53.92	52.30	55.53	53.02	54.90	54.33	53.84	54.73	53.55	49.21	54.96
AAE		1.61	1.31	0.48	0.81	0.34	0.28	0.69	0.34	4.85	0.50
MAE		6.05	2.07	0.90	1.06	0.56	0.85	1.15	0.79	10.60	1.48

Table 5.5: Critical stock prices of short-term American put options on non-dividend paying stocks calculated using eleven different approaches: The binomial tree approach with 15000 time steps (True), the initial guess (IG) in Barone-Adesi and Whaley (1987), the quadratic approximation (QD) of MacMillan (1986) and Barone-Adesi and Whaley (1987), the refined quadratic approximation of Barone-Adesi and Elliott (1991) (BE), the interpolation method (IM), and the quadratic approximations (QD^+ and QD^*) of Li (2010b), the lower bound approximation of Broadie and Detemple (1996) (LB), the tangent approximation proposed by Bunch and Johnson (2000) (TA), the integral approximation of Zhu (2006a) (PS), and the approximate solution in (5.141).

σ, r	True	IG	QD	BE	IM	QD^+	QD^*	LB	TA	PS	Eq. (5.141)
				Mid-term options with $\tau = 1.0$							
0.2, 0.02	74.24	73.38	75.49	73.64	75.19	74.55	74.13	74.85	74.00	71.30	74.85
0.2, 0.04	79.24	77.99	80.12	78.55	80.22	79.41	79.22	79.60	78.58	77.91	79.46
0.2, 0.08	84.60	84.04	85.12	83.91	85.76	84.64	84.57	84.73	83.39	84.21	84.36
0.4, 0.02	49.90	50.18	51.79	49.18	50.88	50.39	49.63	50.90	50.15	42.77	51.18
0.4, 0.04	56.31	54.65	57.88	55.41	57.31	56.70	56.23	57.09	55.86	51.92	57.28
0.4, 0.08	63.88	61.85	64.98	62.81	65.03	64.11	63.90	64.36	62.51	61.75	64.36
0.6, 0.02	33.10	34.26	34.95	32.45	33.73	33.60	32.82	34.13	33.82	24.23	34.59
0.6, 0.04	39.27	38.00	40.93	38.40	39.93	39.71	39.18	40.16	39.40	32.88	40.61
0.6, 0.08	47.01	44.50	48.30	45.88	47.75	47.33	47.05	47.66	46.17	43.16	48.01
				Mid-term options with $\tau = 3.0$							
0.2, 0.02	66.02	64.10	67.21	65.06	67.14	66.28	66.02	66.55	64.97	63.68	66.55
0.2, 0.04	73.77	72.64	74.47	72.73	75.19	73.85	73.75	73.99	71.80	72.95	73.69
0.2, 0.08	81.86	82.08	82.28	81.04	83.79	81.78	81.75	81.87	78.81	81.70	81.20
0.4, 0.02	37.34	35.26	38.77	36.37	37.83	37.72	37.35	38.11	37.22	31.94	38.61
0.4, 0.04	45.95	43.32	46.93	44.68	46.58	46.20	46.03	46.46	44.63	42.90	46.82
0.4, 0.08	56.58	55.06	57.01	55.12	57.80	56.63	56.58	56.77	53.55	55.36	56.77
0.6, 0.02	21.14	19.56	22.16	20.39	20.95	21.44	21.19	21.75	21.60	15.55	22.42
0.6, 0.04	28.23	25.65	28.96	27.16	27.98	28.48	28.36	28.72	27.89	24.28	29.40
0.6, 0.08	37.92	35.63	38.14	36.48	37.90	38.02	38.00	38.18	36.02	35.75	38.69
AAE		1.53	1.06	0.95	0.86	0.26	0.08	0.53	1.06	3.45	0.54
MAE		2.63	1.89	1.46	1.94	0.50	0.29	1.02	3.05	8.87	1.49

Table 5.6: Critical stock prices of mid-term American put options on non-dividend paying stocks calculated using eleven different approaches: The binomial tree approach with 15000 time steps (True), the initial guess (IG) in Barone-Adesi and Whaley (1987), the quadratic approximation (QD) of MacMillan (1986) and Barone-Adesi and Whaley (1987), the refined quadratic approximation of Barone-Adesi and Elliott (1991) (BE), the interpolation method (IM), and the quadratic approximations (QD^+ and QD^*) of Li (2010b), the lower bound approximation of Broadie and Detemple (1996) (LB), the tangent approximation proposed by Bunch and Johnson (2000) (TA), the integral approximation of Zhu (2006a) (PS), and the approximate solution in (5.141).

our formulas and the other analytical approximations as well as numerical procedures are negligible from a practical point of view. Furthermore, the formulas are computationally efficient, robust and very easy to implement. Finally, no distinction between the relationship of r and q is needed.

Chapter 6

Mellin Transforms and the Heston Model

This chapter deals with options written on stocks with a stochastic volatility structure[20]. The Heston (1993) mean reverting square root process is used for the stochastic volatility dynamics. We show how the Mellin transform framework can be applied to solve the problem analytically.

6.1   Introduction and Related Literature

Although the pricing model of Black/Scholes and Merton (BSM) was and still is one of the most influential developments in financial economics, the assumptions underlying the original works were questioned ab initio and became the subject of wide theoretical and empirical study. It quickly became clear that extensions are necessary to fit the empirical data. The main drawback of the model is the assumption of a constant volatility. The inadequacy of a constant volatility for modeling stock returns is mainly based on two empirical observations:

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[20]The main part of this chapter is based on the paper Frontczak (2010b).

- volatilities vary over time

- volatilities are correlated with stock returns.

The first observation can even be strengthened to a persistence of volatilities in a certain level, called the mean reversion or long-run level. Furthermore, most empirical tests describe a negative correlation between volatilities and stock returns. This means that an upward-move in the stock price is accompanied by a downward-move in volatility.

A simple test for a constant volatility is computing implied volatilities from a range of option prices that are observable on the market. If the assumption of a constant volatility were true, implied volatilities should also remain constant. Unfortunately, this is not observed and implied volatilities in the BSM formula present non-constant behavior which depends on moneyness. This pattern is called the "volatility smile" which exhibits either a downward sloping or a down- and upward sloping structure. Recent theoretical work on the volatility smile can be found in the articles of Lee (2001), Balland (2002), Lee (2004b), Fouque et al. (2004), Li (2008) or Gulisashvili and Stein (2009).

Different approaches have been developed to reflect the empirical evidence of a non-constant volatility and to explain the effect of a volatility smile. The first extension was suggested by Merton (1973) himself, who extended the BSM pricing formula to a volatility function that is deterministic in time. Another extension of the original model is the CEV model developed by Cox (1975). Dupire (1994) assumed that volatility dynamics can be modeled as a deterministic function of the stock price and time. Another different approach was proposed by Sircar and Papanicolaou (1999). Based on the PDE framework they developed a methodology that is independent of a particular volatility process. The result is an asymptotic approximation consisting of a BSM-like price plus a Gaussian variable which captures the risk from the volatility component.

The majority of the financial community, however, focuses on stochastic volatility models. These models assume that volatility itself is a random process and fluctuates over time. Stochastic volatility models were first studied by Johnson and Shanno (1987), Hull and White (1987), Scott (1987), and Wiggins (1987). Other models for the volatility dynamics were proposed by Stein and Stein (1991), Heston (1993), Schöbel and Zhu (1999), and Rogers and Veraart (2008). In these models, the stochastic process governing the asset price dynamics is driven by a subordinated stochastic volatility process that may or may not be independent.

While the early models couldn't produce closed-form formulae, it was Stein and Stein (1991) (S&S) who first succeeded in deriving an analytical solution. Assuming that volatility follows a mean reverting Ornstein-Uhlenbeck process and is uncorrelated with asset returns, they present an analytic expression for the density function of asset returns for the purpose of option valuation. Schöbel and Zhu (1999) generalize the S&S model to the case of non-zero correlation between instantaneous volatilities and asset returns. They present a closed-form solution for European options and discuss additional features of the volatility dynamics.

