Typical Macroscopic Behavior of Large Quantum Systems

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Zusammenfassung

In der vorliegenden Dissertation untersuchen wir das Verhalten von typischen reinen Zuständen großer Quantensysteme. Wir verallgemeinern gewisse Typizitätsaussagen von der Gleichverteilung auf der Sphäre auf die viel größere Klasse der sogenannten GAP-Maße und auf realistischere Hamiltonoperatoren und untersuchen die Thermalisierung von Systemen mit hochentarteten Hamiltonoperatoren.

Im ersten Teil dieser Arbeit beweisen wir eine Verallgemeinerung von kanonischer Typizität, der Tatsache, dass für die meisten reinen Zustände ψ eines großen endlich-dimensionalen Hilbertraums $\mathcal{H}$ die reduzierte Dichtematrix eines kleinen Teilsystems näherungsweise ψ-unabhängig und, falls das Teilsystem und seine Umgebung nur schwach wechselwirken, näherungsweise kanonisch ist. Hierbei bezieht sich "die meisten" auf die Gleichverteilung auf der Sphäre $\mathbb{S}(\mathcal{H})$. Wir verallgemeinern kanonische Typizität von der Gleichverteilung auf GAP-Maße als Typizitätsmaße. Grob gesagt ist für jede Dichtematrix ρ auf einem separablen Hilbertraum $\mathcal{H}$ das Maß GAP(ρ) die am meisten ausgebreitete Verteilung auf $\mathbb{S}(\mathcal{H})$ mit Dichtematrix ρ. Ist ρ eine kanonische Dichtematrix, so ergibt sich GAP(ρ) als die thermische Gleichgewichtsverteilung von Wellenfunktionen und kann als ein quantenmechanisches Analogon des kanonischen Ensembles der klassischen statistischen Mechanik aufgefasst werden. Deshalb lassen sich Verallgemeinerungen von Resultaten für die Gleichverteilung auf der Sphäre zu Resultaten für GAP-Maße als Ausdruck einer Version der Äquivalenz der Ensembles ansehen. Die Hauptzutat für den Beweis unseres Resultats ist eine Verallgemeinerung von Lévy's Lemma, einer Aussage über die Konzentration des Maßes für die Gleichverteilung auf der Sphäre, auf GAP-Maße. Des Weiteren verallgemeinern wir dynamische Typizität, die Aussage, dass für jeden Operator B auf $\mathcal{H}$ und jedes $t \in \mathbb{R}$ für die meisten $\psi \in \mathbb{S}(\mathcal{H})$ der Erwartungswert $\langle\psi_t|B|\psi_t\rangle$ näherungsweise ψ-unabhängig ist, auf GAP-Maße. Hierbei folgt ψ_t einer unitären Zeitentwicklung in $\mathcal{H}$. Außerdem zeigen wir, dass für jedes $0 < T < \infty$ die ganze Kurve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ näherungsweise ψ-unabhängig auf dem Zeitintervall $[0, T]$ ist.

Der zweite Teil dieser Arbeit beschäftigt sich mit Aspekten der Zeitentwicklung von typischen reinen Zuständen. Zunächst betrachten wir wieder einen endlich-dimensionalen Hilbertraum $\mathcal{H}$ und eine sich unitär entwickelnde Wellenfunktion $\psi_t \in \mathbb{S}(\mathcal{H})$. Von Neumann folgend nehmen wir als gegeben hin, dass $\mathcal{H}$ orthogonal zerlegt ist in hochdimensionale Unterräume $\mathcal{H}_\nu$, die sogenannten Makroräume, die zu verschiedenen Makrozuständen ν gehören. Normalerweise gibt es in der Zerlegung einen Makroraum, der mit Abstand am größten ist und das thermische Gleichgewicht beschreibt; wir bezeichnen ihn als $\mathcal{H}_{\text{eq}}$. Sei P_ν die Projektion auf $\mathcal{H}_\nu$. Normale Typi-

zität ist die Aussage, dass für alle Anfangszustände $\psi_0 \in \mathbb{S}(\mathcal{H})$ und die meisten Zeiten $t \geq 0$ das Superpositionsgewicht $\|P_\nu \psi_t\|^2$ nahe bei d_ν/D ist, wobei $d_\nu = \dim \mathcal{H}_\nu$ und $D = \dim \mathcal{H} < \infty$ die Dimensionen der entsprechenden Hilberträume sind. Diese Aussage gilt für Hamiltonoperatoren mit gleichverteilter Eigenbasis – eine Annahme, die physikalisch nicht realistisch ist, da sie zu extrem kurzen Thermalisierungszeiten führt. Wir erwarten realistischere Thermalisierungszeiten für Hamiltonoperatoren mit einer Bandstruktur in einer Basis, die die Projektionen P_ν diagonalisiert. Deshalb zeigen wir die folgende Verallgemeinerung von normaler Typizität: Für alle Hamiltonoperatoren mit nicht zu hoch entarteten Eigenwerten und Eigenwertdifferenzen sind die meisten $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ derart, dass für die meisten $t \geq 0$ das Gewicht $\|P_\nu \psi_t\|^2$ ungefähr gegeben ist durch eine ψ_0- und t-unabhängige Größe $M_{\mu\nu}$. Hierbei bezieht sich "die meisten $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$" wieder auf die Gleichverteilung und die beiden Größen sind nah beieinander in dem Sinn, dass der absolute Fehler klein ist. Ist allerdings $M_{\mu\nu}$ selbst klein, so sind kleine absolute Fehler nicht besonders aussagekräftig und wir müssen zeigen, dass auch die relativen Fehler klein sind. Dafür modellieren wir den Hamiltonoperator als eine zufällige Bandmatrix. Für den Beweis, dass in diesem Fall auch die relativen Fehler klein sind, verwenden wir die "Keine-Lücken-Delokalisierung", ein Resultat über die Delokalisierung der Eigenvektoren einer Zufallsmatrix. Des Weiteren zeigen wir auch eine Verallgemeinerung von (verallgemeinerter) normaler Typizität auf GAP-Maße (auf separablen Hilberträumen). Wir bemerken, dass wir auch Resultate für allgemeine Operatoren B statt der Projektionen P_ν und Resultate für endliche Zeitintervalle beweisen.

Im dritten Teil dieser Arbeit untersuchen wir die Thermalisierung von hochentarteten Hamiltonoperatoren. Wir nehmen wieder an, dass der (endlich-dimensionale) Hilbertraum $\mathcal{H}$ des Systems wie oben in Makroräume zerlegt ist. Wir sagen, dass ein abgeschlossenes makroskopisches Quantensystem in einem reinen Zustand ψ im makroskopischen thermischen Gleichgewicht (MATE) ist, falls ψ in oder nahe bei $\mathcal{H}_{\mathrm{eq}}$ liegt. Falls der Hamiltonoperator H des Systems die Eigenzustands-Thermalisierungs-Hypothese (ETH) erfüllt, also falls jede Eigenfunktion von H in MATE ist, dann thermalisiert jeder Anfangszustand $\psi_0 \in \mathbb{S}(\mathcal{H})$, d.h. er erreicht nach einiger Zeit MATE und verbringt dort die meiste Zeit. Motiviert von aktuellen Arbeiten von Shiraishi und Tasaki untersuchen wir die Thermalisierung eines Systems freier Fermionen, dessen Hamiltonoperator hochentartet ist. Während wir zeigen können, dass der zugehörige Hamiltonoperator in einer Dimension die ETH erfüllt (und somit jeder Anfangszustand thermalisiert), können wir in höheren Dimensionen nur zeigen, dass es eine Eigenbasis gibt, deren Eigenvektoren in MATE sind, und es ist möglich, dass die ETH verletzt ist (und somit nicht jeder Anfangszustand thermalisiert). Inspiriert von dieser Situation zeigen wir das folgende Resultat: Ist H_0 ein Hamiltonoperator, der eine Eigenbasis besitzt, deren Eigenvektoren in MATE sind, und addieren wir eine kleine zufällige Störung λV mit $\lambda \ll 1$, dann thermalisiert jeder Anfangszustand $\psi_0 \in \mathbb{S}(\mathcal{H})$ für die meisten Störungen V und für jeden Makrozustand ν sind die meisten Störungen V derart, dass die meisten $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ thermalisieren.

Summary

In this thesis, we study the behavior of typical pure states of large quantum systems. We generalize certain typicality statements from the uniform distribution on the sphere to the much broader class of so-called GAP measures and to more realistic Hamiltonians and study the thermalization of systems with highly degenerate Hamiltonians.

In the first part of this thesis we prove a generalization of canonical typicality, the fact that for most pure states ψ from a large finite-dimensional Hilbert space $\mathcal{H}$ the reduced density matrix of a small subsystem is nearly ψ-independent and, if the subsystem and its environment are only weakly interacting, nearly canonical. Here, "most" refers to the uniform distribution on the sphere $\mathbb{S}(\mathcal{H})$. We generalize canonical typicality from the uniform distribution to GAP measures as measures of typicality. Roughly speaking, for any density matrix ρ on a separable Hilbert space $\mathcal{H}$, the measure $\mathrm{GAP}(\rho)$ is the most spread out distribution on $\mathbb{S}(\mathcal{H})$ with density matrix ρ. If ρ is a canonical density matrix, $\mathrm{GAP}(\rho)$ arises as the thermal equilibrium distribution of wave functions and can be seen as a quantum analogue of the canonical ensemble of classical statistical mechanics. Thus, generalizations of results for the uniform distribution on the sphere to GAP measures can be regarded as expressing a version of equivalence of ensembles. The main tool for the proof of our result is a generalization of Lévy's Lemma, a concentration-of-measure-type result for the uniform distribution on the sphere, to GAP measures. Moreover, we also generalize dynamical typicality, the statement that for any operator B on $\mathcal{H}$ and every $t \in \mathbb{R}$ for most $\psi_0 \in \mathbb{S}(\mathcal{H})$ the expectation $\langle\psi_t|B|\psi_t\rangle$ is nearly ψ_0-independent, to GAP measures. Here, ψ_t evolves unitarily in $\mathcal{H}$. Furthermore, we show that for any $0 < T < \infty$ the whole curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ is approximately ψ_0-independent on the time interval $[0, T]$.

The second part of this thesis is concerned with aspects of the time evolution of typical pure states. At first we again consider a finite-dimensional Hilbert space $\mathcal{H}$ and a unitarily evolving wave function $\psi \in \mathbb{S}(\mathcal{H})$. Following von Neumann, we take for granted an orthogonal decomposition of the Hilbert space $\mathcal{H}$ into high-dimensional subspaces $\mathcal{H}_\nu$, the so-called macro spaces, which correspond to different macro states ν of the system. Usually there is one macro space in the decomposition that is by far the largest one and which is associated with thermal equilibrium; we denote it by $\mathcal{H}_{\mathrm{eq}}$. Let P_ν be the projection to $\mathcal{H}_\nu$. Normal typicality is the statement that for all initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$ and most times $t \geq 0$, the superposition weight $\|P_\nu\psi_t\|^2$ is close to d_ν/D, where $d_\nu = \dim\mathcal{H}_\nu$ and $D = \dim\mathcal{H} < \infty$. This statement holds true for Hamiltonians with uniformly distributed eigenbasis – an assumption that is physically not realistic as it leads to extremely small thermalization times. We expect to obtain more realistic thermalization times for Hamiltonians with a band structure in a basis that diagonalizes the projections P_ν. Therefore we show the following generalization of normal typicality: For all Hamiltonians with not too highly degenerate eigenvalues and eigenvalue gaps most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for most $t \geq 0$ the superposition weight $\|P_\nu \psi_t\|^2$ is close to a ψ_0- and t-independent quantity $M_{\mu\nu}$. Here "most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$" refers again to the uniform distribution on the sphere and the closeness means that the absolute errors are small. However, if $M_{\mu\nu}$ is small itself, small absolute errors are not very meaningful and we have to show that the relative errors are small as well. To this end we model the Hamiltonian by a random band matrix. For the proof of small relative errors we make use of a no-gaps delocalization result concerning the delocalization of the eigenvectors of a random matrix. Moreover, we also prove a generalization of (generalized) normal typicality to GAP measures (on separable Hilbert spaces). Note that we also show statements for arbitrary operators B instead of the projections P_ν and also some finite-time results.

In the third part of this thesis we study the thermalization of highly degenerate Hamiltonians. We again assume that the system's (finite-dimensional) Hilbert space $\mathcal{H}$ is decomposed into macro spaces as above. A closed macroscopic quantum system in a pure state ψ is said to be in macroscopic thermal equilibrium (MATE) if ψ lies in or close to $\mathcal{H}_{\mathrm{eq}}$. If the system's Hamiltonian H satisfies the eigenstate thermalization hypothesis (ETH), i.e., if every eigenfunction of H is in MATE, then every initial state $\psi_0 \in \mathbb{S}(\mathcal{H})$ thermalizes, i.e., after some time it reaches MATE and stays there for most of the time. Motivated by recent works of Shiraishi and Tasaki, we study the thermalization of a system of free fermions whose Hamiltonian is highly degenerate. While we can show that in one dimension the corresponding Hamiltonian satisfies the ETH (and therefore every initial state thermalizes), in higher dimensions we can only show that there is one eigenbasis whose eigenvectors are in MATE and it might be the case that the ETH is violated (and thus not every initial state thermalizes). Inspired by this situation we prove the following result: If H_0 is a Hamiltonian that has an eigenbasis whose eigenvectors are in MATE and we add a small random perturbation λV with $\lambda \ll 1$, then every initial state $\psi_0 \in \mathbb{S}(\mathcal{H})$ thermalizes for most perturbations V and for any macro state ν most perturbations V are such that most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ thermalize.

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List of Publications

a) **Accepted publications**

1. **Time Evolution of Typical Pure States from a Macroscopic Hilbert Subspace**
   Stefan Teufel, Roderich Tumulka, and Cornelia Vogel
   Published in *Journal of Statistical Physics* 190, 69 (2023).
   arXiv:2210.10018
   Cited in the following as [136] and included in Appendix A.1.

2. **Canonical Typicality for Other Ensembles than Micro-canonical**
   Stefan Teufel, Roderich Tumulka, and Cornelia Vogel
   Published in *Annales Henri Poincaré* (2024).
   arXiv:2307.15624
   Cited in the following as [137] and included in Appendix A.2.

3. **Typical Macroscopic Long-Time Behavior for Random Hamiltonians**
   Stefan Teufel, Roderich Tumulka, and Cornelia Vogel
   Published in *Annales Henri Poincaré* (2024).
   arXiv:2303.13242
   Cited in the following as [138] and included in Appendix A.3.

b) **Submitted manuscripts (available as preprints)**

1. **Macroscopic Thermalization for Highly Degenerate Hamiltonians After Slight Perturbation**
   Barbara Roos, Stefan Teufel, Roderich Tumulka, and Cornelia Vogel
   Preprint, arXiv:2408.15832
   Cited in the following as [117] and included in Appendix B.1.

2. **Long-Time Behavior of Typical Pure States from Thermal Equilibrium Ensembles**
   Cornelia Vogel
   Preprint, arXiv:2412.16666
   Cited in the following as [143] and included in Appendix B.2.

Personal Contribution

As customary in mathematics, in all articles below the authors are ordered alphabetically.

a) Accepted publications

1. **Time Evolution of Typical Pure States from a Macroscopic Hilbert Subspace**
   The goal of project [136] was to generalize normal typicality [144, 56, 60, 107] to more realistic Hamiltonians and it was realized in collaboration with Stefan Teufel and Roderich Tumulka. All authors contributed equally to scientific ideas and paper writing. My contributions to the analysis and interpretation were 60% whereas Stefan Teufel and Roderich Tumulka contributed 20% each. We thank both referees for valuable feedback and for pointing out to us reference [5].

   Scientific ideas: 33%. Analysis/Interpretation[1]: 60%. Paper writing: 33%.

2. **Canonical Typicality for Other Ensembles than Micro-canonical**
   The main result of the article [137] is a generalization of canonical typicality [79, 45, 46, 100, 101, 58] to GAP measures [67, 59, 57, 141]. The project was done in collaboration with Stefan Teufel and Roderich Tumulka. The authors contributed equally to scientific ideas and paper writing. Moreover, my contributions to the analysis and interpretation were 60% while the contributions of Stefan Teufel and Roderich Tumulka were 20% each. We thank Tristan Benoist, Andreas Deuchert, Marius Lemm, and Martin Möhle for helpful discussions.

   Scientific ideas: 33%. Analysis/Interpretation: 60%. Paper writing: 33%.

3. **Typical Macroscopic Long-Time Behavior for Random Hamiltonians**
   The motivation for the article [138] was to extend the results from [136], which were only concerned with absolute errors, and to prove bounds for the relative errors as well. This project was joint work with Stefan Teufel

[1] We assign the working on proofs mainly to this category.

and Roderich Tumulka. The authors contributed equally to scientific ideas and paper writing. The contributions to the analysis and interpretation were 60% from myself and 20% each from Stefan Teufel and Roderich Tumulka. We thank László Erdős and Roman Vershynin for helpful discussions.

Scientific ideas: 33%. Analysis/Interpretation: 60%. Paper writing: 33%.

b) Submitted manuscripts (available as preprints)

1. **Macroscopic Thermalization for Highly Degenerate Hamiltonians After Slight Perturbation**
   The project [117] was motivated by works of Shiraishi and Tasaki [123] and Tasaki [134, 133] and was done in collaboration with Barbara Roos, Stefan Teufel, and Roderich Tumulka. The authors contributed equally to scientific ideas and paper writing. The contributions of Barbara Roos and myself to the analysis and interpretation were 35% each, while the contributions of Stefan Teufel and Roderich Tumulka were 15% each. We thank Hal Tasaki and Peter Reimann for very valuable feedback as well as Herbert Spohn for additional references and Hannah Markwig and Thomas Markwig for help with Footnote 7.

   Scientific ideas: 25%. Analysis/Interpretation: 35%. Paper writing: 25%.

2. **Long-Time Behavior of Typical Pure States from Thermal Equilibrium Ensembles**
   I am the single author of the article [143]. The project is a generalization of [136] to GAP measures and the main tool for the proof is a slight improvement of the bounds on the GAP-variance from [106, 137]. Helpful discussions with Stefan Teufel and Roderich Tumulka are gratefully acknowledged.

   Scientific ideas: 100%. Analysis/Interpretation: 100%. Paper writing: 100%.

1. Introduction

A cup of hot coffee cools down to room temperature but never spontaneously heats up, a broken glass never becomes whole again on its own – these two examples from everyday life are instances of the *second law of thermodynamics* which states that the entropy in isolated systems never decreases. The second law is not time-reversal invariant and therefore is in sharp contrast to the time-reversal microscopic laws of physics. So how can a system's macroscopic properties and behavior be obtained from the laws governing its microscopic constituents? Motivated by such questions, the field of *(classical) statistical mechanics* was developed mainly during the second half of the 19th century by its founding fathers Ludwig Boltzmann [13, 14], James Clerk Maxwell [84, 85, 86] and Josiah Willard Gibbs [47].