The maybe most popular stochastic volatility model was introduced by Heston (1993). The model assumes that the dynamics of the squared volatility (variance) are given by a square-root process that exhibits mean reversion. The risk neutral dynamics of the asset price are governed by the following system of stochastic differential equations:

$$dS_t = (r - q)S_t\,dt + \sqrt{V_t}S_t\,dW_t, \qquad (6.1)$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}\,dZ_t, \qquad (6.2)$$

where S_t is the price of a dividend paying stock at time t and V_t its instantaneous variance with initial values $S_0, V_0 \in (0, \infty)$, and where $r, q, \kappa, \theta, \xi > 0$. As usual, r is the constant riskfree interest rate, and q is the constant dividend yield. κ is the speed of mean reversion to the mean reversion level θ,

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and ξ is the so-called volatility of volatility. W_t and Z_t are two correlated Brownian motions with $dW_t dZ_t = \rho dt$ where $\rho \in [-1, 1]$ is the correlation coefficient. The Feller condition $\kappa\theta > \frac{1}{2}\xi^2$ guarantees that the variance process never reaches zero and always stays positive. In the second case if $\kappa\theta \leq \frac{1}{2}\xi^2$ the variance may reach zero but it can subsequently become positive. The natural barrier at $V = 0$ is called reflecting.

The wide-spread recognition and attractiveness of the model comes from important economic, empirical, mathematical, and computational reasons. The model is able to produce a number of different non-Gaussian distributions of stock returns and accounts for a flexible skewness and kurtosis of the density function. Furthermore, it is robust and mathematically and computationally tractable. In his influential paper, Heston presented a new approach for a closed-form valuation of options applying Fourier inversion techniques for the pricing procedure. The characteristic function approach of Heston (1993) turned out to be a very powerful tool. As a natural consequence, it became standard in option pricing theory and was refined and extended in various directions (Bates (1996), Carr and Madan (1999), Bakshi and Madan (2000), Lewis (2000), Lee (2004a), Kahl and Jäckel (2005), Kruse and Nögel (2005), Fahrner (2007) or Lord and Kahl (2007), among others). See also Duffie et al. (2000) and Duffie et al. (2003) for the mathematical foundations of affine processes.

6.2 Option Pricing

The crucial difference between the BSM economy and the Heston model is, that prices of derivatives are now affected by two sources of randomness: the asset price and the volatility. Also, in the BSM economy, the source of randomness is a traded asset which is infinitely divisible and traded continuously. Therefore, a derivative can be hedged continuously by continuously trading the asset. This makes the market complete, i.e. derivatives can be

priced by replication.

In stochastic volatility models, there are two sources of randomness. Only the asset is tradeable. Since volatility is not a traded asset, the market becomes incomplete and a replication argument does not fully work, i.e. the risk associated with the volatility component cannot be completely eliminated.

Let $F(S, V, t)$ be the current price of an option with strike price X and maturity T. Setting up a riskless portfolio of the form

$$\Pi = F(S, V, t) + \triangle_S S + \triangle_{V_1} V_1,$$

with $\triangle_S$ and $\triangle_{V_1}$ being the numbers of the stock and another asset V_1, whose price depends on volatility, it is straightforward to derive a two dimensional PDE that must be satisfied by $F = F(S, V, t)$:

$$F_t + (r - q)SF_S + \frac{1}{2}VS^2 F_{SS} + (\kappa(\theta - V) + \lambda\xi\sqrt{V})F_V$$

$$+\frac{1}{2}\xi^2 V F_{VV} + \rho\xi V S F_{SV} - rF = 0,$$

on $0 < S, V < \infty, 0 < t < T$, and where partial derivatives with respect to the underlying variables are denoted by subscripts. For a detailed derivation, see Lewis (2000) or Gatheral (2002). λ is called the market price of volatility risk and cannot be completely eliminated in an incomplete market environment. Its functional form is difficult to estimate. Heston provides some reasons for the assumption that λ is proportional to volatility, i.e. $\lambda = k\sqrt{V}$ for some constant k. Therefore $\lambda\xi\sqrt{V} = k\xi V = \lambda^* V$ (say). Now it is possible to define the risk-adjusted speed κ^* and mean reversion θ^*, respectively, by $\kappa^* = \kappa - \lambda^*$ and $\theta^* = \kappa\theta/\kappa^*$. The resulting PDE becomes

$$F_t + (r-q)SF_S + \frac{1}{2}VS^2 F_{SS} + \kappa^*(\theta^* - V)F_V + \frac{1}{2}\xi^2 V F_{VV} + \rho\xi V S F_{SV} - rF = 0.$$
(6.3)

Since κ and θ are risk-adjusted, the equation accounts for different risk preferences. The functional form of the pricing equation, however, remains unchanged. In the following we will again write κ and θ, assuming risk-adjusted

parameters. If F is a European call option, i.e. $F(S, V, t) = C^E(S, V, t)$, we have

$$C_t^E + (r-q)SC_S^E + \frac{1}{2}VS^2C_{SS}^E + \kappa(\theta-V)C_V^E + \frac{1}{2}\xi^2VC_{VV}^E + \rho\xi VSC_{SV}^E - rC^E = 0 \tag{6.4}$$

with the terminal condition

$$C^E(S, V, T) = \max(S(T) - X, 0),$$

and where $C^E(S, V, t) : \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T] \to \mathbb{R}^+$. The solution can be expressed as

$$C^E(S, V, t) = Se^{-q(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty f_1(u)du\right) - Xe^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty f_2(u)du\right), \tag{6.5}$$

with $S = S_t$ being the current price and

$$f_1(u) = Re\left(\frac{e^{-iu\ln X}\varphi(u-i)}{iuSe^{(r-q)(T-t)}}\right) \quad \text{and} \quad f_2(u) = Re\left(\frac{e^{-iu\ln X}\varphi(u)}{iu}\right). \tag{6.6}$$

The function $\varphi(u)$ is the characteristic function of the log-stock price at maturity:

$$\varphi(u) = E\left[e^{iu\ln S(T)}\right],$$

where i denotes the imaginary unit. The analytic expression of the characteristic function in the Heston model equals

$$\varphi(u) = e^{C(u,T-t)+D(u,T-t)V+iu(\ln S+(r-q)(T-t))}, \tag{6.7}$$

with $V = V_t$ being the current level of variance, and

$$C(u, T-t) = \frac{\kappa\theta}{\xi^2}\left((\kappa-\rho\xi iu+d(u))(T-t) - 2\ln\left(\frac{1-c(u)e^{d(u)(T-t)}}{1-c(u)}\right)\right), \tag{6.8}$$

$$D(u, T-t) = \frac{\kappa-\rho\xi iu+d(u)}{\xi^2}\left(\frac{1-e^{d(u)(T-t)}}{1-c(u)e^{d(u)(T-t)}}\right), \tag{6.9}$$

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$$c(u) = \frac{\kappa - \rho\xi iu + d(u)}{\kappa - \rho\xi iu - d(u)}, \tag{6.10}$$

and

$$d(u) = \sqrt{iu\xi^2 + \xi^2 u^2 + (\rho\xi iu - \kappa)^2}. \tag{6.11}$$

It seems impossible to evaluate the above integrals exactly. However, they can be approximated with reasonable accuracy by using efficient numerical integration techniques, e.g., Gauss-Legendre or Gauss-Lobatto integration. Similarly, if F is a European put option, i.e. $F(S,V,t) = P^E(S,V,t)$, we have

$$P_t^E + (r-q)SP_S^E + \frac{1}{2}VS^2 P_{SS}^E + \kappa(\theta-V)P_V^E + \frac{1}{2}\xi^2 V P_{VV}^E + \rho\xi V S P_{SV}^E - rP^E = 0, \tag{6.12}$$

where $P^E(S,V,t) : \mathbb{R}^+ \times \mathbb{R}^+ \times [0,T] \to \mathbb{R}^+$. The boundary conditions may be specified as

$$P^E(S,V,T) = \max(X - S(T), 0) \tag{6.13}$$

$$P^E(0,V,t) = Xe^{-r(T-t)}, \tag{6.14}$$

$$P^E(S,0,t) = \max(Xe^{-r(T-t)} - S(t)e^{-q(T-t)}, 0), \tag{6.15}$$

$$\lim_{S\to\infty} P^E(S,V,t) = 0, \tag{6.16}$$

and

$$\lim_{V\to\infty} P^E(S,V,t) = Xe^{-r(T-t)}. \tag{6.17}$$

The first condition is the terminal condition. It specifies the final payoff of the option. The second condition states that for a stock price of zero the put price must equal the discounted strike price. The third condition specifies the payoff for a variance (volatility) of zero. In this case the underlying asset evolves completely deterministic and the put price equals its lower bound derived by arbitrage considerations. The next condition describes the option's price for ever increasing asset prices. Obviously, since a put option gives its holder the right to sell the asset the price will tend to zero if S tends to infinity. Finally, notice that if variance (volatility) becomes infinite the

current asset price contains no information about the terminal payoff of the derivative security, except that the put entitles its holder to sell the asset for X. In this case the put price must equal the discounted strike price, i.e. its upper bound, again derived by arbitrage arguments.

6.3   Alternative Analytic Solution

The objective of this section is to solve equation (6.12) subject to (6.13)-(6.17) in closed-form. Let $\widetilde{P}^E = \widetilde{P}^E(\omega, V, t)$ denote the Mellin transform of $P^E(S, V, t)$. It is easily verified that $\widetilde{P}^E$ exists in the entire halfplane with $Re(\omega) > 0$, where $Re(\omega)$ denotes the real part of ω. A straightforward application to (6.12) gives

$$\widetilde{P}^E_t + (a_1 V + b_1)\widetilde{P}^E_V + (a_2 V + b_2)\widetilde{P}^E_{VV} + (a_0 V + b_0)\widetilde{P}^E = 0, \qquad (6.18)$$

where

$$
\begin{aligned}
a_1 &= -(\omega\rho\xi + \kappa), & b_1 &= \kappa\theta \\
a_2 &= \frac{1}{2}\xi^2, & b_2 &= 0 \\
a_0 &= \frac{1}{2}\omega(\omega+1), & b_0 &= q\omega - r(\omega+1).
\end{aligned} \qquad (6.19)
$$

This is a one dimensional PDE in the complex plane with non-constant coefficients. To provide a unique solution for $0 < V < \infty, 0 < t < T$ we need to incorporate the boundary conditions. The transformed terminal and boundary conditions are given by, respectively,

$$\widetilde{P}^E(\omega, V, T) = X^{\omega+1}\Big(\frac{1}{\omega} - \frac{1}{\omega+1}\Big) \qquad (6.20)$$

$$\widetilde{P}^E(\omega, 0, t) = e^{(q\omega - r(\omega+1))(T-t)} \cdot X^{\omega+1}\Big(\frac{1}{\omega} - \frac{1}{\omega+1}\Big) \qquad (6.21)$$

and condition (6.17) becomes

$$\lim_{V\to\infty} |\widetilde{P}^E(\omega, V, t)| = \infty. \qquad (6.22)$$

Now, similar to the previous sections we change the time variable from t to $\tau = T - t$ and convert the backward in time PDE into a forward in time PDE with solution domain $0 < V, \tau < \infty$. With $\widetilde{P}^E(\omega, V, t) = \widehat{P}^E(\omega, V, \tau)$ the resulting equation is

$$-\widehat{P}^E_\tau + (a_1 V + b_1)\widehat{P}^E_V + (a_2 V + b_2)\widehat{P}^E_{VV} + (a_0 V + b_0)\widehat{P}^E = 0, \qquad (6.23)$$

where the coefficients a_0, a_1, a_2, b_0, b_1 and b_2 are given above and the terminal condition becomes an initial condition

$$\widehat{P}^E(\omega, V, 0) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega + 1}\right). \qquad (6.24)$$

Additionally we have

$$\widehat{P}^E(\omega, 0, \tau) = e^{(q\omega - r(\omega+1))\tau} \cdot X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega + 1}\right), \qquad (6.25)$$

and

$$\lim_{V \to \infty} | \widehat{P}^E(\omega, V, \tau) | = \infty. \qquad (6.26)$$

To simplify the PDE further we assume that the solution $\widehat{P}^E(\omega, V, \tau)$ can be written in the form

$$\widehat{P}^E(\omega, V, \tau) = e^{(q\omega - r(\omega+1))\tau} \cdot h(\omega, V, \tau) \qquad (6.27)$$

with an appropriate function $h(\omega, V, \tau)$. It follows that h must satisfy

$$-h_\tau + (a_1 V + b_1)h_V + a_2 V h_{VV} + a_0 V h = 0, \qquad (6.28)$$

with initial and boundary conditions

$$h(\omega, V, 0) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega + 1}\right) \qquad (6.29)$$

$$h(\omega, 0, \tau) = X^{\omega+1}\left(\frac{1}{\omega} - \frac{1}{\omega + 1}\right) \qquad (6.30)$$

and

$$\lim_{V \to \infty} | h(\omega, V, \tau) | = \infty. \qquad (6.31)$$

Observe that for $\kappa = \theta = \xi = 0$, i.e. if the stock price dynamics are given by the standard BSM model with constant volatility, the PDE for h is solved as

$$h(\omega, V, \tau) = X^{\omega+1}\Big(\frac{1}{\omega} - \frac{1}{\omega+1}\Big)e^{\frac{1}{2}\omega(\omega+1)V\tau}. \tag{6.32}$$

In this case the equation for $\widetilde{P}^E(\omega, V, \tau)$ becomes

$$\widetilde{P}^E(\omega, V, \tau) = X^{\omega+1}\Big(\frac{1}{\omega} - \frac{1}{\omega+1}\Big)e^{(\frac{1}{2}\omega(\omega+1)V+q\omega-r(\omega+1))\tau}, \tag{6.33}$$

and the price of a European put option can be expressed as

$$P^E(S, V, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{P}^E(\omega, V, \tau)\, S^{-\omega}\, d\omega, \tag{6.34}$$

with $0 < c < \infty$. This is exactly the valuation formula for European put options derived in Chapter 5 (equation (5.13) with $n = 1$).

The final step in deriving a general solution for h or equivalently for $\widetilde{P}^E$ for a non-constant volatility is to assume the following functional form of the solution:

$$h(\omega, V, \tau) = \widetilde{c} \cdot H(\omega, \tau) \cdot e^{G(\omega, \tau) \cdot a_0 \cdot V}, \tag{6.35}$$

with $H(\omega, 0) = 1$, $G(\omega, 0) = 0$ and where we have set

$$\widetilde{c} = X^{\omega+1}\Big(\frac{1}{\omega} - \frac{1}{\omega+1}\Big). \tag{6.36}$$

Inserting the functional form for h in (6.28), determining the partial derivatives and simplifying yields two ordinary differential equations. We have

$$G_\tau(\omega, \tau) = A \cdot G^2(\omega, \tau) + B \cdot G(\omega, \tau) + C, \tag{6.37}$$

and

$$H_\tau(\omega, \tau) = a_0 \cdot b_1 \cdot G(\omega, \tau) \cdot H(\omega, \tau) \tag{6.38}$$

where $A = a_0 a_2$, $B = a_1$, and $C = 1$. The ODE for $G(\omega, \tau)$ is identified as a Riccati equation with constant coefficients. These types of equations also appear in frameworks based on Fourier transforms, see Heston (1993), Bates

(1996) or Schöbel and Zhu (1999). Having solved for G, a straightforward calculation shows that $H(\omega, \tau)$ equals

$$H(\omega, \tau) = e^{a_0 b_1 \int_0^\tau G(\omega, x) dx}. \tag{6.39}$$

Thus, we first present the solution for G. The transformation

$$G(\omega, \tau) = \frac{1}{A} u(\omega, \tau) - \frac{B}{2A}$$

gives

$$u_\tau(\omega, \tau) = u^2(\omega, \tau) + \frac{4AC - B^2}{4}. \tag{6.40}$$

Note that this is a special case of the more general class of ODEs given by

$$u_\tau(\omega, \tau) = a u^2(\omega, \tau) + b \tau^n,$$

with $n \in \mathbb{N}$ and a and b constants. This class of ODEs has solutions of the form

$$u(\omega, \tau) = -\frac{1}{a} \frac{F_\tau(\omega, \tau)}{F(\omega, \tau)},$$

where

$$F(\omega, \tau) = \sqrt{\tau} \left(c_1 J_{\frac{1}{2m}} \left(\frac{1}{m} \sqrt{ab}\, \tau^m \right) + c_2 Y_{\frac{1}{2m}} \left(\frac{1}{m} \sqrt{ab}\, \tau^m \right) \right).$$