With the emergence of quantum theory in the twenties and early thirties of the previous century also the field of *quantum statistical mechanics* was established, however, as it is pointed out in an review article by Gogolin and Eisert from 2016 [48], some foundational questions already raised by von Neumann in 1929 [144] were largely forgotten and came back into the focus of research only very recently. More precisely, these questions include the *equilibration* and *thermalization* of large quantum systems. Roughly speaking, we say that a quantity equilibrates if after some time it reaches a certain value and stays close to it for a long time. Moreover, we say that a system *thermalizes* if after some time it appears to be in thermal equilibrium and keeps this appearance for an extended period of time. We will introduce two precise notions of thermal equilibrium in quantum systems later in the introduction. For a detailed review of the literature on equilibration and thermalization in quantum systems see, e.g., [48, 91].

A modern approach to study such questions as the equilibration and thermalization of quantum systems is to consider a closed system in a pure state $\psi \in \mathbb{S}(\mathcal{H}) = \{\phi \in \mathcal{H} : \|\phi\| = 1\}$ that evolves unitarily according to $\psi_t = e^{-iHt}\psi$. Here, $\mathcal{H}$ is a very high but finite dimensional Hilbert space and H is the system's Hamiltonian. Note that throughout this thesis we set $\hbar = 1$. As it is often difficult or even impossible to make statements about *all* initial states, many of the results we discuss in the following are formulated as *typicality statements*, i.e., they are true for "most" initial states, where "most" refers to some probability measure on the sphere $\mathbb{S}(\mathcal{H})$. In the previous literature, the measure of typicality was usually taken to be the uniform distribution on $\mathbb{S}(\mathcal{H})$.

The aim of this thesis is to contribute to the mathematically rigorous study of equilibration, thermalization and thermal equilibrium in macroscopic quantum systems.

Three projects [136, 138, 143] of this thesis are concerned with a generalization of *normal typicality* [144, 56, 60, 107] in several ways. Let $\mathcal{H}_\nu \subset \mathcal{H}$ be a subspace and P_ν the projection to it. We think of $\mathcal{H}_\nu$ as corresponding to some "macro state" ν. Roughly speaking, normal typicality asserts that for all initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$ and most times $t \geq 0$, the superposition weight $\|P_\nu \psi_t\|^2$ is close to d_ν/D provided that $d_\nu = \dim \mathcal{H}_\nu$ and $D = \dim \mathcal{H}$ are sufficiently large. This statement holds true if the Hamiltonian has a uniformly distributed eigenbasis – an assumption that is unfortunately rather unphysical. We generalize this statement to one about all Hamiltonians and most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$, where μ represents a possibly non-equilibrium macro state. More precisely, we show that for most of the time $\|P_\nu \psi_t\|^2$ is close to some quantity $M_{\mu\nu}$ that can also depend on the initial macro state μ. As long as only the absolute errors are concerned, the only assumptions on the Hamiltonian that are needed for small errors are that its eigenvalues and eigenvalue gaps are not too highly degenerate [136]. In order to show that the relative errors are small as well, we have to assume that the Hamiltonian is itself or perturbed by a random matrix with suitably *delocalized* eigenvectors [138]. We argue that the class of Hamiltonians that we can treat includes physically more realistic ones than the ones considered by von Neumann [144]. As another generalization [143], we show that the uniform distribution as a measure of typicality can be replaced by so-called *GAP measures* [67, 59, 57, 141]. For a density matrix ρ on a separable Hilbert space $\mathcal{H}$, the measure $\mathrm{GAP}(\rho)$ is in some sense the most spread out distribution on $\mathbb{S}(\mathcal{H})$ with density matrix ρ. If ρ is a canonical density matrix, the corresponding GAP measure occurs naturally as the distribution of wave functions in thermal equilibrium [59, 57] and can be seen as a quantum analogue of the canonical ensemble of classical statistical mechanics. For this generalization of normal typicality to be valid we require that $\|\rho\|$ is small. We remark that in all situations discussed above we also prove statements for general bounded operators instead of the projections P_ν and we provide finite-time results.

Another project [137] included in this thesis is about a generalization of *canonical typicality* [79, 45, 46, 100, 101, 58] to GAP measures. Canonical typicality states that for most $\psi \in \mathbb{S}(\mathcal{H})$, the reduced density matrix of a sufficiently small subsystem is close to a ψ-independent density matrix, and one can argue that for example if the subsystem and its environment are only weakly interacting, this density matrix is close to a canonical one. Here, "most" refers again to the uniform distribution on $\mathbb{S}(\mathcal{H})$. Then main ingredient of the proof of the generalization of canonical typicality is a generalization of *Lévy's Lemma* [73, 88, 99, 70], a concentration-of-measure type result, from the uniform distribution to GAP measures with small $\|\rho\|$. As a byproduct, we also obtain a generalization of *dynamical typicality* [6, 92, 5, 110, 109, 111] to GAP measures. Dynamical typicality is the statement that for any bounded operator B on $\mathcal{H}$, for all times $t \geq 0$ and most $\psi \in \mathbb{S}(\mathcal{H})$ (most w.r.t. the uniform distribution), the expectation $\langle \psi_t | B | \psi_t \rangle$ is close to a ψ-independent value $w(t)$. Moreover, on any finite time interval we find that the whole curve $t \mapsto \langle \psi_t | B | \psi_t \rangle$ is close to the deterministic curve $t \mapsto w(t)$ for most ψ.

Note that since the uniform distribution can be seen as a quantum analogue of the micro-canonical ensemble and certain GAP measures as a quantum analogue of the canonical ensemble, generalizations of results from the uniform distribution to GAP measures can be viewed as expressing a kind of equivalence of ensembles.

The last project [117] is concerned with the thermalization of a system of free fermions. This work was motivated by recent works by Shiraishi and Tasaki [123] and Tasaki [134, 133] who considered a system of $N \gg 1$ free non-relativistic fermions on a one-dimensional lattice and showed thermalization in the sense that for any initial state and most times, the number of particles in a sublattice is close to its equilibrium value. For their proof they had to introduce a small artificial phase to the hopping terms in the standard Hamiltonian and, using a number-theoretic argument, they showed that the phase lifts the degeneracy of the Hamiltonian's eigenvalues. Another important ingredient of their proof is the *eigenstate thermalization hypothesis* asserting that in all eigenstates of the Hamlitonian the number of particles in any sublattice is close to its equilibrium value. We show that thermalization also takes place for the standard Hamiltonian without the artificial phase. Moreover, by adding a small random perturbation to the Hamiltonian, we generalize these findings to arbitrary dimensions and also prove a thermalization result for arbitrary highly degenerate Hamiltonians after slight perturbation.

In the remainder of the introduction we give some mathematical and physical background to our results; it is organized as follows: In Section 1.1, we discuss the notion of *typicality* as well as important measures of typicality. The next three sections are concerned with different typicality results, more precisely, Section 1.2 is about *normal typicality*, Section 1.3 discusses *dynamical typicality* and Section 1.4 is concerned with *canonical typicality*. In Section 1.5, we introduce *GAP measures* (in finite as well as in infinite dimensions) which can serve as measures of typicality more general than the uniform distribution. In Section 1.6, we discuss the equilibration and thermalization in closed quantum systems. We define different notions of thermal equilibrium, give results on the time scales of equilibration and thermalization and compare to the classical case. In Section 1.7, we present some results concerning the thermalization of the free, non-relativistic Fermi gas in one dimension. Finally, in Section 1.8, we collect some results from random matrix theory that turned out to be very useful for some of the papers attached to thesis.

Furthermore, in Section 2 we summarize the objectives of this thesis and in Section 3 we present and discuss the results obtained in the papers attached to this thesis.

1.1. Typicality

This section is devoted to the notion of *typicality*. We first motivate the idea behind the method of appeal to typicality in Section 1.1.1. Most of the typicality statements that we discuss in later sections are about "most" (initial) wave functions and "most"

Hamiltonians with fixed eigenvalues. Here, "most" refers to the corresponding natural uniform distribution. For wave functions $\psi \in \mathbb{S}(\mathcal{H})$, where $\mathcal{H}$ is a finite-dimensional Hilbert space, this is just the uniform measure on the sphere which we introduce in Section 1.1.2. The uniform measure on the set of Hamiltonians on $\mathcal{H}$ with fixed eigenvalues is derived from the *Haar measure* on the unitary group on $\mathcal{H}$; we discuss this measure in Section 1.1.3.

1.1.1. Appeal to Typicality

The notion of typicality is formulated in the convenient language of probability theory, however, as we explain below, typicality and probability are conceptually different notions. We partially follow the presentations in [60, 61].

We start with making more precise the notion of typicality. To this end, let $\mathbb{P}$ be a probability measure on a measurable space $(X, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra on the set X. Let $A \in \mathcal{F}$ be the subset of X containing the elements of X that fulfill a property P. We say that P is *typical* for the elements in X or that *most* $x \in X$ have P if and only if $\mathbb{P}(A)$ is close to 1. In case it is not clear from the context to which measure "typical" or "most" refers, we write "typical (w.r.t. $\mathbb{P}$)" or "most (w.r.t. $\mathbb{P}$)" or also "$\mathbb{P}$-most" instead. The idea behind typicality is that if we consider a concrete element $x \in X$, knowing that the property P is typical for the elements in X *suggests* that x also has P. Of course, this does *not prove* that x has the property P but we *expect* x to have P unless there is a good reason to believe otherwise.

It is important to note that although we use the language of probability theory we do not mean that x is actually a random object. In probability theory a probability distribution would assign probabilities to x having a certain property and if one draws an element from X repeatedly and independently from each other, after sufficiently many repetitions the relative frequency of the drawn elements with this property is close to the probability of x having this property. This follows immediately from the law of large numbers. In physics, however, the situation is different: Here, for some systems it is impossible to investigate multiple realizations thereof (for example there is only one universe) and even if one considers an experiment that can be repeated (for example the preparation of a (macroscopic) gas), the feasible number of repetitions is only extremely small compared to the dimension of the system's phase space or Hilbert space [61]. Therefore one could not hope to obtain much information about an underlying distribution of such high-dimensional systems from comparably few realizations thereof. For example, one could not conclude whether a certain in the realizations often observed behavior is typical for this system or not which would make it difficult to make predictions for the behavior of future realizations of the system.

The concept of typicality already appeared in the literature almost 100 years ago in a work by Schrödinger [121] and, more prominently, in an article by von Neumann [144], and has been used successfully in physics ever since.

One important example from physics where the method of appeal to typicality turned out to be extremely successful is Wigner's [147] approach to study random matrices as Hamiltonians in order to make predictions about complicated quantum systems in nuclear physics. As Wigner [147, p.3] argues:

> One (...) deals with a specific system, with its proper (though in many cases unknown) Hamiltonian, yet pretends that one deals with a multitude of systems, all with their own Hamiltonians, and averages over the properties of these systems. Evidently, such a procedure can be meaningful only if it turns out that the properties in which one is interested are the same for the vast majority of the admissible Hamiltonians.

Wigner's approach also inspired lots of further research, in particular the mathematical rigorous study of random matrices.

In some of the following sections we will discuss some more recent examples of typicality, namely normal typicality, dynamical typicality and canonical typicality. For a more detailed discussion of the concept of typicality, in particular on its history, see for example [48, 91].

1.1.2. The Uniform Measure on the Sphere

Some of the typicality statement that we discuss in the next sessions are about typical $\psi \in \mathbb{S}(\mathcal{H})$ where "typical" refers to the uniform distribution on the sphere $\mathbb{S}(\mathcal{H})$. Since we assume that $\mathcal{H}$ is finite-dimensional[2], we can identify $\mathcal{H}$ with $\mathbb{C}^D$ where $D = \dim \mathcal{H}$. Moreover, we can identify $\mathbb{C}^D$ with $\mathbb{R}^{2D}$ and therefore it suffices to consider $\mathbb{S}(\mathbb{R}^n)$ for $n \in \mathbb{N}$.

One possible way to define the uniform distribution on $\mathbb{S}(\mathbb{R}^n)$ is with the help of standard Gaussian random variables.

Definition 1.1 (Uniform Distribution on $\mathbb{S}(\mathbb{R}^n)$)**.** Let $X = (X_1, \dots, X_n)$, where $X_1, \dots,$ $X_n \sim \mathcal{N}(0,1)$ are independent and identically distributed (i.i.d.) standard Gaussian random variables. Then,

$$\frac{X}{\|X\|} = \frac{(X_1, \dots, X_n)}{\sqrt{X_1^2 + \dots + X_n^2}} \tag{1.1}$$

is uniformly distributed on $\mathbb{S}(\mathbb{R}^n)$. Here, $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.

This definition agrees with defining the uniform distribution as the surface measure on $\mathbb{S}(\mathbb{R}^n)$ as can be seen from a direct computation using spherical coordinates. To

[2]See Section 1.2 for the physical motivation of this assumption.

this end, let $f : \mathbb{R}^n \to \mathbb{S}(\mathbb{R}^n)$, $x \mapsto x/\|x\|$. Then the distribution of $X/\|X\|$ is given by the pushforward measure $f(X)_*\mathbb{P}$ which is defined as $(f(X)_*\mathbb{P})(A) = \mathbb{P}(f(X) \in A)$ for any Borel-measurable set $A \subseteq \mathbb{S}(\mathbb{R}^n)$. Let $\sigma_{n-1} = \tilde{\sigma}_{n-1}/\tilde{\sigma}_{n-1}(\mathbb{S}(\mathbb{R}^n))$ be the normalized surface measure on $\mathbb{S}(\mathbb{R}^n)$, where $\tilde{\sigma}_{n-1}$ is the not normalized one. Then, for any Borel-measurable set $A \subseteq \mathbb{S}(\mathbb{R}^n)$,

$$\begin{aligned}
(f(X)_*\mathbb{P})(A) &= \int_{\mathbb{S}(\mathbb{R}^n)} \mathbb{1}_A(x)\,(f(X)_*\mathbb{P})(dx) = \int_{\mathbb{R}^n} \mathbb{1}_A(f(x))\,\mathbb{P}(dx) && (1.2a)\\
&= \int_{\mathbb{R}^n} \mathbb{1}_A\left(\frac{x}{\|x\|}\right) \frac{1}{(2\pi)^{n/2}} e^{-\sum_i x_i^2/2}\,dx && (1.2b)\\
&= \int_0^\infty \int_{\mathbb{S}(\mathbb{R}^n)} \mathbb{1}_A(u)\,\tilde{\sigma}_{n-1}(du)\,\frac{1}{(2\pi)^{n/2}} e^{-r^2/2} r^{n-1}\,dr && (1.2c)\\
&= \int_{\mathbb{S}(\mathbb{R}^n)} \mathbb{1}_A(u)\,\sigma_{n-1}(du) = \sigma_{n-1}(A). && (1.2d)
\end{aligned}$$

Here, we used in the fifth line that

$$\int_0^\infty e^{-r^2/2} r^{n-1}\,dr = 2^{n/2-1}\Gamma(n/2) \tag{1.3}$$

and $\tilde{\sigma}_{n-1}(\mathbb{S}(\mathbb{R}^n)) = 2\pi^{n/2}/\Gamma(n/2)$, where Γ denotes the Gamma function.

The surface measure σ_{n-1} can, for example, be defined via the formula

$$\sigma_{n-1}(A) = \lambda^n(\mathbb{S}(\mathbb{R}^n))^{-1}\lambda^n\left(\left\{tx : x \in A,\ 0 \le t \le 1\right\}\right), \quad A \subset \mathbb{S}(\mathbb{R}^n), \tag{1.4}$$

where λ^n denotes the Lebesgue measure on $\mathbb{R}^n$, see (3.6) in [83]. Another way to define it is as the normalized restriction of the $(n-1)$-dimensional Hausdorff measure to the sphere $\mathbb{S}(\mathbb{R}^n)$, see again [83].

We remark that the distribution $f(X)_*\mathbb{P}$ is invariant under orthogonal transformations. This can be seen from the invariance of X under orthogonal transformations and the fact that $\|OX\| = \|X\|$ for every orthogonal matrix $O \in O(n)$, where $O(n)$ denotes the orthogonal group on $\mathbb{R}^n$, as follows: For every Borel-measurable set $A \subseteq \mathbb{S}(\mathbb{R}^n)$ and $O \in O(n)$ we have that

$$\begin{aligned}
(f(X)_*\mathbb{P})(OA) &= \mathbb{P}\left(\frac{X}{\|X\|} \in OA\right) = \mathbb{P}\left(\frac{O^{-1}X}{\|O^{-1}X\|} \in A\right) = \mathbb{P}\left(\frac{X}{\|X\|} \in A\right)\\
&= (f(X)_*\mathbb{P})(A). && (1.5)
\end{aligned}$$

In fact, one can show that there is exactly one probability distribution on $\mathbb{S}(\mathbb{R}^n)$ that is invariant under orthogonal transformations and thus this also serves as a characterization of the uniform distribution on $\mathbb{S}(\mathbb{R}^n)$. To see this, let $\bar{B}_r(x) = \{y \in$

$\mathbb{S}(\mathbb{R}^n) : \|y-x\| \leq r\}$ with $r>0$ and $x \in \mathbb{S}(\mathbb{R}^n)$ be the closed ball in $\mathbb{S}(\mathbb{R}^n)$ around x with radius r. Because of $O\bar{B}_r(x) = \bar{B}_r(Ox)$ for all $O \in O(n)$ it follows that for every orthogonally invariant probability measure μ on $\mathbb{S}(\mathbb{R}^n)$ the measure of a closed ball depends only on the radius r, i.e., $\mu(\bar{B}_r(x)) = \mu(\bar{B}_r(y))$ for all $x, y \in \mathbb{S}(\mathbb{R}^n)$. Moreover, the measure of every ball is clearly finite and due to the compactness of $\mathbb{S}(\mathbb{R}^n)$ strictly positive[3]. Therefore, μ is *uniformly distributed* and since $\mathbb{S}(\mathbb{R}^n)$ is separable, it follows from Theorem 3.4. in [83] that μ is unique up to a constant positive factor. However, this factor has to be equal to 1 due to the normalization constraint and thus μ is unique.