The parameters c_1, c_2 are again constants depending on the underlying boundary conditions, $m = \frac{1}{2}(n+2)$ and J and Y are Bessel functions of the first and second kind, respectively. See Polyanin and Zaitsev (2003) for a reference. Setting $m = 1$ and incorporating the boundary conditions, $u(\omega, \tau)$ is solved as

$$u(\omega, \tau) = \frac{k}{2} \frac{\tan\left(\frac{1}{2}k\tau\right) + \frac{B}{k}}{1 - \frac{B}{k} \tan\left(\frac{1}{2}k\tau\right)}, \tag{6.41}$$

where we have set

$$k = k(\omega) = \sqrt{4AC - B^2} = \sqrt{\xi^2 \omega(\omega + 1) - (\omega\rho\xi + \kappa)^2}. \tag{6.42}$$

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Thus, we immediately get

$$
\begin{aligned}
G(\omega, \tau) &= -\frac{B}{2A} + \frac{k}{2A} \frac{\tan\left(\frac{1}{2}k\tau\right) + \frac{B}{k}}{1 - \frac{B}{k}\tan\left(\frac{1}{2}k\tau\right)} \\
&= -\frac{B}{2A} + \frac{k}{2A} \frac{k\sin\left(\frac{1}{2}k\tau\right) + B\cos\left(\frac{1}{2}k\tau\right)}{k\cos\left(\frac{1}{2}k\tau\right) - B\sin\left(\frac{1}{2}k\tau\right)}.
\end{aligned} \tag{6.43}
$$

Using $k^2 + B^2 = 4A$ it is easily verified that an equivalent expression for $G(\omega, \tau)$ equals

$$
G(\omega, \tau) = \frac{2\sin\left(\frac{1}{2}k\tau\right)}{k\cos\left(\frac{1}{2}k\tau\right) + (\omega\rho\xi + \kappa)\sin\left(\frac{1}{2}k\tau\right)} \tag{6.44}
$$

with $k = k(\omega)$ from above. To solve for $H(\omega, \tau)$ we first mention that (Gradshteyn and Ryzhik (2007))

$$
\int \frac{B\cos x + C\sin x}{b\cos x + c\sin x}dx = \frac{Bc - Cb}{b^2 + c^2}\ln(b\cos x + c\sin x) + \frac{Bb + Cc}{b^2 + c^2}x.
$$

Therefore,

$$
\int_0^\tau G(\omega, x)dx = -\frac{B\tau}{2A} + \frac{1}{A}\ln\left(\frac{k}{k\cos\left(\frac{1}{2}k\tau\right) - B\sin\left(\frac{1}{2}k\tau\right)}\right) \tag{6.45}
$$

and

$$
H(\omega, \tau) = e^{\frac{\kappa\theta}{\xi^2}\left[(\omega\rho\xi+\kappa)\tau + 2\ln\left(\frac{k}{k\cos\left(\frac{1}{2}k\tau\right)+(\omega\rho\xi+\kappa)\sin\left(\frac{1}{2}k\tau\right)}\right)\right]}. \tag{6.46}
$$

Finally, we have arrived at the following result:

Theorem 6.3.1 *A new Mellin-type pricing formula for European put options in Heston's (1993) mean reverting stochastic volatility model given by*

$$
P^E(S, V, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{P}^E(\omega, V, \tau)\, S^{-\omega}\, d\omega, \tag{6.47}
$$

with $0 < c < c^*$ and where

$$
\widetilde{P}^E(\omega, V, \tau) = \widetilde{c} \cdot e^{(q\omega - r(\omega+1))\tau} \cdot H(\omega, \tau) \cdot e^{G(\omega,\tau)a_0 V}. \tag{6.48}
$$

with $G(\omega, \tau)$ and $H(\omega, \tau)$ from above. The parameters $\widetilde{c}$ and k are given in (6.36) and (6.42), respectively. The choice of c^* will be commented below.

Remark 6.3.2 *Note that similar to the approach of Carr and Madan (1999) the final pricing formula only requires a single integration.*

We now consider the issue of specifying c^*. Recall that to guarantee the existence of the inverse Mellin transform of $\widetilde{P}^E(\omega, V, \tau)$ in a vertical strip of the ω-plane, we need $\widetilde{P}^E(c+iy, V, \tau)$ to be integrable, and hence analytic. The analysis shows that $G(\omega, \tau)$ and $H(\omega, \tau)$ have the same points of singularity with

$$
\begin{aligned}
\lim_{\omega \to 0} G(\omega, \tau) &= \frac{2 \sin\left(\frac{1}{2} i \kappa \tau\right)}{i\kappa \cos\left(\frac{1}{2} i \kappa \tau\right) + \kappa \sin\left(\frac{1}{2} i \kappa \tau\right)} \\
&= \frac{2}{i\kappa} \sin\left(\frac{1}{2} i \kappa \tau\right) e^{\frac{1}{2}\kappa\tau} \\
&= \frac{1 - e^{-\kappa\tau}}{\kappa},
\end{aligned}
\tag{6.49}
$$

and

$$
\lim_{\omega \to 0} H(\omega, \tau) = 1.
\tag{6.50}
$$

Furthermore, since

$$
k(\omega) = \sqrt{\xi^2 \omega^2 (1 - \rho^2) + \omega(\xi^2 - 2\rho\xi\kappa) - \kappa^2},
\tag{6.51}
$$

it follows that $k(\omega)$ has two real roots given by

$$
\omega_{1/2} = \frac{-(\xi - 2\rho\kappa) \pm \sqrt{(\xi - 2\rho\kappa)^2 + 4\kappa^2(1 - \rho^2)}}{2\xi(1 - \rho^2)},
\tag{6.52}
$$

where $\rho \neq \pm 1$ and where only the positive root ω_1 is of relevance. For $\rho = \pm 1$ we have a single root

$$
\omega_1 = \frac{\kappa^2}{\xi^2 \mp 2\xi\kappa}.
\tag{6.53}
$$

We deduce that all singular points of G and H are real, starting with ω_1 being a removable singularity. We therefore define c^* as the first non-removable singularity of G and H in $\{\omega \in \mathbb{C} \,|\, 0 < Re(\omega) < \infty, Im(\omega) = 0\}$, i.e. the first real root of $f(\omega)$ except ω_1 where $f(\omega)$ is defined by

$$
f(\omega) = k(\omega) \cos\left(\frac{1}{2} k(\omega)\tau\right) + (\omega\rho\xi + \kappa) \sin\left(\frac{1}{2} k(\omega)\tau\right).
\tag{6.54}
$$

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If $f(\omega)$ has no roots or no other roots except ω_1 in $\{\omega \in \mathbb{C} \mid 0 < Re(\omega) < \infty, Im(\omega) = 0\}$ we set $c^* = \infty$. By definition it follows that $\omega_1 \leq c^*$, with the special cases $\lim_{\tau \to 0} c^* = \infty$, and $\lim_{\tau \to \infty} c^* = \omega_1$.

6.3.1 Further Analysis

In the previous section a Mellin transform approach was used to solve the European put option pricing problem in Heston's mean reverting stochastic volatility model. The outcome is a new characterization of European put prices using an integration along a vertical line segment in a strip of the positive complex half plane. Our solution has a clear and well defined structure. The numerical treatment of the solution is simple and requires a single integration procedure. However, the final expression for the option's price can still be modified to provide further insights on the analytical solution. First we have the following proposition.

Proposition 6.3.3 *An equivalent (and more convenient) way of expressing the solution in Theorem 6.3.1 is:*

$$P^E(S, V, \tau) = Xe^{-r\tau}P_1 - Se^{-q\tau}P_2, \qquad (6.55)$$

with $S = S(t)$ being the current stock price,

$$P_1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{Xe^{-r\tau}}{Se^{-q\tau}}\right)^{\omega} \frac{1}{\omega} H(\omega, \tau) e^{G(\omega, \tau)a_0 V} d\omega, \qquad (6.56)$$

and

$$P_2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{Xe^{-r\tau}}{Se^{-q\tau}}\right)^{\omega+1} \frac{1}{\omega+1} H(\omega, \tau) e^{G(\omega, \tau)a_0 V} d\omega, \qquad (6.57)$$

where $0 < c < c^*$.

PROOF: The statement follows directly from Theorem 6.3.1 by simple rearrangement. □

Remark 6.3.4 *Equation (6.55) together with (6.56) and (6.57) provides a direct connection to Heston's original pricing formula given by*

$$P^E(S, V, \tau) = Xe^{-r\tau}\Pi_1 - Se^{-q\tau}\Pi_2,$$

with

$$\Pi_1 = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\omega \ln X}\varphi(\omega)}{i\omega}\right)d\omega,$$

and

$$\Pi_2 = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\omega \ln X}\varphi(\omega - i)}{i\omega\varphi(-i)}\right)d\omega,$$

where the function $\varphi(\omega)$ is the log-characteristic function of the stock at maturity $S(T)$:

$$\varphi(\omega) = E\left[e^{i\omega \ln S(T)}\right].$$

Remark 6.3.5 *By the fundamental concept of a risk-neutral valuation we have*

$$
\begin{aligned}
P^E(S, V, \tau) &= E_t^Q\left[e^{-r\tau}(X - S(T))\cdot \mathbb{1}_{\{S(T)<X\}}\right] \\
&= Xe^{-r\tau}E_t^Q\left[\mathbb{1}_{\{S(T)<X\}}\right] - Se^{-q\tau}E_t^{Q^*}\left[\mathbb{1}_{\{S(T)<X\}}\right],
\end{aligned}
$$

with E_t being the time t expectation under the corresponding risk-neutral probability measure, while Q^* denotes the equivalent martingale measure given by the Radon-Nikodym derivative

$$\frac{dQ^*}{dQ} = \frac{S(T)e^{-r\tau}}{Se^{-q\tau}}.$$

So the framework allows an expression of the above probabilities as the inverse of Mellin transforms.

A further advantage of the new framework is that hedging parameters, commonly known as Greeks or Greek letters, are easily determined analytically. These quantities represent the sensitivities of the price of a derivative to

a change in the underlying parameters on which the value is dependent. The most popular Greek letters widely used for risk management are delta, gamma, vega, rho, and theta. Two other second-order Greeks commonly used in financial markets are called vanna and vomma, respectively. Vanna is a second order derivative of the option value, once to the underlying spot price and once to volatility. Vomma or Vega Convexity measures the second order sensitivity to volatility. It is the second derivative of the option value with respect to the volatility. Equivalently one can say that vomma measures the rate of change to vega as volatility changes. The results for Greeks are summarized in the next proposition.

Proposition 6.3.6 *Setting*

$$I(\omega, \tau) = H(\omega, \tau)e^{G(\omega,\tau)a_0 V},$$

the analytical expressions for the delta, gamma, vega, rho, and theta in the case of European put options are given by, respectively,

$$P_S^E(S, V, \tau) = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{X}{S}\right)^{\omega+1} \frac{1}{\omega+1} e^{(q\omega-r(\omega+1))\tau} I(\omega, \tau) d\omega, \qquad (6.58)$$

$$P_{SS}^E(S, V, \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{S}\left(\frac{X}{S}\right)^{\omega+1} e^{(q\omega-r(\omega+1))\tau} I(\omega, \tau) d\omega, \qquad (6.59)$$

$$P_V^E(S, V, \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{X}{2}\left(\frac{X}{S}\right)^{\omega} e^{(q\omega-r(\omega+1))\tau} G(\omega, \tau) I(\omega, \tau) d\omega. \qquad (6.60)$$

Recall that the rho of a put option is the partial derivative of P^E with respect to the interest rate and equals

$$P_r^E(S, V, \tau) = \frac{-X\tau}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{X}{S}\right)^{\omega} \frac{1}{\omega} e^{(q\omega-r(\omega+1))\tau} I(\omega, \tau) d\omega. \qquad (6.61)$$

The theta of the put, i.e. the partial derivative of P^E with respect to τ is determined as

$$P_\tau^E(S, V, \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X\left(\frac{X}{S}\right)^{\omega} \frac{1}{\omega(\omega+1)} e^{(q\omega-r(\omega+1))\tau} I(\omega, \tau) J(\omega, \tau) d\omega,$$
$$(6.62)$$

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with

$$J(\omega, \tau) = q\omega - r(\omega + 1) + \frac{1}{2}\omega(\omega + 1)\big(\kappa\theta G(\omega, \tau) + V G_\tau(\omega, \tau)\big), \quad (6.63)$$

where

$$G_\tau(\omega, \tau) = \left(1 - \frac{(\omega\rho\xi + \kappa)^2}{\xi^2\omega(\omega + 1)}\right) \frac{1}{\cos^2\left(\frac{1}{2}k\tau + \tan^{-1}\left(\frac{-(\omega\rho\xi + \kappa)}{k}\right)\right)}. \quad (6.64)$$

Finally, the second-order Greeks Vanna and Vomma equal, respectively

$$P_{SV}^E(S, V, \tau) = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{2}\omega\left(\frac{X}{S}\right)^{\omega+1} e^{(q\omega - r(\omega+1))\tau} G(\omega, \tau) I(\omega, \tau) d\omega, \quad (6.65)$$

and

$$P_{VV}^E(S, V, \tau) = \frac{X}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{4}\omega(\omega + 1)\left(\frac{X}{S}\right)^{\omega} e^{(q\omega - r(\omega+1))\tau} G^2(\omega, \tau) I(\omega, \tau) d\omega. \quad (6.66)$$

PROOF: The expressions follow directly from Theorem 6.3.1 or Proposition 6.3.3. The expression for $J(\omega, \tau)$ follows by straightforward differentiation and (6.38). $\qquad \square$

We point out that instead of using the put call parity relationship for valuing European call options a direct Mellin transform approach is also possible. Recall that the European call option price $C^E(S, V, t)$ is characterized as the unique solution of (6.4) subject to

$$
\begin{aligned}
C^E(S, V, T) &= \max(S(T) - X, 0), \\
C^E(0, V, t) &= 0, \\
C^E(S, 0, t) &= \max(S(t)e^{-q(T-t)} - Xe^{-r(T-t)}, 0), \\
\lim_{S \to \infty} C^E(S, V, t) &= \infty,
\end{aligned}
$$

and

$$\lim_{V \to \infty} C^E(S, V, t) = S(t)e^{-q(T-t)}.$$

Applying the modified definition from Section 5.2. and following the lines of reasoning outlined in the previous section it is straightforward to derive at

Theorem 6.3.7 *The Mellin-type closed-form valuation formula for European call options in the square-root stochastic volatility model of Heston (1993) equals*

$$C^E(S, V, \tau) = Se^{-q\tau} P_2^* - Xe^{-r\tau} P_1^*, \qquad (6.67)$$

where

$$P_2^* = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{Se^{-q\tau}}{Xe^{-r\tau}}\right)^{\omega-1} \frac{1}{\omega-1} H^*(\omega, \tau) e^{G^*(\omega,\tau) a_0^* V} d\omega, \qquad (6.68)$$

and

$$P_1^* = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{Se^{-q\tau}}{Xe^{-r\tau}}\right)^{\omega} \frac{1}{\omega} H^*(\omega, \tau) e^{G^*(\omega,\tau) a_0^* V} d\omega, \qquad (6.69)$$

with

$$H^*(\omega, \tau) = e^{\frac{\kappa\theta}{\xi^2}\left[-(\omega\rho\xi-\kappa)\tau + 2\ln\left(\frac{k^*}{k^*\cos(\frac{1}{2}k^*\tau)-(\omega\rho\xi-\kappa)\sin(\frac{1}{2}k^*\tau)}\right)\right]}, \qquad (6.70)$$

$$G^*(\omega, \tau) = \frac{2\sin\left(\frac{1}{2}k^*\tau\right)}{k^*\cos\left(\frac{1}{2}k^*\tau\right) - (\omega\rho\xi - \kappa)\sin\left(\frac{1}{2}k^*\tau\right)}, \qquad (6.71)$$

$$k^* = k^*(\omega) = \sqrt{\xi^2\omega(\omega-1) - (\omega\rho\xi - \kappa)^2}, \qquad (6.72)$$

and $a_0^* = \frac{1}{2}\omega(\omega-1)$. Furthermore, we have that $1 < c < c^*$ with c^* being characterized equivalently as at the end of the previous section.