For the uniform distribution on the sphere $\mathbb{S}(\mathcal{H})$ of a finite-dimensional Hilbert space $\mathcal{H}$, many averages such as for example moments of components of a uniformly distributed vector can be computed explicitly. We collect some useful identities in Lemma 1.2. Before stating the lemma, we recall that the variance of a complex random variable Z is defined as

$$\operatorname{Var} Z = \mathbb{E}\left[|Z-\mathbb{E}(Z)|^2\right] = \mathbb{E}\left[|Z|^2\right] - |\mathbb{E}(Z)|^2. \tag{1.6}$$

Lemma 1.2. *Let $\mathcal{H}$ be a Hilbert space of dimension $D := \dim \mathcal{H} < \infty$ and let $\psi \in \mathbb{S}(\mathcal{H})$ be uniformly distributed. Then for any operator B on $\mathcal{H}$,*

$$\mathbb{E}\left[\langle\psi|B|\psi\rangle\right] = \frac{1}{D}\operatorname{tr} B, \tag{1.7}$$

$$\operatorname{Var}\left[\langle\psi|B|\psi\rangle\right] = \frac{1}{D(D+1)}\left(\operatorname{tr}(B^*B) - \frac{|\operatorname{tr} B|^2}{D}\right). \tag{1.8}$$

Here, $\mathbb{E}$ denotes the expectation and Var *the variance with respect to the uniform distribution and B^* is the adjoint of B.*

Let $(\varphi_m)_m$ be an orthonormal basis of $\mathcal{H}$ and $a_m := \langle\varphi_m|\psi\rangle$. Then,

$$(i)\quad \mathbb{E}\left(a_k^* a_l a_m^* a_n\right) = 0 \quad \textit{if an index occurs only once}, \tag{1.9a}$$

$$(ii)\quad \mathbb{E}\left(a_k^{*2} a_l^2\right) = 0 \quad \textit{for } k \neq l, \tag{1.9b}$$

$$(iii)\quad \mathbb{E}\left(|a_k|^4\right) = \frac{2}{D(D+1)}, \tag{1.9c}$$

$$(iv)\quad \mathbb{E}\left(|a_k|^2|a_l|^2\right) = \frac{1}{D(D+1)} \quad \textit{for } k \neq l. \tag{1.9d}$$

For a proof of Lemma 1.2 see, for example, [144] and [46, App. A and C].

[3]Suppose there are $r_0 > 0$ and $x \in \mathbb{S}(\mathbb{R}^n)$ such that $\mu(\bar{B}_{r_0}(x)) = 0$. Since all balls of the same radius have the same measure, it follows that $\mu(\bar{B}_{r_0}(y)) = 0$ for all $y \in \mathbb{S}(\mathbb{R}^n)$. The balls $B_{r_0}(y)$ with $y \in \mathbb{S}(\mathbb{R}^n)$ form an open cover of $\mathbb{S}(\mathbb{R}^n)$ and due to the compactness of $\mathbb{S}(\mathbb{R}^n)$ there exists a finite subcover $B_{r_0}(x_1), \ldots, B_{r_0}(x_m)$. Moreover, these open balls obviously also have measure zero. This implies that $\mu(\mathbb{S}(\mathbb{R}^n)) = 0$, a contradiction to the assumption that $\mu(\mathbb{S}(\mathbb{R}^n)) = 1$.

1.1.3. The Haar Measure on the Unitary Group

Let $U(\mathcal{H})$ be the unitary group on a finite-dimensional Hilbert space $\mathcal{H}$. The uniform measure on $U(\mathcal{H})$ is called the *Haar measure* on $U(\mathcal{H})$ and it is invariant under left multiplication with any group element.

More generally, Haar measures can be defined on locally compact groups [42]:

Definition 1.3 (Haar Measure)**.** Let G be a locally compact group. A *left* (resp. *right*) *Haar measure* on G is a nonzero Radon measure μ on G such that

$$\mu(gA) = \mu(A) \quad (\text{resp. } \mu(Ag) = \mu(A)) \tag{1.10}$$

for all Borel-measurable sets $A \subseteq G$ and $g \in G$.

It can be shown that the left and right Haar measure exist and are unique up to a multiplicative positive constant, see Section 2.2 in [42], in particular, Theorem 2.10 and Theorem 2.20. Thus, if we require the left and right Haar measure to be normalized to 1, then they are unique. Moreover, if G is compact (which is the case for the unitary group $U(\mathcal{H})$ if $\mathcal{H}$ is finite-dimensional[4]), the left Haar measure is equal to the right Haar measure [42, Corollary 2.28].

The Haar measure on $U(\mathcal{H})$ induces a probability distribution on the set $\mathrm{ONB}(\mathcal{H})$ of orthonormal bases of $\mathcal{H}$ in the following way [56]: Let $\phi_1, \ldots, \phi_D$ be an orthonormal basis of $\mathcal{H}$. The elements of any other orthonormal basis $\chi_1, \ldots, \chi_D$ of $\mathcal{H}$ can be expanded in the basis of the ϕ_j as

$$\chi_k = \sum_{j=1}^{D} \langle \phi_j | \chi_k \rangle \phi_j =: \sum_{j=1}^{D} U_{kj} \phi_j, \tag{1.11}$$

where we defined $U_{kj} := \langle \phi_j | \chi_k \rangle$. The coefficients U_{kj} form a unitary matrix U and via (1.11) every $U \in U(\mathcal{H})$ defines an orthonormal basis of $\mathcal{H}$, i.e., we have a one-to-one correspondence between unitary matrices on $\mathcal{H}$ and orthonormal bases of $\mathcal{H}$. In this way the Haar measure on $U(\mathcal{H})$ induces a probability measure on $\mathrm{ONB}(\mathcal{H})$ which we call the uniform distribution on $\mathrm{ONB}(\mathcal{H})$ and denote it by u_{ONB}. Note that as the Haar measure is invariant under right multiplication by group elements, the uniform distribution on $\mathrm{ONB}(\mathcal{H})$ is independent of the choice of the basis $\phi_1, \ldots, \phi_D$.

[4]The unitary group is obviously bounded as each element has operator norm equal to 1. Let $g_1 : U(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $U \mapsto U^*U$ and $g_2 : U(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $U \mapsto UU^*$, where $\mathcal{L}(\mathcal{H})$ denotes the set of linear operators on $\mathcal{H}$. The functions g_1 and g_2 are continuous and $U(\mathcal{H}) = g_1^{-1}(I) \cap g_2^{-1}(I)$, where I denotes the identity on $\mathcal{H}$, i.e., $U(\mathcal{H})$ is the intersection of two closed sets and therefore closed. As $\mathcal{H}$ and therefore also $U(\mathcal{H})$ is finite-dimensional, this implies the compactness of $U(\mathcal{H})$.

1.2. Normal Typicality

In this section we present and discuss the phenomenon of *normal typicality*. The first result concerning normal typicality goes back to von Neumann [144] and it was more elaborated on in [56, 55].

For the results concerning normal typicality we always assume that the Hilbert space $\mathcal{H}$ is finite-dimensional. The physical background is that one often restricts the considerations to a micro-canonical energy shell, i.e., the subspace $\mathcal{H}_{\mathrm{mc}} \subset \mathcal{H}$ spanned by the eigenvectors of the system's Hamiltonian H corresponding to eigenvalues in a micro-canonical energy interval $[E - \Delta E, E]$. Here, $E \in \mathbb{R}$, ΔE is the macroscopic resolution of the energy and we assumed that the Hamiltonian has pure point spectrum. If further there are only finitely many eigenvalues which are smaller than any arbitrary value and if each eigenvalue has finite multiplicity, then it follows that $\dim \mathcal{H}_{\mathrm{mc}} < \infty$. These assumptions are for example fulfilled if the system is enclosed in a finite volume, see, e.g., [39, Theorem 1, p. 355].

In the setting in [144, 56, 55, 60] the Hilbert space is assumed to be decomposed into mutually orthogonal subspaces ("macro spaces"). We introduce and discuss the motivation for this decomposition in Section 1.2.1. Then, in Section 1.2.2, we present von Neumann's *Quantum Ergodic Theorem* and discuss his assumptions in Section 1.2.3. Throughout this section we closely follow the presentation of von Neumann's result [144] and its refinements in [56, 55, 60].

1.2.1. Decomposition of the Hilbert Space

Following von Neumann [144], we take as given a decomposition of $\mathcal{H}$ into mutually orthogonal subspaces $\mathcal{H}_\nu$ ("macro spaces") corresponding to different macro states ν,

$$\mathcal{H} = \bigoplus_\nu \mathcal{H}_\nu. \tag{1.12}$$

Usually there is one macro space $\mathcal{H}_{\mathrm{eq}}$ in the decomposition that corresponds to thermal equilibrium, see also Section 1.6, and fulfills

$$d_{\mathrm{eq}}/D \approx 1. \tag{1.13}$$

Von Neumann's motivation for considering such a decomposition was the following: Let $M_1, \dots, M_K$ be a set of pairwise commuting self-adjoint operators on $\mathcal{H}$, i.e., $[M_i, M_j] = M_i M_j - M_j M_i = 0$ for all i, j. Moreover, we assume that the M_i have pure point spectrum, that the eigenvalues of the M_i are highly degenerate and that the distance between neighboring eigenvalues of M_i is of the order of the macroscopic resolution of the measurement of M_i. We call the M_i "macroscopic observables" or simply "macro observables"[5]. Since the M_i commute pairwise, they can be diagonalized simultaneously. We define the macro states as lists of eigenvalues of the M_i, i.e., $\nu = (m_1, \ldots, m_K)$, where m_i is an eigenvalue of M_i. Then the macro space $\mathcal{H}_\nu$ is defined as the joint eigenspace of the M_i corresponding to the list ν, that is,

$$\mathcal{H}_\nu = \bigcap_{i=1}^{K} \mathcal{H}_{M_i, m_i}, \tag{1.14}$$

where $\mathcal{H}_{M_i,m_i}$ is the eigenspace of the macro observable M_i with eigenvalue m_i. By assumption, each $\mathcal{H}_{M_i,m_i}$ is high-dimensional and it is expected that also the $\mathcal{H}_\nu$ are usually high-dimensional. Von Neumann argued that such macro observables can be obtained by "rounding" observables with small commutators to approximate them by exactly commuting ones.[6] As an example, von Neumann considered the position and momentum operators Q_k and P_k whose commutator is small and approximated them by exactly commuting operators. However, as it is pointed out in [56], it is not clear why the difference between the old and new operators should be small in operator norm.

Von Neumann's considerations raised the mathematical question whether it is always possible to approximate a set of almost commuting self-adjoint operators by exactly commuting ones. As we assume that $\mathcal{H}$ is finite-dimensional, the observables are self-adjoint matrices; for results in the case that $\mathcal{H}$ is separable and therefore possibly infinite-dimensional, see [76]. For two self-adjoint matrices whose commutator is small in the operator norm, the answer is affirmative, as was proved by Lin in 1995 [75]. More precisely, he showed the following theorem:

Theorem 1.4 (Lin (1995) [75])**.** *Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and any two self-adjoint $n \times n$ matrices A and B with $\|A\|, \|B\| \leq 1$ and*

$$\|[A, B]\| \leq \delta, \tag{1.15}$$

where $[A, B] = AB - BA$ is the commutator of A and B, there are two self-adjoint $n \times n$ matrices A', B' with $[A', B'] = 0$ and

$$\|A - A'\| + \|B - B'\| < \varepsilon. \tag{1.16}$$

Unfortunately, already for three almost commuting self-adjoint matrices the answer is in general negative, see [18, 64] for counterexamples. However, the macro observables we are interested in are often of a special form and one might hope that for such operators it is nevertheless possible to approximate them by commuting ones.

[5]Natural choices of macro observables are, e.g., the number of particles, the total energy, the total momentum or the total magnetization in cells of a partition of the available volume of the system.

[6]It is also possible to construct macro spaces without approximating non-commuting observables by commuting operators, see [116] for such a construction.

One positive result in this direction is from Ogata [95]: She showed that for certain averaged operators in quantum spin systems exactly commuting approximations exist.

Theorem 1.5 (Ogata (2013) [95])**.** *Let $m, n \in \mathbb{N}$ and let $A_1, \ldots, A_m$ be self-adjoint operators on $\mathbb{C}^n$. For every $N \in \mathbb{N}$, $1 \leq j \leq m$, $1 \leq k \leq N$, let $\mathcal{H}_N = (\mathbb{C}^n)^{\otimes N}$ be the N-fold tensor product of $\mathbb{C}^n$ and let $A_{jk} : \mathcal{H}_N \to \mathcal{H}_N$ be the operator A_j acting on the k-th factor of $\mathcal{H}_N$. For $j = 1, \ldots, m$ define*

$$H_{jN} := \frac{1}{N} \sum_{k=1}^{N} A_{jk}. \tag{1.17}$$

Then there exist sequences of self-adjoint operators Y_{jN}, $j = 1, \ldots, m$, $N \in \mathbb{N}$, such that

$$[Y_{iN}, Y_{jN}] = 0 \qquad \forall i, j = 1, \ldots, m \tag{1.18}$$

and

$$\lim_{N \to \infty} \|H_{jN} - Y_{jN}\| = 0 \qquad \forall\, j = 1, \ldots, m. \tag{1.19}$$

1.2.2. Von Neumann's Quantum Ergodic Theorem

In this section we discuss von Neumann's quantum ergodic theorem [144] and we closely follow its presentation in [56]. We start with introducing some notations and notions needed for stating the theorem.

Definition 1.6 (Most times)**.** Let $\delta > 0$. A statement $s(t)$ is true for $(1-\delta)$-most $t \in [0, \infty)$ if

$$\liminf_{T \to \infty} \frac{1}{T} \lambda \Big\{ t \in [0, T] : s(t) \text{ holds} \Big\} \geq 1 - \delta, \tag{1.20}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$.

Definition 1.7 (Infinite time average)**.** Let $f : \mathbb{R}_+ \to \mathbb{C}$. The infinite time average of f is given by

$$\overline{f(t)} := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t)\, dt \tag{1.21}$$

whenever this limit exists.

Let $\mathcal{D} := \{\mathcal{H}_\nu\}$ denote the decomposition (1.12) and let P_ν be the projection to the macro space $\mathcal{H}_\nu$. Roughly speaking, normal typicality states that for all initial states

$\psi_0 \in \mathbb{S}(\mathcal{H})$ and most times $t \in [0, \infty)$, $\|P_\nu \psi_t\|^2 \approx d_\nu/D$, where $d_\nu = \dim \mathcal{H}_\nu$ and $D = \dim \mathcal{H}$. We are going to discuss two statements, one for most decompositions $\mathcal{D}$ and another one for most Hamiltonians. Here, "most" refers again to the suitable uniform distributions; they can be obtained from the measure u_{ONB} defined in Section 1.1.3 as follows: For the measure on the set of decompositions $\mathcal{D}$ we regard the number K of subspaces, their dimensions d_ν and a partition of $\{1, \ldots, D\}$ into sets I_ν with $|I_\nu| = d_\nu$ as fixed. Let $\chi_1, \ldots, \chi_D$ be an orthonormal basis of $\mathcal{H}$ distributed according to the uniform measure u_{ONB}. Then we set $\mathcal{H}_\nu = \operatorname{span}\{\chi_j : j \in I_\nu\}$ and in this way we obtain a random decomposition $\{\mathcal{H}_\nu\}$ whose distribution we call uniform. For Hamiltonians, we regard its eigenvalues $E_1, \ldots, E_D$ as fixed and take its eigenbasis to be uniformly distributed according to u_{ONB}.

In the following we assume that the eigenvalues $E_1, \ldots, E_D$ of H are non-degenerate and, moreover, that also its gaps are non-degenerate, i.e.,

$$E_j - E_k \neq E_{j'} - E_{k'} \text{ unless } \begin{cases} \text{either } j = j' \text{ and } k = k' \\ \text{or } j = k \text{ and } j' = k'. \end{cases} \tag{1.22}$$

We first assume that the Hamiltonian H is fixed and the decomposition $\{\mathcal{H}_\nu\}$ is random. The idea of the proof of normal typicality is to compute the expectation with respect to the uniform distribution on the set of decompositions $\{\mathcal{H}_\nu\}$ of the quantity

$$\overline{\left| \|P_\nu \psi_t\|^2 - \frac{d_\nu}{D} \right|^2} = \overline{\|P_\nu \psi_t\|^4} - 2\frac{d_\nu}{D}\overline{\|P_\nu \psi_t\|^2} + \frac{d_\nu^2}{D^2} \tag{1.23}$$

for arbitrary $\psi_0 \in \mathbb{S}(\mathcal{H})$ and ν. If we show that this quantity is small, it follows from Markov's inequality that for most $\{\mathcal{H}_\nu\}$ and most $t \geq 0$, $\|P_\nu \psi_t\|^2 \approx d_\nu/D$.

To this end, let $\phi_1, \ldots, \phi_D$ be an orthonormal eigenbasis of H corresponding to the eigenvalues $E_1, \ldots, E_D$. Then we can write

$$\psi_0 = \sum_{j=1}^{D} c_j \phi_j, \qquad \psi_t = \sum_{j=1}^{D} e^{-iE_j t} c_j \phi_j \tag{1.24}$$

with coefficients $c_j = \langle \phi_j | \psi_0 \rangle$. Since the eigenvalues of H are non-degenerate, we find that

$$\overline{\|P_\nu \psi_t\|^2} = \sum_{j,k=1}^{D} \overline{e^{i(E_j - E_k)t}} c_j^* c_k \langle \phi_j | P_\nu | \phi_k \rangle = \sum_{j=1}^{D} |c_j|^2 \langle \phi_j | P_\nu | \phi_j \rangle. \tag{1.25}$$

Here we used that for $x \in \mathbb{R}$ it holds that $\overline{e^{ixt}} = \delta_{x0}$, i.e., the time average of the complex exponential e^{ixt} is zero unless $x = 0$ and in this case it is equal to 1.

Due to the assumption that the gaps of H are also non-degenerate, we obtain

$$\overline{\|P_\nu\psi_t\|^4} = \sum_{j,k,j',k'=1}^{D} \overline{e^{i(E_j-E_{j'}-E_k+E_{k'})t}} c_j^* c_{j'} c_k c_{k'}^* \langle\phi_j|P_\nu|\phi_k\rangle\langle\phi_{j'}|P_\nu|\phi_{k'}\rangle^* \tag{1.26a}$$

$$= \sum_{j,k,j',k'=1}^{D} \left(\delta_{jj'}\delta_{kk'} + \delta_{jk}\delta_{j'k'} - \delta_{jj'}\delta_{kk'}\delta_{jk}\right) c_j^* c_{j'} c_k c_{k'}^* \langle\phi_j|P_\nu|\phi_k\rangle\langle\phi_{j'}|P_\nu|\phi_{k'}\rangle^* \tag{1.26b}$$

$$= \sum_{\substack{j,k=1\\ j\neq k}}^{D} |c_j|^2|c_k|^2 \left|\langle\phi_j|P_\nu|\phi_k\rangle\right|^2 + \left|\sum_{j=1}^{D} |c_j|^2\langle\phi_j|P_\nu|\phi_j\rangle\right|^2 . \tag{1.26c}$$

Putting everything together, we arrive at

$$\overline{\left|\|P_\nu\psi_t\|^2 - \frac{d_\nu}{D}\right|^2} = \sum_{\substack{j,k=1\\ j\neq k}}^{D} |c_j|^2|c_k|^2 \left|\langle\phi_j|P_\nu|\phi_k\rangle\right|^2 + \left|\sum_{j=1}^{D} |c_j|^2\langle\phi_j|P_\nu|\phi_j\rangle - \frac{d_\nu}{D}\right|^2 \tag{1.27a}$$

$$\leq \max_{j\neq k} \left|\langle\phi_j|P_\nu|\phi_k\rangle\right|^2 + \max_j \left|\langle\phi_j|P_\nu|\phi_j\rangle - \frac{d_\nu}{D}\right|^2 , \tag{1.27b}$$

where we used that $\sum_{j=1}^{D}|c_j|^2 = 1$. Von Neumann [144] showed that the expectations of the maxima on the right-hand side of (1.27b) can be bounded from above by terms of the order $\log D/D$ and are therefore small for large D, see Lemma 4 in [56]. The dimensions d_ν of the macro spaces are assumed to be larger than $C_1 \log D$ but smaller than D/C_1 where $C_1 > 1$ is a large constant. Let $\varepsilon > 0$. After suitably increasing the constant in the lower bound for the d_ν it can be shown that for any $\psi_0 \in \mathbb{S}(\mathcal{H})$ and any macro state ν, most decompositions $\{\mathcal{H}_\mu\}$ are such that for most $t \in [0,\infty)$,

$$\left|\|P_\nu\psi_t\|^2 - \frac{d_\nu}{D}\right| < \varepsilon\sqrt{\frac{d_\nu}{KD}}, \tag{1.28}$$

where K denotes the number of subspaces in the decomposition $\{\mathcal{H}_\nu\}$.[7] As already mentioned in Section 1.2.1, in the decomposition (1.12) there usually is one macro space $\mathcal{H}_{\text{eq}}$ that is associated with thermal equilibrium and that compromises most of the dimensions, i.e., $d_{\text{eq}}/D \approx 1$. Unfortunately, von Neumann's QET as formulated above does not cover the case that d_ν is very close to D. Moreover, (1.28) shows that while the absolute error is small, the relative error need not be small. However, as it is pointed out in [56], von Neumann actually proved a stronger statement. More

[7] More precisely, the constant has to be changed to $\max\{C_1, 10K^2/(\varepsilon^2\delta'\delta)\}$ and the bound is true for $(1-\delta)$ decompositions for $(1-\delta')$-most of the time.

precisely, he showed that for $C_2 < d_\nu < D - C_2$, where $C_2 > 1$ is a big constant, the probability that the maxima in (1.27b) are large is exponentially small in the dimension D, see Lemma 5 in [56]. With the help of these bounds, Goldstein, Lebowitz, Mastrodonato, Tumulka, and Zanghì (2010) [56] showed the following theorem:

Theorem 1.8 (Normal typicality (for most $\mathcal{D}$) [56]). *Let $\varepsilon, \delta, \delta' > 0$ and let $\mathcal{H}$ be a Hilbert space of dimension D. Suppose that the numbers D, K, and $d_1, \dots, d_K \in \mathbb{N}$ are such that $d_1 + \cdots + d_K = D$ and*

$$\max\left\{C_2, \sqrt{(3K/\varepsilon^2\delta')D\log D}\right\} < d_\nu < D - C_2, \tag{1.29}$$

$$\varepsilon^2\delta' < 2K/C_2, \quad D/\log D > 100K/\varepsilon^2\delta', \quad \text{and} \quad D > 1/\delta \tag{1.30}$$

for all ν, where $C_2 > 1$ is a large constant. Moreover, suppose that H is a Hamiltonian on $\mathcal{H}$ with non-degenerate eigenvalues and eigenvalue gaps. Then, $(1-\delta)$-most orthogonal decompositions $\mathcal{D} = \{\mathcal{H}_\nu\}$ of $\mathcal{H}$ with $\dim \mathcal{H}_\nu = d_\nu$ for all ν are such that for every initial wave function $\psi_0 \in \mathbb{S}(\mathcal{H})$, for $(1-\delta')$-most $t \in [0,\infty)$,

$$\left| \|P_\nu \psi_t\|^2 - \frac{d_\nu}{D} \right| < \varepsilon \frac{d_\nu}{D}. \tag{1.31}$$

Roughly speaking, Theorem 1.8 shows that for most orthogonal decompositions $\mathcal{D} = \{\mathcal{H}_\nu\}$ and all $\psi_0 \in \mathbb{S}(\mathcal{H})$, the curve $t \mapsto \|P_\nu\psi_t\|^2$ is approximately constant in the long run and its approximate value is given by d_ν/D, provided that D and d_ν are sufficiently large. Note that due to the phenomenon of *recurrence* we cannot expect the value $\|P_\nu\psi_t\|^2$ to converge for $t \to \infty$; after a sufficiently long time, it will return arbitrarily close to its initial value. We also remark that in contrast to (1.28), the bound (1.31) shows that not only the absolute but also the relative error is small.

Theorem 1.8 can also be translated into a statement about all orthogonal decompositions $\{\mathcal{H}_\nu\}$ of $\mathcal{H}$ and most Hamiltonians H with non-degenerate eigenvalues and eigenvalue gaps. To this end, let $\phi_1, \dots, \phi_D$ be an eigenbasis of a fixed H with corresponding eigenvalues $E_1, \dots, E_D$ and let $\chi_1, \dots, \chi_D$ denote an orthonormal basis of $\mathcal{H}$ distributed according to u_{ONB} from which the random decomposition $\mathcal{D} = \{\mathcal{H}_\nu\}$ is constructed. Then we can write

$$\langle\phi_j|P_\nu|\phi_k\rangle = \sum_{m\in I_\nu} \langle\phi_j|\chi_m\rangle\langle\chi_m|\phi_k\rangle. \tag{1.32}$$

The coefficients $(\langle\phi_j|\chi_m\rangle)_{j,m}$ form a unitary matrix and it is Haar-distributed, see Section 1.1.3. Since the Haar measure is invariant under inversion, the joint distribution of the $\langle\phi_j|\chi_m\rangle$ is the same if we regard the orthonormal basis $\phi_1, \dots, \phi_D$ as random and $\chi_1, \dots, \chi_D$ as fixed. This is called the "unitary inversion trick" in [56]. Due to this trick, also the distribution of (1.32) is the same regardless of whether we fix the basis of the ϕ_j and take the χ_j as a random basis or vice versa. With this the following theorem is an immediate consequence of Theorem 1.8:

Theorem 1.9 (Normal typicality (for most H) [56])**.** *Let $\varepsilon, \delta, \delta' > 0$ and let $\mathcal{H}$ be a Hilbert space of dimension D. Suppose that the numbers D, K, and $d_1, \dots, d_K$ with $d_1 + \cdots + d_K = D$ fulfill the assumptions in Theorem 1.8 and let $\mathcal{D} = \{\mathcal{H}_\nu\}$ be an orthogonal decomposition of $\mathcal{H}$ with $\dim \mathcal{H}_\nu = d_\nu$ for all ν. Moreover, suppose that the real numbers $E_1, \dots, E_D$ are non-degenerate and have non-degenerate gaps. Then, $(1-\delta)$-most Hamiltonians with eigenvalues $E_1, \dots, E_D$ are such that for every $\psi_0 \in \mathbb{S}(\mathcal{H})$, for $(1-\delta')$-most $t \in [0, \infty)$,*

$$\left| \|P_\nu \psi_t\|^2 - \frac{d_\nu}{D} \right| < \varepsilon \frac{d_\nu}{D} \tag{1.33}$$

Theorem 1.9 shows that for all orthogonal decompositions of the Hilbert space, most Hamiltonians (with non-degenerate eigenvalues and eigenvalue gaps) are such that for all initial wave functions $\psi_0 \in \mathbb{S}(\mathcal{H})$ it holds that $\|P_\nu \psi_t\|^2 \approx d_\nu / D$ for most of the time (in the sense that the relative error is small).

1.2.3. Discussion

We have already mentioned that von Neumann's original assumptions do not allow that one of the dimensions d_ν is very close to D, i.e., that one of the macro spaces $\mathcal{H}_\nu$ represents thermal equilibrium. We say that a wave function $\psi \in \mathbb{S}(\mathcal{H})$ is in thermal equilibrium if it is close to $\mathcal{H}_{\text{eq}}$ in the sense that $\|P_{\text{eq}} \psi\|^2$ is close to 1. As it is pointed out in [55], the reason von Neumann did not consider this particularly interesting case is that he thought of thermal equilibrium in a different way: He had in mind that a wave function $\psi \in \mathbb{S}(\mathcal{H})$ is in thermal equilibrium if $\|P_\nu \psi\|^2 \approx d_\nu / D$ for all ν.

While the theorem concerning normal typicality in [56] (that followed from what von Neumann [144] had actually proved) covers the case that one of the $\mathcal{H}_\nu$ is the thermal equilibrium subspace, the proof in this case can be simplified and it is carried out in [55]. Then main ingredient for the simpler proof is the observation that the set

$$\mathcal{S}_\varepsilon = \left\{ U \in U(D) \middle| \forall j : \sum_{k=1}^{d_{\text{eq}}} |U_{jk}|^2 > 1 - \varepsilon \right\}, \tag{1.34}$$

where $\varepsilon > 0$ is arbitrary, has Haar measure close to 1 provided that D and d_{eq} are large enough, see Lemma 1 in [55]. The motivation for studying this set is that

$$\langle \phi_j | P_{\text{eq}} | \phi_j \rangle = \sum_{k=1}^{d_{eq}} |U_{jk}|^2, \tag{1.35}$$

see also (1.32), where $U_{jk} = \langle \chi_k | \phi_j \rangle$, $\{\phi_j\}$ is an orthonormal basis of $\mathcal{H}$ consisting of the Hamiltonian's eigenvectors and $\{\chi_k\}$ is a random orthonormal basis of $\mathcal{H}$ such that $\mathcal{H}_{\mathrm{eq}}$ is spanned by its first d_{eq} vectors. This quantity appears naturally in the computations, see also Section 1.2.2.

With the help of the statement regarding $\mathcal{S}_\varepsilon$, Goldstein, Lebowitz, Mastrodonato, Tumulka, and Zanghì proved in [55] the following theorem:

Theorem 1.10 (Normal typicality for $\mathcal{H}_{\mathrm{eq}}$ [55])**.** *Let $\eta, \delta, \delta' \in (0,1)$ and let*

$$D_0(\eta\delta', \delta) = \max\Big\{10^3(\eta\delta')^{-2}\log(4/\delta), 10^6(\eta\delta')^{-4}\Big\}. \tag{1.36}$$

Let $\mathcal{H}$ be a Hilbert space of dimension $D > D_0(\eta\delta', \delta)$ and let $\mathcal{H}_{\mathrm{eq}}$ be a subspace of $\mathcal{H}$ of dimension $d_{\mathrm{eq}} > (1-\eta\delta'/2)D$. Moreover, let $E_1, \ldots, E_D$ be distinct real numbers and let H be a Hamiltonian on $\mathcal{H}$ with eigenvalues $E_1, \ldots, E_D$ and a uniformly distributed eigenbasis. Then, with probability at least $1-\delta$, every initial state $\psi_0 \in \mathbb{S}(\mathcal{H})$ is such that for $(1-\delta')$-most of the time,

$$\langle\psi_t|P_{\mathrm{eq}}|\psi_t\rangle > 1 - \eta. \tag{1.37}$$

It follows from Theorem 1.9 that every initial state spends most of its time in thermal equilibrium. If the initial state is not in thermal equilibrium, this tells us that it approaches thermal equilibrium and stays there for most of the time. Unfortunately, the theorem does not tell us anything about the time it takes a non-equilibrium initial state to reach thermal equilibrium, put another way, it does not prove anything concerning the *thermalization time*.

Here and also in the theorems in Section 1.2.2 it is important that the statements hold true for *all* rather than *most* initial states because most initial states are in thermal equilibrium anyways and therefore a theorem concerning only most initial states would not allow us to conclude anything about the approach to thermal equilibrium. The claim that most states are in thermal equilibrium can be proved very easily, see Theorem 1.31 and its short proof.

We also remark that in contrast to the theorems in Section 1.2.2 it is only required that the eigenvalues of H are non-degenerate, the assumption that also the eigenvalue gaps are non-degenerate is not needed here. Furthermore, note that Theorem 1.9 and Theorem 1.10 are concerned with *most* Hamiltonians and it cannot be expected that they are true for all Hamiltonians. Several counterexamples can be found in [55]. One of them is given by *Anderson localization* [4, 94]: There are some (physically relevant) Hamiltonians for which there are eigenstates ϕ whose spatial energy density is not uniform on the macroscopic scale. However, wave functions in thermal equilibrium should have a uniform energy density, i.e., such eigenstates are not in or close to $\mathcal{H}_{\mathrm{eq}}$ and $\langle\phi|P_{\mathrm{eq}}|\phi\rangle$ is not close to 1. Because of $\langle\phi_t|P_{\mathrm{eq}}|\phi_t\rangle = \langle\phi|P_{\mathrm{eq}}|\phi\rangle$ for all times t, the eigenstate ϕ never approaches thermal equilibrium.

While the assumption that the eigenbasis of H is uniformly distributed might seem natural at a first glance, a closer look reveals that it is physically rather unrealistic. The reason is the following: If the energy eigenbasis of H is uniformly distributed, it is completely unrelated to the orthogonal decomposition (1.12) of the Hilbert space into macro spaces and this should not be the case for physically realistic systems. If the eigenbasis of H and the decomposition (1.12) are unrelated, the system goes almost immediately from any (possibly very far from thermal equilibrium) macro space into $\mathcal{H}_{\mathrm{eq}}$ [50, 51, 52], see Section 1.6.5 for more details. In more realistic systems, an initial state far from thermal equilibrium with for example a very non-uniform energy density on the macroscopic scale should evolve into one with a uniform energy density and transporting the energy in the system in order to achieve the uniform distribution of energy requires time and the passage through other macro spaces. To see such a behavior, the Hamiltonian has to have more structure, for example, we expect a system to show such a behavior if its Hamiltonian has a band structure in a basis that diagonalizes the projections P_ν to the macro spaces $\mathcal{H}_\nu$. This motivated our works on a generalization of normal typicality to more realistic Hamiltonians in [136, 138], see also Section 3.2.

1.3. Dynamical Typicality

The name "dynamical typicality" was first introduced in a paper by Bartsch and Gemmer [6] in (2009). It describes the phenomenon that given an observable A on a Hilbert space $\mathcal{H}$ of dimension $D = \dim \mathcal{H} < \infty$, a Hamiltonian H on $\mathcal{H}$ and $a \in \mathbb{R}$, there is a (real-valued) function $a(t)$ such that for every $t \in \mathbb{R}$ and most wave functions $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle \approx a$ it holds that $\langle\psi_t|A|\psi_t\rangle \approx a(t)$.

For their proof they assumed without loss of generality that the moments $c_j = \mathrm{tr}(A^j)/D$ of the observables A fulfill $c_1 = 0$, $c_2 = 1$, and that $c_3, \dots, c_8$ are of order 1. Given $a \in \mathbb{R}$, they introduced an ensemble of pure states $|\psi\rangle$ by requiring them to have the same quantum expectation value a, i.e., $\langle\psi|A|\psi\rangle = a$, and to be uniformly distributed otherwise. This means that the ensemble is invariant under all unitary operators on $\mathcal{H}$ that do not change the quantum expectation value of A. Then they introduced a "substitute" ensemble $\{|\omega\rangle\}$ which was easier to handle and they computed the average and variance of $\langle\omega|\omega\rangle$, the average of $\langle\omega|A|\omega\rangle$ and the variance of $\langle\omega|A(t)|\omega\rangle$. Here, the time evolved operator A is defined as $A(t) = e^{iHt}Ae^{-iHt}$. From the results of these computations they concluded that the substitute ensemble indeed approximates well the original ensemble as long as a is not too far away from zero and provided that the dimension D is sufficiently large. Moreover, they argued that the curve $t \mapsto a_\omega(t) = \langle\omega|A(t)|\omega\rangle$ and therefore also the curve $t \mapsto \langle\phi|A(t)|\phi\rangle$ is nearly deterministic and they provided numerical examples to support their findings.

A rigorous version was proved two years later by Müller, Gross, and Eisert [92]. They also considered a finite-dimensional Hilbert space of dimension D, identified it with $\mathbb{C}^D$ and analyzed the set

$$M_a := \left\{ |\psi\rangle \in \mathbb{C}^D \middle| \langle\psi|A|\psi\rangle = a \text{ and } \|\psi\| = 1 \right\}, \tag{1.38}$$

where A is an observable on $\mathbb{C}^D$. They showed that M_a is a submanifold of $\mathbb{R}^{2D}$ and pointed out that this implies that M_a inherits the "Hausdorff measure" from $\mathbb{R}^{2D}$ which, after normalizing it, yields a natural probability measure on M_a. The main result in [92] is a concentration-of-measure-type result for M_a from which dynamical typicality follows almost immediately. More precisely, they showed the following theorem:

Theorem 1.11 (Concentration of measure for M_a [92]). *Let A be a self-adjoint operator on $\mathbb{C}^D$ with eigenvalues $a_1, \ldots, a_D$ and let $a_{\min} := \min_k a_k$, $a_{\max} := \max_k a_k$. Moreover, let $a_{\mathrm{AM}} := \frac{1}{D}\sum_k a_k$ be their arithmetic mean and $a > a_{\min}$ any value which is not too close to a_{AM}, i.e.,*

$$a \leq a_{\mathrm{AM}} - \frac{\pi(a_{\max} - a_{\min})}{\sqrt{2(D-1)}}. \tag{1.39}$$

Let $|\psi\rangle \in M_a$ be a random state according to the natural probability distribution on M_a described above. Then, for any $\varepsilon > 0$ and any Lipschitz-continuous function $f : M_a \to \mathbb{R}$ with Lipschitz constant λ, we have that

$$\mathbb{P}\left\{|f(\psi) - \bar{f}| > \lambda\varepsilon\right\} \leq bD^{2/3} e^{-cD\left(\varepsilon - \frac{1}{4D}\right)^2 + 2\delta\sqrt{D}}, \tag{1.40}$$

where $\bar{f}$ is the median of f on M_a. The constants b, c and δ can be obtained in the following way:

- *Shift the eigenvalues a_k by some offset s to $a'_k = a_k + s$ such that $a'_{\min} > 0$ and*

$$a' = \left(1 + \frac{1}{D}\right)\left(1 + \frac{\delta}{\sqrt{D}}\right) a'_H, \tag{1.41}$$

  with $\delta > 0$, where a'_H denotes the harmonic mean of the a'_k. The offset can be chosen arbitrarily with the only constraint that the constant b below is positive.