Remark 6.3.8 *Again, a direct analogy to Heston's original pricing formula is provided. The corresponding expressions for the Greeks follow immediately either by direct differentiation of the price formula or using the put-call parity.*

6.3.2 Numerical Examples

In this section we evaluate the results of the previous sections for the purpose of computing and comparing option prices for a range of different parameter combinations. Since our numerical calculations are not based on a calibration procedure we will use notional parameter specifications. As a benchmark we choose the pricing formula due to Heston based on Fourier

inversion (H). From the previous analysis it follows that the numerical inversion in both integral transform approaches requires the calculation of logarithms with complex arguments. As pointed out by Schöbel and Zhu (1999) and Kahl and Jäckel (2005) this calculation may cause problems especially for options with long maturities or high mean reversion levels. We therefore additionally implement the rotation count algorithm proposed by the second authors to overcome these possible inconsistencies (H(RCA)). The Mellin transform solution (MT) is based on equations (6.47) for puts and (6.67) for calls, respectively. The limits of integration $c \pm i\infty$ are truncated at $c \pm i500$. Although any other choice of truncation is possible this turned out to provide comparable results. To assess the accuracy of the alternative solutions we determine the absolute difference between H(RCA) and MT (Diff). Table 6.1 gives a first look at the results for different asset prices and expiration dates. We distinguish between in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) options. Fixed parameters are $X = 100$, $r = 0.04$, $q = 0.02$, $V = 0.09$, $\kappa = 3$, $\theta = 0.12$, $\xi = 0.2$, and $\rho = -0.5$, whereas S and τ vary from 80 to 120 currency units, and three months to three years, respectively. Using these values we have for the European put $\omega_1 = 9.6749$ constant, while c^* varies over time from 54.7066 ($\tau = 0.25$) to 11.7046 ($\tau = 3.0$) and for the European call $\omega_1 = 31.0082$ with c^* changing from 116.7385 ($\tau = 0.25$) to 33.7810 ($\tau = 3.0$). We therefore use $c = 2$ for the calculations (in both cases). Our major finding is that the pricing formulae derived in this paper provide comparable results for all parameter combinations. The absolute differences are very small (of order 10^{-6} to 10^{-8} for puts and 10^{-5} to 10^{-8} for calls). They can be neglected from a practical point of view. In addition, since the numerical integration is accomplished in both frameworks equivalently efficient, the calculations are done very quickly.

Next, we examine how the option prices vary if the correlation between the underlying asset and its instantaneous variance changes. Although from a

(S, τ)	Puts				Calls			
	H	H(RCA)	MT	Diff	H	H(RCA)	MT	Diff
$(80 \,;\, 0.25)$	19.8379	19.8379	19.8379	$1.7 \cdot 10^{-6}$	0.4339	0.4339	0.4339	$1.7 \cdot 10^{-6}$
$(90 \,;\, 0.25)$	11.6806	11.6806	11.6806	$1.1 \cdot 10^{-6}$	2.2267	2.2268	2.2268	$1.1 \cdot 10^{-6}$
$(100 \,;\, 0.25)$	5.9508	5.9508	5.9508	$4.9 \cdot 10^{-7}$	6.4471	6.4471	6.4471	$4.9 \cdot 10^{-7}$
$(110 \,;\, 0.25)$	2.6708	2.6708	2.6708	$6.4 \cdot 10^{-6}$	13.1172	13.1173	13.1173	$6.4 \cdot 10^{-5}$
$(120 \,;\, 0.25)$	1.0870	1.0870	1.0870	$7.5 \cdot 10^{-6}$	21.4835	21.4835	21.4835	$7.4 \cdot 10^{-6}$
$(80 \,;\, 0.5)$	20.5221	20.5221	20.5221	$3.4 \cdot 10^{-6}$	1.7062	1.7062	1.7062	$3.4 \cdot 10^{-6}$
$(90 \,;\, 0.5)$	13.5342	13.5342	13.5342	$2.2 \cdot 10^{-6}$	4.6188	4.6188	4.6188	$2.2 \cdot 10^{-6}$
$(100 \,;\, 0.5)$	8.4302	8.4302	8.4302	$1.1 \cdot 10^{-6}$	9.4153	9.4153	9.4153	$1.1 \cdot 10^{-6}$
$(110 \,;\, 0.5)$	5.0232	5.0232	5.0232	$3.0 \cdot 10^{-7}$	15.9088	15.9088	15.9088	$3.0 \cdot 10^{-7}$
$(120 \,;\, 0.5)$	2.8995	2.8995	2.8995	$9.7 \cdot 10^{-7}$	23.6856	23.6856	23.6856	$9.7 \cdot 10^{-7}$
$(80 \,;\, 1.0)$	22.1413	22.1413	22.1413	$6.7 \cdot 10^{-6}$	4.4783	4.4782	4.4783	$6.7 \cdot 10^{-6}$
$(90 \,;\, 1.0)$	16.2923	16.2923	16.2923	$4.7 \cdot 10^{-6}$	8.4312	8.4312	8.4312	$4.7 \cdot 10^{-6}$
$(100 \,;\, 1.0)$	11.7819	11.7819	11.7819	$2.3 \cdot 10^{-6}$	13.7229	13.7229	13.7229	$2.3 \cdot 10^{-6}$
$(110 \,;\, 1.0)$	8.4207	8.4207	8.4207	$2.5 \cdot 10^{-7}$	20.1636	20.1636	20.1636	$2.5 \cdot 10^{-7}$
$(120 \,;\, 1.0)$	5.9755	5.9755	5.9755	$2.3 \cdot 10^{-6}$	27.5204	27.5204	27.5204	$2.3 \cdot 10^{-6}$
$(80 \,;\, 2.0)$	24.5972	24.5972	24.5972	$1.3 \cdot 10^{-6}$	9.1487	9.1487	9.1487	$1.3 \cdot 10^{-5}$
$(90 \,;\, 2.0)$	19.8041	19.8041	19.8041	$8.2 \cdot 10^{-6}$	13.9635	13.9635	13.9635	$8.2 \cdot 10^{-6}$
$(100 \,;\, 2.0)$	15.9136	15.9136	15.9136	$3.6 \cdot 10^{-6}$	19.6809	19.6809	19.6809	$3.6 \cdot 10^{-6}$
$(110 \,;\, 2.0)$	12.7852	12.7852	12.7852	$7.2 \cdot 10^{-7}$	26.1604	26.1604	26.1604	$7.2 \cdot 10^{-7}$
$(120 \,;\, 2.0)$	10.2833	10.2833	10.2833	$5.2 \cdot 10^{-6}$	33.2664	33.2664	33.2664	$5.2 \cdot 10^{-6}$
$(80 \,;\, 3.0)$	26.1731	26.1731	26.1731	$1.4 \cdot 10^{-6}$	12.8222	12.8222	12.8222	$1.4 \cdot 10^{-6}$
$(90 \,;\, 3.0)$	21.9865	21.9865	21.9865	$7.3 \cdot 10^{-6}$	18.0533	18.0533	18.0533	$7.3 \cdot 10^{-7}$
$(100 \,;\, 3.0)$	18.5011	18.5011	18.5011	$2.3 \cdot 10^{-8}$	23.9855	23.9855	23.9855	$2.3 \cdot 10^{-8}$
$(110 \,;\, 3.0)$	15.6055	15.6055	15.6055	$6.9 \cdot 10^{-6}$	30.5076	30.5076	30.5076	$6.9 \cdot 10^{-6}$
$(120 \,;\, 3.0)$	13.2004	13.2004	13.2004	$1.2 \cdot 10^{-6}$	37.5201	37.5201	37.5201	$1.2 \cdot 10^{-6}$

Table 6.1: European option prices in Heston's stochastic volatility model for different asset prices S and maturities τ. Fixed parameters are $X = 100$, $r = 0.04$, $q = 0.02$, $V = 0.09$, $\kappa = 3$, $\theta = 0.12$, $\xi = 0.2$, $\rho = -0.5$, and $c = 2$.

practical point of view it may be less realistic to allow for a positive correlation we do not make any restrictions on ρ and let it range from -1.00 to 1.00. We fix time to maturity to be 6 months. Also, to provide a variety of parameter combinations we change some of the remaining parameters slightly: $X = 100$, $r = 0.05$, $q = 0.02$, $V = 0.04$, $\kappa = 2$, $\theta = 0.05$, and $\xi = 0.2$. We

abstain from presenting the numerical values of ω_1 and c^* in this case and choose again $c = 2$ for the integration. Our findings are reported in Table 6.2.