- *Compute $c = 3a'_{\min}/(32a')$ and*

$$a'_Q := \left(\frac{1}{D}\sum_k a'^{-2}_k\right)^{-1/2}, \qquad b = 3040 a'^2_{\max}\left[a'^2\left(1 - \frac{a'^2}{\delta^2 a'^2_Q}\right)\right]^{-1}. \tag{1.42}$$

We remark that this theorem can be seen as a kind of Lévy Lemma, a concentration-of-measure-type result for the sphere which we discuss in the next section, but for M_a instead of the sphere.

Since the function $f_{A(t)} : M_a \to \mathbb{R}$, $f_{A(t)}(\psi) = \langle\psi|A(t)|\psi\rangle$ is Lipschitz-continuous for any observable A on $\mathbb{C}^D$ and any $t \in \mathbb{R}$, see, e.g., Lemma 5 in [100], we can apply Theorem 1.11 to f and this immediately provides us with a rigorous version of dynamical typicality.

Another closely related result was proved by Reimann [110]: He considered two observables A and B on a Hilbert space $\mathcal{H}$ with $D = \dim\mathcal{H} < \infty$ and showed that for every $t \in \mathbb{R}$ and most $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle \approx a$, it holds that $\langle\psi_t|B|\psi_t\rangle \approx b(t)$ where $b(t)$ is a suitable function. He started from a real number $a \in (a_{\min}, a_{\max})$, where $a_{\min}$ and $a_{\max}$ again denote the smallest and largest eigenvalue of A, constructed an ensemble $\{|\varphi\rangle\}$ of wave functions such that for most $|\varphi\rangle$ its norm squared $\|\varphi\|^2$ is close to 1 and $\langle\varphi|A|\varphi\rangle$ is close to a. Then he showed that also for $B_t = e^{iHt}Be^{-iHt}$, the expectation value $\langle\varphi|B_t|\varphi\rangle$ is close to its ensemble average which serves as $b(t)$. Reimann also compares his result to the ones from Bartsch and Gemmer [6] and Müller, Gross, and Eisert [92], see Section IV. and Section VI. in [110].

In view of Section 1.2.1 it is of interest to consider the case $A = P_\nu$ where P_ν is a projection to some macro space $\mathcal{H}_\nu$. Then $|\varphi\rangle$ is uniformly distributed over $\mathbb{S}(\mathcal{H}_\nu)$. However, this case is not covered in the proofs of Reimann [110] and Müller, Gross, and Eisert [92] as $a_{\max} = 1$ and, for technical reasons, it is required that $a < a_{\max}$.

A more general result concerning dynamical typicality which also covers the aforementioned case was proved by Balz, Richter, Gemmer, Steinigeweg, and Reimann [5] in 2019:

Theorem 1.12 (Dynamical typicality [5]). *Let $\mathcal{H}$ be a Hilbert space of dimension $D = \dim\mathcal{H} < \infty$ decomposed into K mutually orthogonal subspaces $\mathcal{H}_\nu$ as in* (1.12). *Let H be a (possibly time-dependent) Hamiltonian on $\mathcal{H}$ and let A be a self-adjoint operator on $\mathcal{H}$ with maximum eigenvalue $a_{\max}$ and minimum eigenvalue $a_{\min}$ and set $\Delta_A := a_{\max} - a_{\min}$. Moreover, let $t \in \mathbb{R}$ and let $A_t := \mathcal{U}_t^* A \mathcal{U}_t$ where $\mathcal{U}_t$ is the quantum-mechanical time evolution operator. Furthermore, let $p_1, \dots, p_K \geq 0$ be such that $\sum_\nu p_\nu = 1$ and let ρ be any density matrix on $\mathcal{H}$ that satisfies $p_\nu = \mathrm{tr}(\rho P_\nu)$ for all ν. Moreover, for every ν, let U_ν be an operator on $\mathcal{H}$ that satisfies $U_\nu^* U_\nu = P_\nu$ and $U_\nu P_\mu = P_\mu U_\nu = \delta_{\mu\nu} U_\nu$ and define the unitary operator $U := \sum_\nu U_\nu$ and the density matrix $\rho_U = U^*\rho U$. Let each U_ν be uniformly distributed on $\mathcal{H}_\nu$. Then,*

$$\mathbb{E}_U\left(\mathrm{tr}(\rho_U A_t)\right) = \sum_{\nu=1}^{K} \frac{p_\nu}{d_\nu}\,\mathrm{tr}(A_t P_\nu), \tag{1.43}$$

$$\mathrm{Var}_U\left(\mathrm{tr}(\rho_U A_t)\right) \leq 5\Delta_A^2 \max_\nu \left(\frac{p_\nu}{d_\nu}\right), \tag{1.44}$$

where $\mathbb{E}_U$ and Var_U denote the expectation and variance with respect to the distribution of U.

Theorem 1.12 shows that as soon as the dimensions d_ν of the subspaces $\mathcal{H}_\nu$ are all large and Δ_A is not too large, then $\mathrm{tr}(\rho_U A_t)$ is close to its expectation value.

This theorem can be applied to the aforementioned setting as follows: Considering $\rho = |\psi\rangle\langle\psi|$ with $\psi \in \mathbb{S}(\mathcal{H})$ and setting $p_\mu = 1$ and $p_\nu = 0$ for $\nu \neq \mu$ amounts to considering $\rho = |\psi\rangle\langle\psi|$ with $\psi \in \mathbb{S}(\mathcal{H}_\mu)$. Moreover, averaging $\operatorname{tr}(\rho_U A_t)$ with respect to U is equal to taking the expectation of $\langle\psi|A_t|\psi\rangle$ over ψ uniformly distributed in the sphere $\mathbb{S}(\mathcal{H}_\mu)$ (and the same holds true for the variance). Therefore, it follows from Theorem 1.12 that for every $t \in \mathbb{R}$ and most $\psi \in \mathbb{S}(\mathcal{H}_\mu)$,

$$\langle\psi_t|A|\psi_t\rangle \approx \sum_{\nu=1}^{K} \frac{p_\nu}{d_\nu} \operatorname{tr}(A_t P_\nu) = \frac{1}{d_\mu} \operatorname{tr}(\mathcal{U}_t^* A \mathcal{U}_t P_\mu). \tag{1.45}$$

Moreover, if the Hamiltonian H is time-independent, the right-hand side in (1.45) becomes $\operatorname{tr}(e^{iHt} A e^{-iHt} P_\mu)/d_\mu$.

We remark that similar conclusions can be drawn from a result by Reimann and Gemmer [111] who use a different strategy of proof. Furthermore, we note that the proof simplifies a lot when only the special case mentioned above is considered; we give this rather simple proof in [136]. Finally, we remark that dynamical typicality for projections, i.e., the statement that for every $t \in \mathbb{R}$ and most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$, $\langle\psi_t|P_\nu|\psi_t\rangle \approx \operatorname{tr}(e^{itH} P_\nu e^{-itH} P_\mu)/d_\mu$ (in the sense that the absolute error is small) can also be obtained from Eq. (13) in a paper of Reimann [104] (by choosing the distribution of the coefficients c_n to be the centered Gaussian distribution with covariance P_μ, $K = 1$, and $A = e^{iHt} P_\nu e^{-iHt}$).

1.4. Canonical Typicality

We consider a macroscopic quantum system $S = a \cup b$ with finite-dimensional Hilbert space $\mathcal{H}_S = \mathcal{H}_a \otimes \mathcal{H}_b$. Canonical typicality is the statement that for most $\psi \in \mathbb{S}(\mathcal{H}_R)$, where $\mathcal{H}_R \subset \mathcal{H}_S$ is a high-dimensional subspace, the reduced density matrix

$$\rho_a^\psi := \operatorname{tr}_b |\psi\rangle\langle\psi|, \tag{1.46}$$

is close to $\operatorname{tr}_b \rho_R$ with $\rho_R = P_R/d_R$ being the normalized projection to $\mathcal{H}_R$, i.e.,

$$\rho_a^\psi \approx \operatorname{tr}_b \rho_R, \tag{1.47}$$

provided that $d_R = \dim \mathcal{H}_R$ is large and $d_a = \dim \mathcal{H}_a$ is not too large. Here, tr_b denotes the partial trace over the environment b of the subsystem a.

The motivation for calling this phenomenon "canonical typicality" is the following: Suppose that $\mathcal{H}_R$ is a micro-canonical subspace, i.e., $\mathcal{H}_R = \mathcal{H}_{\mathrm{mc}}$ and $\rho_R = \rho_{\mathrm{mc}}$. If b is large and the interaction between a and b is weak (and thus it can be assumed that the system's Hamiltonian is of the form $H = H_a + H_b$), it can be argued that $\operatorname{tr}_b \rho_{\mathrm{mc}}$

is close to a canonical density matrix

$$\rho_{a,\mathrm{can}} = \frac{1}{Z_a} e^{-\beta H_a} \tag{1.48}$$

with suitable inverse temperature $\beta > 0$, where $Z_a = \operatorname{tr} e^{-\beta H_a}$ is a normalization constant [58].

The term *canonical typicality* first appeared in a paper of Goldstein, Lebowitz, Tumulka, and Zanghì [58] in 2006 and their argument is based on ideas of Schrödinger [122] which he presented in his book on *Statistical Thermodynamics* in 1952.

Around the same time when the preprint of the paper of Goldstein, Lebowitz, Tumulka, and Zanghì [58] appeared, Popescu, Short, and Winter [100, 101] gave a rigorous proof of canonical typicality in the more general form discussed in the introduction to this section using Lévy's Lemma, a concentration-of-measure-type result for the uniform measure on the sphere of high-dimensional Hilbert spaces.

In the more than 50 years between the work of Schrödinger [122] and the publication of the two results we just mentioned, there have been found some closely related results. Already Lloyd [79] computed in his PhD thesis from 1988 the reduced density matrix of a typical state in the setting described above and obtained that it is approximately canonical; in his words: "We show that the exact state for a system in contact with a thermostat at temperature T is likely to be a mixture that has the same form as the canonical ensemble for the system."

In 2003, Gemmer and Mahler [45] obtained canonical typicality by computing the size of suitable Hilbert space regions and assuming that the degeneracies of the energy eigenvalues are huge.

Another related result was proved by Tasaki [131] in 1998. He considered a system consisting of a subsystem coupled to a heat bath and argued that under the "hypothesis of equal weights for eigenstates" and assuming a special form of the coupling Hamiltonian, for initial states ψ with small energy fluctuation at a sufficiently large and typical time t the expectation of any operator A which acts on the subsystem is close to its expectation in a canonical state. It follows that the reduced density matrix $\rho_a^{\psi_t}$ is approximately canonical. However, this does not imply that for $t = 0$ and a typical initial state the reduced density matrix is canonical.

Canonical typicality can be proved in several ways. In the following we present the argument of Goldstein, Lebowitz, Tumulka, and Zanghì [58] based on ideas of Schrödinger in Section 1.4.1. Then, in Section 1.4.2 and Section 1.4.4, we discuss the proofs of polynomial resp. exponential bounds. Here, "polynomial" and "exponential" refers to the behavior in d_R of upper bounds on the probability that the reduced density matrix ρ_a^ψ differs from $\operatorname{tr}_b \rho_R$ significantly. While the proof of polynomial bounds makes use of the identities in Lemma 1.2, the main tool for the proof of exponential bounds due to Popescu, Short, and Winter [100, 101] is Lévy's Lemma, which we discuss in Section 1.4.3.

1.4.1. An Argument Based on Ideas of Schrödinger

In [58], some ideas of Schrödinger [122] were extended and the term "canonical typicality" was introduced. This section closely follows the presentation in [58].

Let $\mathcal{H}_{\mathrm{mc}} \subset \mathcal{H}_S$ be a micro-canonical subspace corresponding to a small energy interval $[E, E + \Delta E]$. Let $|E_1^{(a)}\rangle, \dots, |E_{d_a}^{(a)}\rangle$ and $|E_1^{(b)}\rangle, \dots, |E_{d_b}^{(b)}\rangle$ be orthonormal eigenbases of H_a and H_b respectively. Then,

$$\operatorname{tr}_b \rho_{\mathrm{mc}} = \frac{1}{d_{\mathrm{mc}}} \sum_i \dim(\mathcal{H}_i^{(b)}) |E_i^{(a)}\rangle\langle E_i^{(a)}|, \tag{1.49}$$

where $\mathcal{H}_i^{(b)} := \mathcal{H}^{(b)}_{[E-E_i^{(a)}, E-E_i^{(a)}+\Delta E]}$ is the subspace of $\mathcal{H}_b$ spanned by the eigenvectors of H_b with eigenvalues in the interval $[E - E_i^{(a)}, E - E_i^{(a)} + \Delta E]$.

Let $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ be uniformly distributed. Then we can write $\psi = \phi/\|\phi\|$ where ϕ is a complex-valued Gaussian random vector in $\mathbb{C}^{d_{\mathrm{mc}}}$, see Section 1.1.2, i.e,

$$\phi = \sum_i \sum_j c_{ij} |E_i^{(a)}\rangle |E_j^{(b)}\rangle =: \sum_i |E_i^{(a)}\rangle |\phi_i\rangle, \tag{1.50}$$

where $|\phi_i\rangle := \sum_j c_{ij} |E_j^{(b)}\rangle$ and the coefficients c_{ij} are independent complex-valued Gaussian random variables for i, j such that $E_i^{(a)} + E_i^{(b)} \in [E, E + \Delta E]$ and zero otherwise. With this, the reduced density matrix ρ_a^ψ becomes

$$\rho_a^\psi = \frac{1}{\|\phi\|^2} \sum_{i,i'} \langle \phi_i | \phi_{i'} \rangle |E_i^{(a)}\rangle\langle E_{i'}^{(a)}|. \tag{1.51}$$

Then they argue that

$$\langle \phi_i | \phi_{i'} \rangle \approx \|\phi_i\|^2 \delta_{ii'} = \delta_{ii'} \sum_j |c_{ij}|^2 \tag{1.52}$$

because if $i \neq i'$ and if there are only few j that appear in $|\phi_i\rangle$ as well as in $|\phi_{i'}\rangle$, the two vectors are obviously approximately orthogonal. If on the other hand there are many j appearing in both vectors, the corresponding parts of $|\phi_i\rangle$ and $|\phi_{i'}\rangle$ are (after suitable normalization) two independent uniformly distributed vectors on the sphere of a high-dimensional Hilbert space and therefore with high probability approximately orthogonal[8].

From (1.52) we see that $\|\phi_i\|^2$ is a sum of $\dim \mathcal{H}_i^{(b)}$ independent identically distributed random variables with mean 1. Therefore, it follows from the law of large

[8] This can easily be seen as follows: Let $\psi = (\psi_1, \dots, \psi_D)$ and $\phi = (\phi_1, \dots, \phi_D)$ be two independent uniformly distributed vectors on the sphere $\mathbb{S}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ of dimension $D \in \mathbb{N}$ and

numbers that $\|\phi_i\|^2 \approx \dim \mathcal{H}_i^{(b)}$ and similarly $\|\phi\|^2 \approx d_{\mathrm{mc}}$. This finally implies

$$\rho_a^\psi \approx \frac{1}{d_{\mathrm{mc}}} \sum_i \dim(\mathcal{H}_i^{(b)}) |E_i^{(a)}\rangle\langle E_i^{(a)}| = \operatorname{tr}_b \rho_{\mathrm{mc}} \tag{1.53}$$

as desired. Moreover, as already discussed in the introduction to this section, if a and b are weakly interacting and b is large enough, $\operatorname{tr}_b \rho_{\mathrm{mc}}$ is close to $\rho_{a,\mathrm{can}}$ and thus $\rho_a^\psi \approx \rho_{a,\mathrm{can}}$.

1.4.2. Proof of Polynomial Bounds

Canonical typicality can be proved in several ways; in this section, we discuss how an upper polynomial bound on the probability that ρ_a^ψ and $\operatorname{tr}_b \rho_R$ differ significantly can be obtained. These Chebyshev-type bounds were first shown by Sugita [130], see also [140].

As customary, the distance between two density matrices is measured in the *trace norm* $\|\cdot\|_{\mathrm{tr}}$. Recall that the trace norm of an operator A is defined as

$$\|A\|_{\mathrm{tr}} := \operatorname{tr}|A| = \operatorname{tr}\sqrt{A^*A}. \tag{1.54}$$

If A is self-adjoint, then $\|A\|_{\mathrm{tr}}$ is the sum of the absolute eigenvalues of A.

Theorem 1.13 (Canonical typicality – polynomial bounds [130, 140])**.** *Let $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, where $\mathcal{H}_a$ and $\mathcal{H}_b$ are Hilbert spaces of dimensions d_a and d_b respectively. Let $\mathcal{H}_R \subset \mathcal{H}$ be a subspace of dimension d_R and let $\rho_R = P_R/d_R$ where P_R denotes the orthogonal projection to $\mathcal{H}_R$. Moreover, let u_R be the uniform distribution over $\mathbb{S}(\mathcal{H}_R)$. Then for every $\varepsilon > 0$,*

$$u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \big\|\rho_a^\psi - \operatorname{tr}_b \rho_R\big\|_{\mathrm{tr}} > \varepsilon\Big\} \leq \frac{d_a^4}{\varepsilon^2 d_R}. \tag{1.55}$$

Theorem 1.13 shows that as soon as $d_a^4 \ll d_R$, i.e., as soon as $\mathcal{H}_R$ is large and the subsystem a it not too large, for typical $\psi \in \mathbb{S}(\mathcal{H}_R)$ the reduced density matrix ρ_a^ψ is

---

let $\varepsilon > 0$. Then we find with the help of Lemma 1.2 that

$$\mathbb{E}\,|\langle\psi,\phi\rangle|^2 = \sum_{j,k} \mathbb{E}\left(\psi_j^*\phi_j\phi_k^*\psi_k\right) = \sum_{j,k} \mathbb{E}\left(\psi_j^*\psi_k\right)\mathbb{E}\left(\phi_k^*\phi_j\right) = \sum_j \frac{1}{D^2} = \frac{1}{D}.$$

Now Markov's inequality implies

$$\mathbb{P}\left(|\langle\psi,\phi\rangle| > \varepsilon\right) = \mathbb{P}\left(|\langle\psi,\phi\rangle|^2 > \varepsilon^2\right) \leq \frac{\mathbb{E}\,|\langle\psi,\phi\rangle|^2}{\varepsilon^2} = \frac{1}{\varepsilon^2 D}.$$

Thus, as soon as D is sufficiently large, ψ and ϕ are nearly orthogonal with very high probability.

approximately given by $\operatorname{tr}_b \rho_R$ in the sense that the trace norm of $\rho_a^\psi - \operatorname{tr}_b \rho_R$ is small. Put another way, the reduced density matrix ρ_a^ψ is nearly constant on $\mathbb{S}(\mathcal{H}_R)$.