	Puts				Calls			
(S, ρ)	H	H(RCA)	MT	Diff	H	H(RCA)	MT	Diff
$(80 \,; \text{-}1.00)$	18.4620	18.4620	18.4620	$1.7 \cdot 10^{-6}$	0.1350	0.1350	0.1350	$1.7 \cdot 10^{-6}$
$(100 \,; \text{-}1.00)$	5.0431	5.0431	5.0431	$2.1 \cdot 10^{-6}$	6.5170	6.5170	6.5170	$2.1 \cdot 10^{-6}$
$(120 \,; \text{-}1.00)$	1.0353	1.0353	1.0353	$2.6 \cdot 10^{-5}$	22.3103	22.3103	22.3103	$2.6 \cdot 10^{-5}$
$(80 \,; \text{-}0.75)$	18.5533	18.5533	18.5533	$1.3 \cdot 10^{-6}$	0.2263	0.2263	0.2263	$1.3 \cdot 10^{-6}$
$(100 \,; \text{-}0.75)$	5.0403	5.0403	5.0403	$4.1 \cdot 10^{-6}$	6.5143	6.5143	6.5143	$4.1 \cdot 10^{-6}$
$(120 \,; \text{-}0.75)$	0.9541	0.9541	0.9541	$6.6 \cdot 10^{-6}$	22.2291	22.2291	22.2291	$6.6 \cdot 10^{-6}$
$(80 \,; \text{-}0.50)$	18.6413	18.6413	18.6413	$1.0 \cdot 10^{-6}$	0.3143	0.3143	0.3143	$1.0 \cdot 10^{-6}$
$(100 \,; \text{-}0.50)$	5.0371	5.0371	5.0371	$4.4 \cdot 10^{-6}$	6.5111	6.5111	6.5111	$4.4 \cdot 10^{-6}$
$(120 \,; \text{-}0.50)$	0.8695	0.8695	0.8695	$2.5 \cdot 10^{-6}$	22.1445	22.1445	22.1445	$2.5 \cdot 10^{-6}$
$(80 \,; \text{-}0.25)$	18.7269	18.7269	18.7269	$7.9 \cdot 10^{-6}$	0.3999	0.3999	0.3999	$7.9 \cdot 10^{-6}$
$(100 \,; \text{-}0.25)$	5.0332	5.0332	5.0332	$4.7 \cdot 10^{-6}$	6.5072	6.5072	6.5072	$4.7 \cdot 10^{-6}$
$(120 \,; \text{-}0.25)$	0.7812	0.7812	0.7812	$1.5 \cdot 10^{-6}$	22.0562	22.0562	22.0562	$1.5 \cdot 10^{-6}$
$(80 \,; 0.00)$	18.8104	18.8104	18.8104	$4.9 \cdot 10^{-5}$	0.4834	0.4834	0.4834	$4.9 \cdot 10^{-5}$
$(100 \,; 0.00)$	5.0285	5.0285	5.0285	$2.7 \cdot 10^{-5}$	6.5025	6.5025	6.5025	$3.0 \cdot 10^{-5}$
$(120 \,; 0.00)$	0.6887	0.6887	0.6887	$6.0 \cdot 10^{-5}$	21.9637	21.9637	21.9637	$6.0 \cdot 10^{-5}$
$(80 \,; 0.25)$	18.8921	18.8921	18.8921	$1.1 \cdot 10^{-6}$	0.5651	0.5651	0.5651	$1.1 \cdot 10^{-6}$
$(100 \,; 0.25)$	5.0229	5.0229	5.0229	$5.3 \cdot 10^{-6}$	6.4969	6.4969	6.4969	$5.3 \cdot 10^{-6}$
$(120 \,; 0.25)$	0.5913	0.5913	0.5913	$9.6 \cdot 10^{-6}$	21.8663	21.8663	21.8663	$9.5 \cdot 10^{-6}$
$(80 \,; 0.50)$	18.9721	18.9721	18.9721	$2.2 \cdot 10^{-6}$	0.6451	0.6451	0.6450	$2.2 \cdot 10^{-6}$
$(100 \,; 0.50)$	5.0166	5.0166	5.0166	$5.7 \cdot 10^{-6}$	6.4906	6.4906	6.4906	$5.7 \cdot 10^{-6}$
$(120 \,; 0.50)$	0.4882	0.4881	0.4881	$1.2 \cdot 10^{-6}$	21.7931	21.7630	21.7630	$1.2 \cdot 10^{-6}$
$(80 \,; 1.00)$	19.1275	19.1275	19.1275	$9.60 \cdot 10^{-6}$	0.8005	0.8005	0.8005	$1.4 \cdot 10^{-5}$
$(100 \,; 1.00)$	5.0027	5.0027	5.0027	$4.2 \cdot 10^{-6}$	6.4767	6.4767	6.4767	$5.7 \cdot 10^{-6}$
$(120 \,; 1.00)$	0.2566	0.2566	0.2566	$1.3 \cdot 10^{-6}$	21.5316	21.5316	21.5316	$2.0 \cdot 10^{-6}$

Table 6.2: European option prices in Heston's stochastic volatility model for different asset prices S and correlations ρ. Fixed parameters are $X = 100$, $r = 0.05$, $q = 0.02$, $V = 0.04$, $\kappa = 2$, $\theta = 0.05$, $\xi = 0.2$, and $c = 2$.

The Mellin transform approach gives satisfactory results as the absolute differences show. For both puts and calls they are of order 10^{-5} to 10^{-6}. An-

alyzing the results in detail one basically observes two different kinds of behavior. For ITM put options we have an increase in value for increasing values of ρ. The maximum difference is 0.6655 or 3.60%. The opposite is true for OTM puts. Here we have a strict decline in price if ρ is increased. The magnitude of price reactions to changes in ρ increases, too. The maximum change in the downward move is 0.7787 or equivalently 75.21%. The same behavior is observed for ATM options. However, the changes are much more moderate with a maximum percentage change of 0.80%. For European calls the situation is different. OTM calls increase significantly in value if ρ is increased whereas ITM and ATM call prices decrease for an increasing ρ. The maximum percentage changes are 492.96% (OTM), 3.49% (ITM), and 0.62% (ATM), respectively.

Finally, we compare the values of delta for different $(S; \tau)$ combinations. For the calculation of the delta of a European put we use equation (6.58). The corresponding delta value for a call is easily determined from (6.65). S and τ vary from 80 to 120 currency units, and three months to three years, respectively. Again, the remaining parameters are slightly altered and equal $X = 100$, $r = 0.06$, $q = 0.03$, $V = 0.16$, $\kappa = 3$, $\theta = 0.16$, $\xi = 0.10$, $\rho = 0.75$, and $c = 2$. The results are summarized in Table 6.3. Once more, we observe a high consistency with Heston's framework based on Fourier inversion. For all parameter combinations our results agree with Heston's with a great degree of precision.

In summary, our numerical experiments suggest that the new framework is able to compete with Heston's solution based on Fourier inversion. The accuracy of the results is very satisfying and the framework is flexible enough to account for all the pricing features inherent in the model. The findings justify the assessment of the Mellin transform approach as a very competitive alternative.