Besides the trace norm we also need the often easier computable *Hilbert-Schmidt norm* $\|\cdot\|_2$ for the proof of Theorem 1.13. For a $d_a \times d_a$-matrix $A = (A_{ij})$ it is defined as

$$\|A\|_2 := \sqrt{\operatorname{tr}(A^* A)} = \sqrt{\sum_{i,j=1}^{d_a} |A_{ij}|^2}. \tag{1.56}$$

If A is self-adjoint, then $\|A\|_2^2$ is the sum of the squares of the eigenvalues of A.

The trace norm and the Hilbert-Schmidt norm can be related via the inequality

$$\|A\|_{\mathrm{tr}} \leq \sqrt{d_a} \|A\|_2, \tag{1.57}$$

see, e.g., Lemma 6 in [100] for a proof.

In the rest of this section we give the proof of Theorem 1.13 and closely follow the presentation in [140], see also Section 4.6 in [137].

Proof of Theorem 1.13. We first observe that it follows from Lemma 1.2 and Chebyshev's inequality that for any operator A on $\mathcal{H}$ and $\varepsilon > 0$,

$$\begin{aligned} u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \big|\langle\psi|A|\psi\rangle - \operatorname{tr}(A\rho_R)\big| > \varepsilon\Big\} \\ &\leq \frac{1}{\varepsilon^2 d_R(d_R+1)} \left(\operatorname{tr}(P_R A^* P_R A) - \frac{\left|\operatorname{tr}(P_R A)\right|^2}{d_R}\right) && (1.58a) \\ &\leq \frac{\operatorname{tr}(A^* \rho_R A)}{\varepsilon^2 (d_R+1)}. && (1.58b) \end{aligned}$$

In the last step we used the inequality $|\operatorname{tr}(BC)| \leq \|B\| \operatorname{tr}(|C|)$ for any operators B and C, see, e.g. [126, Theorem 3.7.6], and the fact that $A^*\rho_R A$ is a positive operator.

Let $\{|l\rangle_a : l = 1, \ldots, d_a\}$ be an orthonormal basis of $\mathcal{H}_a$ and define the operators

$$A_{lm} := (|l\rangle_{aa}\langle m|) \otimes I_b, \tag{1.59}$$

where I_b denotes the identity operator on $\mathcal{H}_b$. Because of $\langle\psi|A_{lm}|\psi\rangle = {}_a\langle m|\rho_a^\psi|l\rangle_a$ and $\operatorname{tr}(A_{lm}\rho_R) = {}_a\langle m| \operatorname{tr}_b \rho_R |l\rangle_a$ it follows from (1.58b) that

$$u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \big|{}_a\langle m|\rho_a^\psi|l\rangle_a - {}_a\langle m| \operatorname{tr}_b \rho_R|l\rangle_a\big| > \varepsilon\Big\} \leq \frac{\operatorname{tr}(A_{lm}^* \rho_R A_{lm})}{\varepsilon^2(d_R+1)} \leq \frac{{}_a\langle l| \operatorname{tr}_b \rho_R |l\rangle_a}{\varepsilon^2 d_R}. \tag{1.60}$$

With (1.57) we get

$$\left\|\rho_a^\psi - \operatorname{tr}_b \rho_R\right\|_{\mathrm{tr}}^2 \le d_a \sum_{l,m=1}^{d_a} \left|{}_a\langle m|\rho_a^\psi - \operatorname{tr}_b \rho_R|l\rangle_a\right|^2 \le d_a^3 \max_{l,m} \left|{}_a\langle m|\rho_a^\psi - \operatorname{tr}_b \rho_R|l\rangle_a\right|^2 \tag{1.61}$$

and together with (1.60) we finally obtain

$$u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \left\|\rho_a^\psi - \operatorname{tr}_b \rho_R\right\|_{\mathrm{tr}} > \varepsilon\Big\}$$

$$\le u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \exists l, m : \left|{}_a\langle m|\rho_a^\psi|l\rangle_a - {}_a\langle m|\operatorname{tr}_b \rho_R|l\rangle_a\right| > \varepsilon d_a^{-3/2}\Big\} \tag{1.62a}$$

$$\le \sum_{l,m=1}^{d_a} u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \left|{}_a\langle m|\rho_a^\psi|l\rangle_a - {}_a\langle m|\operatorname{tr}_b \rho_R|l\rangle_a\right| > \varepsilon d_a^{-3/2}\Big\} \tag{1.62b}$$

$$\le \frac{d_a^3}{\varepsilon^2 d_R} \sum_{l,m} {}_a\langle l|\operatorname{tr}_b \rho_R|l\rangle_a = \frac{d_a^4}{\varepsilon^2 d_R}. \tag{1.62c}$$

□

1.4.3. Lévy's Lemma

Another way to prove canonical typicality and obtain stronger bounds is to apply *Lévy's Lemma* in a suitable way. This was done by Popescu, Short, and Winter [100, 101] and we discuss their proof in Section 1.4.4. The present section is devoted to a discussion of a proof of Lévy's Lemma in the version due to Maurey and Pisier [99] and here we closely follow the proof given by Milman and Schechtman [88].

Theorem 1.14 (Lévy's Lemma [88])**.** *Let $\mathcal{H}$ be a Hilbert space of dimension $D = \dim \mathcal{H} < \infty$, let $f : \mathbb{S}(\mathcal{H}) \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant η and let u be the uniform distribution over $\mathbb{S}(\mathcal{H})$. Then for every $\varepsilon > 0$,*

$$u\Big\{\psi \in \mathbb{S}(\mathcal{H}) : |f(\psi) - u(f)| > \varepsilon\Big\} \le 4 \exp\left(-\frac{2D\varepsilon^2}{9\pi^3\eta^2}\right), \tag{1.63}$$

where $u(f) = \int_{\mathbb{S}(\mathcal{H})} f(\psi)\, u(d\psi)$.

Roughly speaking, Lévy's Lemma states that Lipschitz continuous functions on spheres in high-dimensional Hilbert spaces are approximately constant.

We remark that the statement of Theorem 1.14 remains true for complex-valued functions f if we replace the factor 4 by 8 and divide the term in the exponential by 2. This follows immediately from applying Theorem 1.14 to the real and imaginary part of f separately.

As it is pointed out in [137], Lévy's original statement from 1922 (reprinted in [73, Section 3.I.9]) is somewhat different. He showed that if a hypersurface $S \subset \mathbb{S}(\mathbb{R}^d)$ divides $\mathbb{S}(\mathbb{R}^d)$ into two regions of equal size, then the size of its ε-neighborhood is at least as large as the ε-neighborhood of an equator. Moreover, he showed that if d is large, the size of the ε-neighborhood of an equator is close to the full size of the sphere. Milman and Schechtman [88] argue that this implies that for f as in Theorem 1.14 and large d most $\psi \in \mathbb{S}(\mathbb{R}^d)$ are such that $f(\psi)$ is close to the median of f.

We now give a proof of Theorem 1.14 and closely follow Section V.1. and Section V.2. in [88]. The first step is to prove the following lemma about Gaussian distributions [88]:

Lemma 1.15. *Let $F : \mathbb{R}^D \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant η. Let $X = (X_1, \ldots, X_D)$ be a vector of independent real-valued Gaussian random variables with mean zero and variance 1. Then for every $\varepsilon > 0$,*

$$\mathbb{P}\Big\{|F(X) - \mathbb{E}F(X)| > \varepsilon\Big\} \leq 2\exp\left(-\frac{2\varepsilon^2}{\pi^2\eta^2}\right). \tag{1.64}$$

Proof. The function F can be approximated uniformly by continuously differentiable functions and therefore we assume without loss of generality that F itself is continuously differentiable. Let $Y = (Y_1, \ldots, Y_D)$ have the same distribution as X and let it be independent of X. Since the Gaussian distribution is invariant under orthogonal transformations, it follows that X and Y have the same joint distribution as $X_\theta := X\sin\theta + Y\cos\theta$ and $\frac{d}{d\theta}X_\theta = X\cos\theta - Y\sin\theta$ where $0 \leq \theta \leq \pi/2$.

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a convex non-negative function. With the help of Jensen's inequality and Fubini's theorem we obtain

$$\begin{aligned}
\mathbb{E}\varphi(F(X) - \mathbb{E}F(X)) &\leq \mathbb{E}\varphi(F(X) - F(Y)) && \text{(1.65a)}\\
&= \mathbb{E}\left[\varphi\left(\int_0^{\pi/2} \frac{d}{d\theta}F(X_\theta)\, d\theta\right)\right] && \text{(1.65b)}\\
&= \mathbb{E}\left[\varphi\left(\int_0^{\pi/2} \left\langle \nabla F(X_\theta), \frac{d}{d\theta}X_\theta \right\rangle d\theta\right)\right] && \text{(1.65c)}\\
&\leq \frac{2}{\pi}\,\mathbb{E}\left[\int_0^{\pi/2} \varphi\left(\frac{\pi}{2}\left\langle \nabla F(X_\theta), \frac{d}{d\theta}X_\theta\right\rangle\right) d\theta\right] && \text{(1.65d)}\\
&= \mathbb{E}\varphi\left(\frac{\pi}{2}\left\langle \nabla F(X), Y\right\rangle\right). && \text{(1.65e)}
\end{aligned}$$

Choosing $\varphi(x) = e^{\lambda x}$ with $\lambda \in \mathbb{R}$ gives

$$\mathbb{E}\exp\left(\lambda(F(X) - \mathbb{E}F(X))\right) \leq \mathbb{E}\exp\left(\frac{\lambda\pi}{2}\langle \nabla F(X), Y\rangle\right) \tag{1.66a}$$

$$= \mathbb{E}\exp\left(\frac{\lambda\pi}{2}\sum_{j=1}^{D}\frac{\partial F}{\partial x_j}(X)Y_j\right). \tag{1.66b}$$

We first perform the expectation over Y. To this end note that if G is a real-valued Gaussian random variable with mean μ and variance σ^2, its moment generating function $M_G(t) = \mathbb{E}e^{tG}$ where $t \in \mathbb{R}$ is given by

$$M_G(t) = \exp\left(t\mu + \frac{1}{2}\sigma^2 t^2\right). \tag{1.67}$$

With this we get

$$\mathbb{E}\exp\left(\frac{\lambda\pi}{2}\sum_{j=1}^{D}\frac{\partial F}{\partial x_j}(X)Y_j\right) = \mathbb{E}\exp\left(\frac{\lambda^2\pi^2}{8}\sum_{j=1}^{D}\left(\frac{\partial F}{\partial x_j}(X)\right)^2\right) \tag{1.68a}$$

$$= \mathbb{E}\exp\left(\frac{\lambda^2\pi^2}{8}\|\nabla F(X)\|^2\right) \leq \exp\left(\frac{\lambda^2\pi^2\eta^2}{8}\right), \tag{1.68b}$$

where we used that $\|\nabla f(X)\| \leq \eta$ in the last step. An application of Markov's inequality shows that

$$\mathbb{P}\Big\{\,|F(X) - \mathbb{E}F(X)| > \varepsilon\Big\} \tag{1.69a}$$

$$\leq \mathbb{P}\Big\{F(X) - \mathbb{E}F(X) > \varepsilon\Big\} + \mathbb{P}\Big\{\mathbb{E}F(X) - F(X) > \varepsilon\Big\} \tag{1.69b}$$

$$= \mathbb{P}\Big\{\exp(\lambda[F(X) - \mathbb{E}F(X)])) > e^{\lambda\varepsilon}\Big\} + \mathbb{P}\Big\{\exp(-\lambda[F(X) - \mathbb{E}F(X)])) > e^{\lambda\varepsilon}\Big\}$$

$$\leq 2\exp\left(-\lambda\varepsilon + \frac{\lambda^2\pi^2\eta^2}{8}\right). \tag{1.69c}$$

Minimizing the right-hand side with respect to λ shows that the minimum is attained for $\lambda_{\min} = 4\varepsilon/(\pi^2\eta^2)$ for this choice of λ we immediately obtain (1.64).

□

Proof of Theorem 1.14. We first note that we can identify $\mathcal{H}$ with $\mathbb{R}^{2D}$. Let $X = (X_1, \ldots, X_{2D})$ be a vector of independent real-valued Gaussian random variables with mean zero and variance 1 as in Lemma 1.15. Then $X/\|X\|$ is uniformly distributed on $\mathbb{S}(\mathbb{R}^{2D})$, see Section 1.1.2.

Without loss of generality we assume that $u(f) = 0$. Let $\tilde{f} : \mathbb{R}^{2D} \to \mathbb{R}$, $\tilde{f}(x) = \|x\| f(x/\|x\|)$. The function $\tilde{f}$ is Lipschitz continuous with constant 3η; this can be seen as follows: For any $x, y \in \mathbb{R}^{2D}$ with $\|x\| \leq \|y\|$ we have that

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| = \left|\|x\| f\left(\frac{x}{\|x\|}\right) - \|y\| f\left(\frac{y}{\|y\|}\right)\right| \tag{1.70a}$$

$$\leq \|x\| \left| f\left(\frac{x}{\|x\|}\right) - f\left(\frac{y}{\|y\|}\right) \right| + \left| f\left(\frac{y}{\|y\|}\right) \right| \Big| \|x\| - \|y\| \Big| \tag{1.70b}$$

$$\leq \eta \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + \|f\|_\infty \|x - y\| \tag{1.70c}$$

$$\leq (\eta + \|f\|_\infty) \|x - y\|, \tag{1.70d}$$

where $\|f\|_\infty = \sup_s |f(s)|$ is the sup-norm of f and the last inequality follows from

$$\|x\|^2 \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = 2\|x\|^2 - 2\operatorname{Re}\langle x, y\rangle \left(1 - \frac{\|x\|}{\|y\|}\right) - 2\operatorname{Re}\langle x, y\rangle \tag{1.71a}$$

$$\leq 2\|x\|\|y\| - 2\operatorname{Re}\langle x, y\rangle \leq \|x - y\|^2. \tag{1.71b}$$

Moreover, f is bounded by 2η: Because of $u(f) = 0$ and the continuity of f there exists a $z \in \mathbb{R}^{2D}$ such that $f(z/\|z\|) = 0$ and therefore we find that

$$\left| f\left(\frac{x}{\|x\|}\right) \right| = \left| f\left(\frac{x}{\|x\|}\right) - f\left(\frac{z}{\|z\|}\right) \right| \leq \eta \left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| \leq 2\eta. \tag{1.72}$$

Taking the supremum over $x \in \mathbb{R}^{2D}$ immediately yields $\|f\|_\infty \leq 2\eta$. As the argument is symmetric in x and y, we obtain the same estimate if $\|y\| \leq \|x\|$ and thus we conclude from (1.70d) that $\tilde{f}$ is Lipschitz continuous with constant 3η.

Let $\delta > 0$. We compute

$$u\Big\{\psi \in \mathbb{S}(\mathbb{R}^{2D}) : |f(\psi)| > \varepsilon\Big\} = \mathbb{P}\left(\left| f\left(\frac{X}{\|X\|}\right) \right| > \varepsilon\right) = \mathbb{P}\left(\left|\tilde{f}(X)\right| > \varepsilon\|X\|\right) \tag{1.73a}$$

$$\leq \mathbb{P}\left(\left|\tilde{f}(X)\right| > \delta\varepsilon\sqrt{2D}\right) + \mathbb{P}\left(\|X\| < \delta\sqrt{2D}\right) \tag{1.73b}$$

$$\leq \mathbb{P}\left(\left|\tilde{f}(X)\right| > \delta\varepsilon\sqrt{2D}\right) + \mathbb{P}\left(\Big|\|X\| - \mathbb{E}\|X\|\Big| > \mathbb{E}\|X\| - \delta\sqrt{2D}\right) \tag{1.73c}$$

and

$$\mathbb{E}\|X\| \geq \frac{1}{\sqrt{2D}} \sum_j \mathbb{E}|X_j| = \sqrt{\frac{2}{\pi}}\sqrt{2D}. \tag{1.74}$$

We choose $\delta = \frac{1}{\sqrt{2\pi}}$ and by applying Lemma 1.15 to $\tilde{f}$ and $\|\cdot\|$ we obtain

$$u\Big\{\psi \in \mathbb{S}(\mathbb{R}^{2D}) : |f(\psi)| > \varepsilon\Big\} \leq 2\exp\left(-\frac{2\varepsilon^2 D}{9\pi^3\eta^2}\right) + 2\exp\left(-\frac{2D}{9\pi^3}\right). \tag{1.75}$$

Since we can safely assume that $\varepsilon \leq \eta$, the first term dominates and this finishes the proof. □

1.4.4. Proof of Exponential Bounds

The bound in Theorem 1.13 for the measure of the set of wave functions $\psi \in \mathbb{S}(\mathcal{H}_R)$ for which the reduced density matrix ρ_a^ψ differs significantly from $\operatorname{tr}_b \rho_R$ can be strengthened substantially: While the bound in Theorem 1.13 is polynomially small in d_R, it can be improved to one that is even exponentially small in d_R. The corresponding proof was given by Popescu, Short, and Winter [100, 101] and it is based on Lévy's Lemma which we discussed in the previous section.

Theorem 1.16 (Canonical typicality – exponential bounds [100, 101])**.** *Let $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, where $\mathcal{H}_a$ and $\mathcal{H}_b$ are Hilbert spaces of dimensions d_a and d_b respectively. Let $\mathcal{H}_R \subset \mathcal{H}$ be a subspace of dimension d_R and let $\rho_R = P_R/d_R$, where P_R denotes the orthogonal projection to $\mathcal{H}_R$, and let u_R be the uniform distribution over $\mathbb{S}(\mathcal{H}_R)$. Then for every $\varepsilon > 0$,*

$$u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \big\|\rho_a^\psi - \operatorname{tr}_b \rho_R\big\|_{\operatorname{tr}} > \varepsilon\Big\} \leq 8d_a^2 \exp\left(-\frac{d_R\varepsilon^2}{36 d_a^2 \pi^3}\right). \tag{1.76}$$

Theorem 1.16 shows again that for most $\psi \in \mathbb{S}(\mathcal{H}_R)$ the reduced density matrix ρ_a^ψ is approximately given by $\operatorname{tr}_b \rho_R$. However, in contrast to Theorem 1.13, the bound is exponentially small in d_R and therefore for large d_R usually much smaller. We remark that Popescu, Short, and Winter [100, 101] also prove the slightly stronger bound

$$u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \big\|\rho_a^\psi - \operatorname{tr}_b \rho_R\big\|_{\operatorname{tr}} > \varepsilon + \sqrt{d_a \operatorname{tr}_b(\operatorname{tr}_a \rho_R)^2}\Big\} \leq 4 \exp\left(-\frac{d_R\varepsilon^2}{18\pi^3}\right). \tag{1.77}$$

While the proofs of both bounds make use of Lévy's Lemma, in the proof of Theorem 1.16 it plays a more central role and this proof can easily be adapted to another class of interesting distributions, the so-called *GAP measures* (see Section 1.5), once a Lévy Lemma for them is proven, which is precisely what we did in [137], see Section 3.1 for a discussion of our result. Therefore, in the following we only give the proof of the somewhat weaker bound and we closely follow [100].

Proof of Theorem 1.16. Let U_a be a unitary operator on the Hilbert space $\mathcal{H}_a$. We consider the function $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$,

$$f(\psi) = \operatorname{tr}_a(U_a \rho_a^\psi) = \operatorname{tr}((U_a \otimes I_b)|\psi\rangle\langle\psi|). \tag{1.78}$$

By Lemma 5 in [100], f is Lipschitz continuous with constant $\eta \leq 2\|U_a \otimes I_b\| = 2$. Moreover, we have that

$$\mathbb{E}_R \operatorname{tr}_a(U_a \rho_a^\psi) = \mathbb{E}_R \langle\psi|U_a \otimes I_b|\psi\rangle = \frac{\operatorname{tr}(P_R(U_a \otimes I_b))}{d_R} = \operatorname{tr}_a(U_a \operatorname{tr}_b \rho_R), \tag{1.79}$$

where $\mathbb{E}_R$ denotes the expectation with respect to u_R. Therefore it follows from Theorem 1.14 together with the remark concerning complex-valued functions below it that

$$u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \big|\mathrm{tr}_a(U_a\rho_a^\psi) - \mathrm{tr}_a(U_a \,\mathrm{tr}_b\,\rho_R)\big| > \varepsilon\Big\} \le 8 \exp\left(-\frac{d_R\varepsilon^2}{36\pi^3}\right). \tag{1.80}$$

Let $(U_a^j)_{j=0}^{d_a^2-1}$ be unitary operators on $\mathcal{H}_a$ that form a basis of the space of operators on $\mathcal{H}_a$ and that satisfy the normalization[9]

$$\mathrm{tr}_a\left(U_a^{j*}U_a^k\right) = d_a\delta_{jk}. \tag{1.82}$$

From (1.80) we immediately obtain

$$u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \exists j : \big|\mathrm{tr}_a(U_a^j\rho_a^\psi) - \mathrm{tr}_a(U_a^j \,\mathrm{tr}_b\,\rho_R)\big| > \varepsilon\Big\} \le 8d_a^2 \exp\left(-\frac{d_R\varepsilon^2}{36\pi^3}\right). \tag{1.83}$$

We expand ρ_a^ψ in the basis of the U_a^j,

$$\rho_a^\psi = \frac{1}{d_a}\sum_j C_j(\rho_a^\psi)U_a^j, \tag{1.84}$$

where $C_j(\rho_a^\psi) = \mathrm{tr}_a(U_a^{j*}\rho_a^\psi)$ and similarly for $\mathrm{tr}_b\,\rho_R$. Therefore (1.83) is an upper bound on the measure of the set of $\psi \in \mathbb{S}(\mathcal{H}_R)$ such that there exists a j with $|C_j(\rho_a^\psi) - C_j(\mathrm{tr}_b\,\rho_R)| > \varepsilon$. Next we use the relation between the trace norm and the Hilbert-Schmidt norm in (1.57) to relate the quantity of interest, $\|\rho_a^\psi - \mathrm{tr}_b\,\rho_R\|_{\mathrm{tr}}$, to the distance $|C_j(\rho_a^\psi) - C_j(\mathrm{tr}_b\,\rho_R)|$ of the coefficients in the expansions of ρ_a^ψ and $\mathrm{tr}_b\,\rho_R$. To this end, suppose that $|C_j(\rho_a^\psi) - C_j(\mathrm{tr}_b\,\rho_R)| \le \varepsilon$ for all j. Then we get

$$\begin{aligned}\left\|\rho_a^\psi - \mathrm{tr}_b\,\rho_R\right\|_{\mathrm{tr}}^2 &\le d_a\left\|\rho_a^\psi - \mathrm{tr}_b\,\rho_R\right\|_2^2 &\text{(1.85a)}\\ &= \frac{1}{d_a}\left\|\sum_j\left(C_j(\rho_a^\psi) - C_j(\mathrm{tr}_b\,\rho_R)\right)U_a^j\right\|_2^2 &\text{(1.85b)}\\ &= \frac{1}{d_a}\mathrm{tr}_a\left|\sum_j\left(C_j(\rho_a^\psi) - C_j(\mathrm{tr}_b\,\rho_R)\right)U_a^j\right|^2 &\text{(1.85c)}\end{aligned}$$

---

[9] One possibility to define this basis is given by [100]

$$U_a^j = \sum_{k=0}^{d_a-1} e^{2\pi ik(j-(j \bmod d_a))/d_a^2}|(k+j) \bmod d_a\rangle\langle k|, \tag{1.81}$$

where $(|k\rangle)_{k=0}^{d_a-1}$ is an orthonormal basis of $\mathcal{H}_a$.

$$= \sum_j \left|C_j(\rho_a^\psi) - C_j(\operatorname{tr}_b \rho_R)\right|^2 \leq d_a^2 \varepsilon^2. \tag{1.85d}$$

Putting everything together we obtain

$$\begin{aligned} u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \big\|\rho_a^\psi - \operatorname{tr}_b \rho_R\big\|_{\mathrm{tr}} > d_a\varepsilon\Big\} \\ &\leq u_R\Big\{\psi \in \mathbb{S}(\mathcal{H}_R) : \exists j : \big|C_j(\rho_a^\psi) - C_j(\operatorname{tr}_b \rho_R)\big| > \varepsilon\Big\} && (1.86a) \\ &\leq 8d_a^2 \exp\left(-\frac{d_R\varepsilon^2}{36\pi^3}\right). && (1.86b) \end{aligned}$$

Finally, replacing ε by εd_a^{-1} gives the result. □

1.5. GAP Measures

The typicality results in the previous sections were formulated for the uniform measure on the sphere as a measure of typicality. Here, we introduce a much more general class of distributions on the sphere, the so-called *GAP measures*; the acronym stands for *Gaussian adjusted projected* measure. We first give some motivation for studying GAP measures in Section 1.5.1 and then provide four definitions of GAP measures on finite-dimensional Hilbert spaces including a construction to which the acronym refers in Section 1.5.2. We list a couple of important properties of GAP measures in Section 1.5.3. Finally, in Section 1.5.4, we discuss how GAP measures can also be defined on separable Hilbert spaces.

1.5.1. Motivation

In this section we motivate why the *GAP measures*, a class of probability distributions on the sphere, are interesting and important objects to investigate. Throughout this section we closely follow [59, 57]. In [59] the main motivation for studying probability distributions of wave functions was "to exploit the analogy between classical and quantum statistical mechanics". More precisely, it was shown in [59, 57] that certain GAP measures can be seen as a quantum analogue of the canonical ensemble in classical statistical mechanics. We elaborate more on these results later in this section.

Let $\mathcal{H}$ be a Hilbert space of dimension $D = \dim \mathcal{H} < \infty$ and let $\mathcal{H}_{\mathrm{mc}} \subset \mathcal{H}$ be a micro-canonical subspace as in the beginning of Section 1.2. The corresponding *micro-canonical density matrix* is given by

$$\rho_{\mathrm{mc}} = \frac{1}{d_{\mathrm{mc}}} P_{\mathrm{mc}}, \tag{1.87}$$

where $d_{\mathrm{mc}} = \dim \mathcal{H}_{\mathrm{mc}}$ and P_{mc} denotes the orthogonal projection to $\mathcal{H}_{\mathrm{mc}}$. Moreover, the *micro-canonical measure* is given by the uniform distribution on the sphere $\mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ and we denote it by u_{mc}; this distribution has already been proposed long ago by Schrödinger [121, 122] and Bloch [145].

The micro-canonical density matrix ρ_{mc} and the measure u_{mc} are related via

$$\rho_{\mathrm{mc}} = \int_{\mathbb{S}(\mathcal{H}_{\mathrm{mc}})} |\psi\rangle\langle\psi|\, u_{\mathrm{mc}}(d\psi). \tag{1.88}$$

This identity can be seen as follows: First note that the right-hand side of (1.88) is self-adjoint. As the uniform distribution on the sphere is invariant under unitary transformations, the right-hand side of (1.88) is invariant under conjugation with unitary operators and thus has to be a multiple of the identity on $\mathcal{H}_{\mathrm{mc}}$, i.e., of P_{mc}. The trace of the right-hand side of (1.88) is equal to 1 and because of $\mathrm{tr}(P_{\mathrm{mc}}) = d_{\mathrm{mc}}$ we conclude that the right-hand side of (1.88) is given by $P_{\mathrm{mc}}/d_{\mathrm{mc}} = \rho_{\mathrm{mc}}$.

More generally, let μ be a probability distribution on $\mathbb{S}(\mathcal{H})$. Then we can associate to it a density matrix ρ_μ via

$$\rho_\mu = \int_{\mathbb{S}(\mathcal{H})} |\psi\rangle\langle\psi|\, \mu(d\psi). \tag{1.89}$$

We say that "the distribution μ has density matrix ρ_μ". Note that if μ has mean zero, ρ_μ is just the covariance matrix of μ. Moreover, note that while every probability measure μ on $\mathbb{S}(\mathcal{H})$ uniquely defines a density matrix ρ_μ via (1.89), the converse need not be true, i.e., there can be multiple probability measures on $\mathbb{S}(\mathcal{H})$ that give rise to the same density matrix. An example is given by ρ_{mc} [59]: Take any orthonormal basis of $\mathcal{H}_{\mathrm{mc}}$ and consider the uniform distribution over its d_{mc} elements. This measure is a discrete measure concentrated on the d_{mc} basis elements in contrast to u_{mc} which is spread out over the whole sphere $\mathbb{S}(\mathcal{H}_{\mathrm{mc}})$. However, as one easily sees, both measures have the same density matrix ρ_{mc}.

For any density matrix ρ on $\mathcal{H}$, $\mathrm{GAP}(\rho)$ is the most spread out distribution over $\mathbb{S}(\mathcal{H})$ with density matrix ρ, i.e., such that $\rho_{\mathrm{GAP}(\rho)} = \rho$. This measure was first introduced by Jozsa, Robb, and Wootters [67] who named it *Scrooge measure*[10]. They were interested in a quantity called the "accessible information" and they showed that given a density matrix ρ it is minimized by $\mathrm{GAP}(\rho)$.

As mentioned above, Goldstein, Lebowitz, Tumulka, and Zanghì [59] had a different motivation to study GAP measures. They argued and later proved together with Mastrodonato in [57] that if ρ is a canonical density matrix, the measure $\mathrm{GAP}(\rho)$ describes the thermal equilibrium distribution of the wave function. There, the setting

[10] The name refers to Ebenezer Scrooge, the protagonist of Charles Dickens' novella *A Christmas Carol* (1843), who is very stingy. Josza, Robb, and Wootters argue that also $\mathrm{GAP}(\rho)$ is in some sense very stingy: It is the most spread out distribution on $\mathbb{S}(\mathcal{H})$ with density matrix ρ and therefore "particularly stingy with its information".

is as it was for canonical typicality, i.e., we have a large system S which consists of a subsystem a and its environment b and the Hilbert space of the full system is written as $\mathcal{H}_S = \mathcal{H}_a \otimes \mathcal{H}_b$. The authors considered the system to be in a pure state $\psi \in \mathbb{S}(\mathcal{H}_S)$ and they needed a notion for the wave function of the subsystem a. This notion was provided by the *conditional wave function*[26, 44, 59]:[11]

Definition 1.17 (Conditional wave function)**.** Let $B = (|m\rangle_b)_{m=1,\dots,\dim \mathcal{H}_b}$ be an orthonormal basis of $\mathcal{H}_b$. Let $\psi \in \mathbb{S}(\mathcal{H})$ and let $|M\rangle_b$ be a random element of the basis $\{|m\rangle_b\}$ which is chosen with the Born distribution

$$\mathbb{P}^{\psi,B}(M = m) = \|{}_b\langle m|\psi\rangle\|_a^2\,, \tag{1.90}$$

where $\|\cdot\|_a$ denotes the norm in $\mathcal{H}_a$. The *conditional wave function* of the system a is defined as

$$\psi_a := \frac{{}_b\langle M|\psi\rangle}{\|{}_b\langle M|\psi\rangle\|_a}. \tag{1.91}$$

Note that ψ_a is well-defined as the event that its denominator is equal to zero has probability 0 due to (1.90). The motivation for calling the distribution of the conditional wave function ψ_a the *Born distribution*, denoted by $\text{Born}_a^{\psi,B}$, is that we can also think of ψ_a as resulting from a quantum measurement leading to the collapsed state $\psi_a \otimes |m\rangle_b$ with probability $\|{}_b\langle m|\psi\rangle\|_a^2$, see Footnote 2 in [57] for details. We also remark that the Born measure of a set $A \subset \mathbb{S}(\mathcal{H}_a)$ is given by

$$\text{Born}_a^{\psi,B}(A) = \mathbb{P}(\psi_a \in A) = \sum_m \|{}_b\langle m|\psi\rangle\|_a^2\, \mathbb{1}_A\left(\frac{{}_b\langle m|\psi\rangle}{\|{}_b\langle m|\psi\rangle\|_a}\right). \tag{1.92}$$

Before coming to the main theorem of this section, we have to introduce some notation. For $0 < \gamma < 1/D$ let $\mathcal{D}_{\geq\gamma}(\mathcal{H})$ be the set of density matrices on $\mathcal{H}$ whose eigenvalues are all greater or equal to γ. Moreover, for any measure μ on $\mathbb{S}(\mathcal{H})$ and any measurable function $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$ we define

$$\mu(f) := \int_{\mathbb{S}(\mathcal{H})} f(\psi)\, \mu(d\psi). \tag{1.93}$$

[11] The definition of the conditional wave function is inspired by Bohmian mechanics which is a formulation of quantum mechanics based on the particle picture, see, e.g. [28]. There, for $\psi \in \mathbb{S}(\mathcal{H})$, the (non-normalized) conditional wave function ψ_a of the subsystem a is defined by

$$\psi_a(x) = \psi(x, Y),$$

where x is the variable for the configuration of the subsystem a and Y is the *actual* configuration of the environment b. In case the particles have spin, the position basis is no longer a basis and one has to proceed differently; one natural possibility is to trace out the spin degrees of freedom and then to consider the resulting *conditional density matrix*, see [97] for details.

We are now ready to formulate and discuss one of the main results in [59, 57] from which it follows that for certain density matrices ρ on $\mathcal{H}_a$ the measure $\mathrm{GAP}(\rho)$ can be regarded as the thermal equilibrium distribution of wave functions.

Theorem 1.18 (Goldstein, Lebowitz, Mastrodonato, Tumulka, Zanghì (2016) [57])**.** *Let $0 < \varepsilon, \delta < 1$, $d_a \in \mathbb{N}$ and $0 < \gamma < 1/d_a$. Then there are numbers $D_R = D_R(\varepsilon, \delta, d_a, \gamma) > 0$ and $r = r(\varepsilon, d_1, \gamma) > 0$ such that for every $d_R \in \mathbb{N}$ with $d_R > D_R$, for every Hilbert space $\mathcal{H}_a$ with $\dim \mathcal{H}_a = d_a$, for every density matrix $\Omega \in \mathcal{D}_{\geq\gamma}(\mathcal{H}_a)$, for every Hilbert space $\mathcal{H}_b$ and $\mathcal{H}_R \subseteq \mathcal{H}_a \otimes \mathcal{H}_b$ with $\dim \mathcal{H}_R = d_R$ satisfying*

$$\|\mathrm{tr}_b(\rho_R) - \Omega\|_{\mathrm{tr}} < r \tag{1.94}$$

and for every bounded measurable function $f : \mathbb{S}(\mathcal{H}_a) \to \mathbb{R}$,

$$u_R \times u_{\mathrm{ONB}}\Big\{(\psi, B) \in \mathbb{S}(\mathcal{H}_R) \times \mathrm{ONB}(\mathcal{H}_b) : \\ \Big|\mathrm{Born}_a^{\psi,B}(f) - \mathrm{GAP}(\Omega)(f)\Big| < \varepsilon\|f\|_\infty\Big\} \geq 1 - \delta. \tag{1.95}$$

Theorem 1.18 shows that if a density matrix Ω on $\mathcal{H}_a$ with not too small eigenvalues is close to the reduced density matrix $\mathrm{tr}_b\, \rho_R$, then for most $\psi \in \mathbb{S}(\mathcal{H}_R)$ and most orthonormal bases B of $\mathcal{H}_b$, the Born distribution $\mathrm{Born}_a^{\psi,B}$ of the conditional wave function ψ_a is close to $\mathrm{GAP}(\Omega)$.

Now suppose as in the introduction of Section 1.4 that the interaction between a and b is weak, b is large and $\mathcal{H}_R = \mathcal{H}_{\mathrm{mc}}$, i.e., $\mathcal{H}_R$ is a micro-canonical subspace. Then $\mathrm{tr}_b\, \rho_{\mathrm{mc}}$ is close to a canonical density matrix $\rho_{a,\mathrm{can}}$ with a suitable inverse temperature $\beta > 0$ [58]. In this special case, Theorem 1.18 shows that for most wave functions $\psi \in \mathbb{S}(\mathcal{H}_R)$ and most orthonormal bases B of $\mathcal{H}_b$,

$$\mathrm{Born}_a^{\psi,B} \approx \mathrm{GAP}(\rho_{a,\mathrm{can}}), \tag{1.96}$$

i.e., the Born distribution of the conditional wave function ψ_a is close to the GAP measure with a canonical density matrix. Thus, starting from a micro-canonical ensemble for the full system, we obtain $\mathrm{GAP}(\rho_{a,\mathrm{can}})$ for the subsystem a. This justifies to regard $\mathrm{GAP}(\rho_{a,\mathrm{can}})$ as the thermal equilibrium distribution of the wave function of the subsystem and we can think of these GAP measures as quantum analogues of the canonical ensemble of classical statistical mechanics. Therefore generalizing typicality results for the uniform measure on the sphere to GAP measures can be regarded as expressing a version of equivalence of ensembles.

1.5.2. Definition in Finite Dimensions

In this section we give four different definitions of the measure $\mathrm{GAP}(\rho)$ in the case that the Hilbert space $\mathcal{H}$ is finite-dimensional. We again closely follow [59, 57]. While three of the definitions require that $\mathcal{H}$ is finite-dimensional, one of them can also be generalized to separable Hilbert spaces, see Section 1.5.4, and we start with this one. It is also the definition to which the acronym GAP refers. As the name *Gaussian adjusted projected measure* suggests, the starting point is a Gaussian measure that is then adjusted and finally projected to the sphere.