	Puts				Calls			
$(S\,;\tau)$	$\triangle_H$	$\triangle_{H(RCA)}$	$\triangle_{MT}$	Diff	$\triangle_H$	$\triangle_{H(RCA)}$	$\triangle_{MT}$	Diff
$(80\,;0.25)$	-0.8318	-0.8318	-0.8318	$2.4\cdot10^{-7}$	0.1607	0.1607	0.1607	$2.4\cdot10^{-7}$
$(90\,;0.25)$	-0.6422	-0.6422	-0.6422	$2.4\cdot10^{-7}$	0.3503	0.3503	0.3503	$2.4\cdot10^{-7}$
$(100\,;0.25)$	-0.4348	-0.4348	-0.4348	$2.4\cdot10^{-7}$	0.5578	0.5578	0.5578	$2.4\cdot10^{-7}$
$(110\,;0.25)$	-0.2625	-0.2625	-0.2625	$2.4\cdot10^{-7}$	0.7300	0.7300	0.7300	$2.4\cdot10^{-7}$
$(120\,;0.25)$	-0.1447	-0.1447	-0.1447	$2.5\cdot10^{-7}$	0.8479	0.8479	0.8479	$2.5\cdot10^{-7}$
$(80\,;0.5)$	-0.7118	-0.7118	-0.7118	$4.8\cdot10^{-7}$	0.2734	0.2734	0.2734	$4.8\cdot10^{-7}$
$(90\,;0.5)$	-0.5558	-0.5558	-0.5558	$4.8\cdot10^{-7}$	0.4294	0.4294	0.4294	$4.8\cdot10^{-7}$
$(100\,;0.5)$	-0.4085	-0.4085	-0.4085	$4.8\cdot10^{-7}$	0.5766	0.5766	0.5766	$4.8\cdot10^{-7}$
$(110\,;0.5)$	-0.2863	-0.2863	-0.2863	$4.8\cdot10^{-7}$	0.6988	0.6988	0.6988	$4.7\cdot10^{-7}$
$(120\,;0.5)$	-0.1936	-0.1936	-0.1936	$4.8\cdot10^{-7}$	0.7915	0.7915	0.7915	$4.8\cdot10^{-7}$
$(80\,;1.0)$	-0.5892	-0.5892	-0.5892	$8.6\cdot10^{-8}$	0.3812	0.3812	0.3812	$8.6\cdot10^{-8}$
$(90\,;1.0)$	-0.4738	-0.4738	-0.4738	$8.7\cdot10^{-8}$	0.4966	0.4966	0.4966	$8.7\cdot10^{-8}$
$(100\,;1.0)$	-0.3723	-0.3723	-0.3723	$8.6\cdot10^{-8}$	0.5981	0.5981	0.5981	$8.6\cdot10^{-8}$
$(110\,;1.0)$	-0.2878	-0.2878	-0.2878	$8.0\cdot10^{-8}$	0.6827	0.6827	0.6827	$8.0\cdot10^{-8}$
$(120\,;1.0)$	-0.2199	-0.2199	-0.2199	$8.6\cdot10^{-8}$	0.7505	0.7505	0.7505	$8.6\cdot10^{-8}$
$(80\,;2.0)$	-0.4684	-0.4684	-0.4684	$1.7\cdot10^{-7}$	0.4733	0.4733	0.4733	$1.7\cdot10^{-7}$
$(90\,;2.0)$	-0.3895	-0.3895	-0.3895	$1.7\cdot10^{-7}$	0.5523	0.5523	0.5523	$1.7\cdot10^{-7}$
$(100\,;2.0)$	-0.3222	-0.3222	-0.3222	$1.7\cdot10^{-7}$	0.6196	0.6196	0.6196	$1.7\cdot10^{-7}$
$(110\,;2.0)$	-0.2659	-0.2659	-0.2659	$1.7\cdot10^{-7}$	0.6758	0.6758	0.6758	$1.7\cdot10^{-7}$
$(120\,;2.0)$	-0.2193	-0.2193	-0.2193	$1.7\cdot10^{-7}$	0.7224	0.7224	0.7224	$1.7\cdot10^{-7}$
$(80\,;3.0)$	-0.3969	-0.3969	-0.3969	$2.4\cdot10^{-7}$	0.5170	0.5170	0.5170	$2.4\cdot10^{-7}$
$(90\,;3.0)$	-0.3361	-0.3361	-0.3361	$2.4\cdot10^{-7}$	0.5779	0.5779	0.5779	$2.4\cdot10^{-7}$
$(100\,;3.0)$	-0.2847	-0.2847	-0.2847	$2.4\cdot10^{-7}$	0.6292	0.6292	0.6292	$2.4\cdot10^{-7}$
$(110\,;3.0)$	-0.2417	-0.2417	-0.2417	$2.4\cdot10^{-7}$	0.6723	0.6723	0.6723	$2.4\cdot10^{-7}$
$(120\,;3.0)$	-0.2056	-0.2056	-0.2056	$2.4\cdot10^{-7}$	0.7083	0.7083	0.7083	$2.4\cdot10^{-7}$

Table 6.3: Delta values of European option prices in Heston's stochastic volatility model for different asset prices S and maturities τ. Fixed parameters are $X = 100$, $r = 0.06$, $q = 0.03$, $V = 0.16$, $\kappa = 3$, $\theta = 0.16$, $\xi = 0.1$, $\rho = -0.75$, and $c = 2$.

Chapter 7

Conclusion

This thesis is the outcome of my research activities at the Eberhard Karls University in Tuebingen. The thesis was concerned with the pricing of options based on a Mellin integral transform approach. The interest in this subject was motivated by the articles of Panini and Srivastav (2004) and Panini and Srivastav (2005). Several extensions were derived by using the new methodology.

Firstly, we have established a new formula for European power put options on dividend-paying stocks consisting of a single integral. Focusing on plain vanilla American put options on dividend-paying stocks, we have used the Mellin transform approach to derive the valuation formulas for the price and the free boundary. To emphasize the generality of the results, we have proved the equivalence of the new pricing formula to the integral characterizations of Kim (1990), Jacka (1991), and Carr et al. (1992). Also, we have recovered important theoretical properties of the pricing function. Finally, it was shown how to use the expressions to derive the price of a perpetual American put. This is a straightforward extension of Panini and Srivastav (2005).

The second contribution of the thesis is the introduction of a modified version of the integral transform that allows an analytical valuation of European and American call options. The outcome of this modification are new inte-

gral representations of American call options and free boundaries. For the modified transform, it was also shown how Gauss-Laguerre quadrature may be applied for an accurate numerical evaluation. Additionally, we have recovered important theoretical properties of American call options using the new method.

The integral expressions for American options were used subsequently to provide simple analytical approximations for the free boundary. It was shown that the approximations are correct in the sense that they satisfy all important and financially meaningful asymptotic requirements. Our approximative solutions are easy to implement and numerically robust. To provide a sufficient numerical analysis, we have compared our solutions to eleven other numerical and analytical approaches found in the literature. The numerical experiments have shown that the approximations produce accurate prices for the critical stock price for a wide range of parameter combinations. Many of the alternative frameworks have been outperformed by our solutions.

Finally, we have left the log-normal model behind us and have applied the new integral transform approach for a closed-form valuation of European options within the mean reverting stochastic volatility model of Heston (1993). Here, the main results are new analytical characterizations of options' prices and hedging parameters. The new solutions can be written as a single integral which allows a fast and efficient computation. Numerical tests have demonstrated the flexibility and accuracy of the sophisticated alternative solutions.

Further research can explore extensions of the methodology. In the case of American options, the analysis presented in this thesis was restricted to plain vanilla options in the Black/Scholes and Merton economy. A natural extension is therefore the valuation of American-style contracts on two (or more) assets or options with more complicated payoffs. Prominent candidates are for instance exchange options or American straddles and strangles. The more complicated payoff will affect the non-homogeneous part of the early exercise

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premium. However, it could be possible to solve the equations explicitly in some cases. For European options, extensions may be possible to other price processes, such as jump diffusions, other stochastic volatility, and/or interest rate models. Also two-dimensional or even n-dimensional pricing problems could be studied using the double or n-dimensional Mellin transform. Finally, one could try to combine the different aspects and use this approach to value path dependent American options, American options in jump diffusion, or stochastic volatility models.

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