Let ρ be a density matrix on $\mathcal{H}$ with eigen-ONB $(|n\rangle)_{n=1,\dots,D}$ and corresponding eigenvalues p_n, i.e.,

$$\rho = \sum_n p_n |n\rangle\langle n|. \tag{1.97}$$

Let $(Z_n)_{n=1,\dots,D}$ be a sequence of independent complex-valued Gaussian random variables with mean zero and variances[12]

$$\mathbb{E}|Z_n|^2 = p_n. \tag{1.98}$$

We define the Gaussian measure $G(\rho)$ to be the distribution of the random vector

$$\Psi^G := \sum_n Z_n |n\rangle, \tag{1.99}$$

i.e., $G(\rho)$ has mean zero and covariance matrix ρ. Note that $G(\rho)$ is not a distribution on the sphere $\mathbb{S}(\mathcal{H})$ as $\|\Psi^G\|^2 \neq 1$ in general. However, the norm square of Ψ^G is close to 1. To see this note that

$$\mathbb{E}\left\|\Psi^G\right\|^2 = \sum_n \mathbb{E}|Z_n|^2 = \sum_n p_n = 1 \tag{1.100}$$

and

$$\begin{aligned}\mathbb{E}\left\|\Psi^G\right\|^4 &= \sum_{n\neq m} \mathbb{E}|Z_n|^2\,\mathbb{E}|Z_m|^2 + \sum_n \mathbb{E}|Z_n|^4 = \sum_{n\neq m} p_n p_m + 3\sum_n p_n^2 && \text{(1.101a)}\\ &= 1 + 2\sum_n p_n^2 \leq 1 + 2\|\rho\|, && \text{(1.101b)}\end{aligned}$$

where $\|\rho\|$ is the largest eigenvalue of ρ. From this it follows that

$$\operatorname{Var}\left\|\Psi^G\right\|^2 = \mathbb{E}\left\|\Psi^G\right\|^4 - \left|\mathbb{E}\left\|\Psi^G\right\|^2\right|^2 \leq 2\|\rho\| \tag{1.102}$$

and we conclude that as soon as all eigenvalues of ρ are sufficiently small, most Ψ^G

[12] Recall that Z is a complex Gaussian random variable with mean zero and variance $\sigma^2 > 0$ if and only if $\operatorname{Re} Z$ and $\operatorname{Im} Z$ are independent real Gaussian random variables with mean zero and variance $\sigma^2/2$.

have norm close to 1 and are therefore close to the sphere $\mathbb{S}(\mathcal{H})$.

We remark that with probability 1, $\|\Psi^G\|^2 < \infty$ (since otherwise $\mathbb{E}\|\Psi^G\|^2$ would not be finite) and Ψ^G lies in the subspace of $\mathcal{H}$ spanned by the eigenvectors $|n\rangle$ of ρ corresponding to $p_n > 0$. Moreover, note that the distribution $G(\rho)$ of Ψ^G does not depend on the eigenbasis $(|n\rangle)_n$ of ρ but only on ρ.

Since we are interested in a measure on $\mathbb{S}(\mathcal{H})$, it would be natural to project $G(\rho)$ to $\mathbb{S}(\mathcal{H})$. However, projecting $G(\rho)$ directly to the sphere would lead to a distribution that does not have density matrix ρ. Therefore, some adjustment is necessary, and we define the Gaussian adjusted measure $\mathrm{GA}(\rho)$ on $\mathcal{H}$ as

$$\mathrm{GA}(\rho)(d\psi) = \|\psi\|^2 G(\rho)(d\psi). \tag{1.103}$$

Note that $\mathrm{GA}(\rho)$ is a probability measure on $\mathcal{H}$ as $\mathbb{E}\|\Psi^G\|^2 = 1$. We shall see below that $\|\psi\|^2$ is the right factor to ensure that $\rho_{\mathrm{GAP}(\rho)} = \rho$ as desired.

Let Ψ^{GA} be a $\mathrm{GA}(\rho)$-distributed random vector. We define $\mathrm{GAP}(\rho)$ as the distribution of the random vector

$$\Psi^{\mathrm{GAP}} := \frac{\Psi^{\mathrm{GA}}}{\|\Psi^{\mathrm{GA}}\|}. \tag{1.104}$$

The vector Ψ^{GAP} is well-defined as the denominator is non-zero with probability one. This immediately follows from the fact that Ψ^{GA} is continuously distributed and therefore the probability that Ψ^{GA} is equal to a certain value, in this case 0, is zero.

Now we can easily verify the claim that $\rho_{\mathrm{GAP}(\rho)} = \rho$:

$$\begin{aligned} \rho_{\mathrm{GAP}(\rho)} &= \int_{\mathbb{S}(\mathcal{H})} |\psi\rangle\langle\psi| \, \mathrm{GAP}(\rho)(d\psi) = \int_{\mathcal{H}} \frac{1}{\|\psi\|^2} |\psi\rangle\langle\psi| \, \mathrm{GA}(\rho)(d\psi) \\ &= \int_{\mathcal{H}} |\psi\rangle\langle\psi| \, G(\rho)(d\psi) = \rho. \end{aligned} \tag{1.105}$$

Note that for every density matrix ρ on $\mathcal{H}$ the measure $\mathrm{GAP}(\rho)$ exists and is unique, see [141] and also Section 1.5.4 where we discuss this result which is true for separable Hilbert spaces.

Another possibility to characterize $\mathrm{GAP}(\rho)$ on finite-dimensional Hilbert spaces is by giving its density with respect to the uniform distribution u on $\mathbb{S}(\mathcal{H})$. To this end, we have to assume that all eigenvalues of ρ are positive (otherwise we restrict to the subspace of $\mathcal{H}$ spanned by the eigenvectors of ρ corresponding to its positive eigenvalues). Let λ be the Lebesgue measure on $\mathbb{C}^D$. Then we have that [59]

$$\frac{dG(\rho)}{d\lambda}(\psi) = \frac{1}{\pi^D \det\rho} \exp\left(-\langle\psi|\rho^{-1}|\psi\rangle\right), \tag{1.106}$$

$$\frac{d\mathrm{GA}(\rho)}{d\lambda}(\psi) = \frac{\|\psi\|^2}{\pi^D \det\rho} \exp\left(-\langle\psi|\rho^{-1}|\psi\rangle\right), \tag{1.107}$$

$$\frac{d\text{GAP}(\rho)}{du}(\psi) = \frac{1}{\pi^D \det\rho}\int_0^\infty r^{2D-1} r^2 \exp\left(-r^2\langle\psi|\rho^{-1}|\psi\rangle\right)\, dr$$
$$= \frac{D!}{2\pi^D \det\rho}\langle\psi|\rho^{-1}|\psi\rangle^{-D-1}. \tag{1.108}$$

The third way to define GAP measures on finite-dimensional Hilbert spaces goes back to Josza, Robb, and Wootters [67] and it does not make use of Gaussian measures. Let $\Psi^u \sim u$, i.e., Ψ^u is uniformly distributed on the sphere $\mathbb{S}(\mathcal{H})$. Let $uD(\rho)$ be the distribution of the random vector

$$\Psi^{uD(\rho)} := D^{1/2}\rho^{1/2}\Psi^u. \tag{1.109}$$

We also call $uD(\rho)$ the *ρ-distortion* of the measure u. Note that this measure is concentrated on the ellipsoid $D^{1/2}\rho^{1/2}\mathbb{S}(\mathcal{H})$. The measure $\text{GAP}(\rho)$ can now be obtained by adjusting the measure $uD(\rho)$ and then projecting it to the sphere.

For any measure μ on $\mathcal{H}$ we define the "adjustment" procedure which results in the adjusted measure $A\mu$ via

$$A\mu(d\psi) := \|\psi\|^2\mu(d\psi). \tag{1.110}$$

Moreover, the "projection" procedure is defined with the help of

$$P : \mathcal{H}\backslash\{0\} \to \mathbb{S}(\mathcal{H}), \quad P(\psi) = \frac{\psi}{\|\psi\|}. \tag{1.111}$$

Applying first the adjustment procedure to μ and then projecting it to the sphere results in the measure $\mu AP := P_*(A\mu) = A\mu \circ P^{-1}$ where $P_*(A\mu)$ is the pushforward measure of $A\mu$ under P. Note that if $\int_{\mathcal{H}} \|\psi\|^2\,\mu(d\psi) = 1$, then μAP is a probability measure on $\mathbb{S}(\mathcal{H})$. Adopting this language, we find that $u\text{DAP}(\rho) = \text{GAP}(\rho)$, see [57] for a proof.

A fourth possible way to define GAP measures was suggested to the authors of [57] by an anonymous referee. As this definition makes use of the uniform distribution on $\mathbb{S}(\mathcal{H})$, it also only works for $D = \dim\mathcal{H} < \infty$. Let $\mathcal{H}_2$ be another Hilbert space of dimension D and let $\Phi \in \mathbb{S}(\mathcal{H}\otimes\mathcal{H}_2)$ with $\operatorname{tr}_2|\Phi\rangle\langle\Phi| = \rho$. Moreover, let $\Psi_2 \sim \mu_2$, where

$$\mu_2(d\psi_2) = D\left\|{}_2\langle\psi_2|\Phi\rangle\right\|^2\, u_2(d\psi_2). \tag{1.112}$$

Here, u_2 denotes the uniform distribution over $\mathbb{S}(\mathcal{H}_2)$, $\|\cdot\|$ is the norm in $\mathcal{H}$ and ${}_2\langle\cdot|\cdot\rangle$ denotes the partial inner product in $\mathcal{H}_2$. Note that μ_2 is a probability measure on $\mathbb{S}(\mathcal{H}_2)$; this follows from

$$\mu_2(\mathbb{S}(\mathcal{H}_2)) = \int_{\mathbb{S}(\mathcal{H}_2)} D\langle\Phi|\psi_2\rangle\langle\psi_2|\Phi\rangle\, u_2(d\psi_2) = D\langle\Phi|I_2/D|\Phi\rangle = 1, \tag{1.113}$$

where I_2 denotes the identity on $\mathcal{H}_2$. The measure $\mathrm{GAP}(\rho)$ can now be defined to be the distribution of the random vector

$$\Psi = \frac{{}_2\langle\Psi_2|\Phi\rangle}{\|{}_2\langle\Psi_2|\Phi\rangle\|}. \tag{1.114}$$

For the proof that Ψ is indeed $\mathrm{GAP}(\rho)$-distributed and more details, see again [57].

1.5.3. Properties

In this section we collect some important properties of GAP measures on finite-dimensional Hilbert spaces.

Proposition 1.19 (Properties of $\mathrm{GAP}(\rho)$)**.** *Let ρ be a density matrix on a finite dimensional Hilbert space $\mathcal{H}$ of dimension $D = \dim \mathcal{H}$. Then,*

(a) $\mathrm{GAP}(\rho)$ has density matrix ρ, i.e., $\rho_{\mathrm{GAP}(\rho)} = \rho$.

(b) The map $\rho \mapsto \mathrm{GAP}(\rho)$ is covariant: For any unitary operator U on $\mathcal{H}$,

$$U_*\mathrm{GAP}(\rho) = \mathrm{GAP}(U\rho U^*). \tag{1.115}$$

(c) Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be two finite-dimensional Hilbert spaces and let ρ_a and ρ_b be density matrices on $\mathcal{H}_a$ and $\mathcal{H}_b$ respectively. Then, if $\psi \in \mathcal{H}_a \otimes \mathcal{H}_b$ has distribution $\mathrm{GAP}(\rho_a \otimes \rho_b)$, for any basis $\{|m\rangle_b\}$ of $\mathcal{H}_b$, the conditional wave function ψ_a has distribution $\mathrm{GAP}(\rho_a)$.

(d) For $B \sim u_{\mathrm{ONB}}$,

$$\mathbb{E}\,\mathrm{Born}_a^{\psi,B} = \mathrm{GAP}(\rho_a^\psi), \tag{1.116}$$

where $\mathbb{E}$ is the expectation with respect to u_{ONB}.

(e) If $\rho = P_\nu/d_\nu$, where P_ν is the orthogonal projection to a subspace $\mathcal{H}_\nu \subset \mathcal{H}$, then

$$\mathrm{GAP}(\rho) = u_\nu, \tag{1.117}$$

where u_ν denotes the uniform distribution on $\mathbb{S}(\mathcal{H}_\nu)$.

(f) Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of density matrices on $\mathcal{H}$ such that $\|\rho_n - \rho\|_{\mathrm{tr}} \to 0$. Then, $\mathrm{GAP}(\rho_n) \Rightarrow \mathrm{GAP}(\rho)$, i.e., the sequence $(\mathrm{GAP}(\rho_n))_{n\in\mathbb{N}}$ converges weakly to $\mathrm{GAP}(\rho)$.[13]

[13] Recall that a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathcal{H}$ converges weakly to a measure μ on $\mathcal{H}$ if $\mu_n(f) \to \mu(f)$ for every bounded continuous function f.

(*g*) *For every* $0 < \varepsilon < 1$ *and every continuous function* $f : \mathbb{S}(\mathcal{H}) \to \mathbb{R}$ *there is* $r = r(\varepsilon, D, F) > 0$ *such that for all density matrices* ρ_1 *and* ρ_2,

$$\text{if } \|\rho_1 - \rho_2\|_{\mathrm{tr}} < r \text{ then } |\mathrm{GAP}(\rho_1)(f) - \mathrm{GAP}(\rho_2)(f)| < \varepsilon. \tag{1.118}$$

(*h*) *For every* $0 < \varepsilon < 1$ *and every* $0 < \gamma < 1/D$ *there is* $r = r(\varepsilon, D, \gamma) > 0$ *such that for all* $\rho_1, \rho_2 \in \mathcal{D}_{\geq\gamma}(\mathcal{H})$,

$$\text{if } \|\rho_1 - \rho_2\|_{\mathrm{tr}} < r \text{ then } \left\| \frac{d\mathrm{GAP}(\rho_1)}{du} - \frac{d\mathrm{GAP}(\rho_2)}{du} \right\|_{\infty} < \varepsilon. \tag{1.119}$$

As a consequence, for such ρ_1 *and* ρ_2 *every* $f \in L^1(\mathbb{S}(\mathcal{H}), u)$ *is such that*

$$\big|\mathrm{GAP}(\rho_1)(f) - \mathrm{GAP}(\rho_2)(f)\big| < \varepsilon \|f\|_1. \tag{1.120}$$

Proof. Property (a) was already proved in the previous section. For the proof of (b) and (c) see [59], whereas (d) is Lemma 1 in [57]. Concerning (e), note that $\mathrm{GAP}(\rho)$ is a probability measure on $\mathbb{S}(\mathcal{H}_\nu)$ that is invariant under all unitaries U_ν on $\mathcal{H}_\nu$ as $U_\nu P_\nu U_\nu^* = P_\nu$ and as there is only one such measure, namely the uniform distribution on $\mathbb{S}(\mathcal{H}_\nu)$, it follows that $\mathrm{GAP}(\rho) = u_\nu$. Finally, the proof of (f) is given in [141, Theorem 3], whereas the proofs of (g) and (h) can be found in [57, Lemma 5 and Lemma 6]. □

Note that because of (*e*) in Proposition 1.19 results proved for GAP measures also cover the uniform distribution on the sphere as a special case.

Next we state two lemmas concerning moments of components of $\mathrm{GAP}(\rho)$-distributed vectors and the expectation and variance of $\langle\psi|A|\psi\rangle$ for operators A on $\mathcal{H}$ and $\mathrm{GAP}(\rho)$-distributed $\psi \in \mathbb{S}(\mathcal{H})$. These computations were first done by Reimann [106]. Note that an analogous statement for the uniform distribution was already given in Lemma 1.2.

Lemma 1.20 (Reimann [106])**.** *Let* ρ *be a density matrix with positive eigenvalues* p_n *on a Hilbert space* $\mathcal{H}$ *of dimension* $D = \dim\mathcal{H} < \infty$. *Moreover, let* $\dim\mathcal{H} \geq 4$ *and* $p_{\max} = \|\rho\| < 1/4$. *Then for any self-adjoint operator* A *on* $\mathcal{H}$,

$$\mathbb{E}_\rho\left[\langle\psi|A|\psi\rangle\right] = \mathrm{tr}(\rho A), \tag{1.121}$$

$$\mathrm{Var}_\rho\left[\langle\psi|A|\psi\rangle\right] \leq \frac{\Delta_A^2 \,\mathrm{tr}\,\rho^2}{1 - p_{\max}} \left(1 + \frac{4\sqrt{\mathrm{tr}\,\rho^2} + 2\,\mathrm{tr}\,\rho^2}{(1 - 2p_{\max})(1 - 3p_{\max})}\right), \tag{1.122}$$

where $\mathbb{E}_\rho$ *and* Var_ρ *denote the expectation and variance with respect to* $\mathrm{GAP}(\rho)$ *and* $\Delta_A := a_{\max} - a_{\min}$, *where* $a_{\max}$ *and* $a_{\min}$ *are the largest and smallest eigenvalue of* A.

An important ingredient for the proof of Lemma 1.20 are formulas for certain moments of coefficients of GAP(ρ)-distributed vectors.

Lemma 1.21 (Reimann [106]). *Let $\mathcal{H}$ and ρ be as in Lemma 1.20 and let $\psi \in \mathbb{S}(\mathcal{H})$ be* GAP(ρ)*-distributed. Let $(|n\rangle)_{n=1,\dots,D}$ be an orthonormal basis of eigenvectors of ρ such that $\rho|n\rangle = p_n|n\rangle$ and define $a_n := \langle n|\psi\rangle$. Then,*

$$\mathbb{E}_\rho\left(a_m^* a_n\right) = p_n \delta_{nm}, \tag{1.123}$$

$$\mathbb{E}_\rho\left(|a_m|^2 |a_n|^2\right) = p_m p_n (1+\delta_{mn}) K_{mn}, \tag{1.124}$$

where

$$K_{mn} = \int_0^\infty (1+xp_m)^{-1}(1+xp_n)^{-1}\prod_{k=1}^{D}(1+xp_k)^{-1}\,dx. \tag{1.125}$$

Moreover, all other fourth moments vanish.

As we prove a generalization and slight improvement of the upper bound on the GAP(ρ)-variance from Lemma 1.20 in [137, 143], we now give the proof of Lemma 1.20 and closely follow the proof in [106].

Proof of Lemma 1.20. The expression for the expectation is an immediate consequence of $\mathbb{E}_\rho|\psi\rangle\langle\psi| = \rho$ as

$$\mathbb{E}_\rho\langle\psi|A|\psi\rangle = \mathbb{E}_\rho \operatorname{tr}(|\psi\rangle\langle\psi|A) = \operatorname{tr}(\mathbb{E}_\rho|\psi\rangle\langle\psi|A) = \operatorname{tr}(\rho A). \tag{1.126}$$

Next we turn to the computation of $\mathrm{Var}_\rho\langle\psi|A|\psi\rangle$. Since the variance is invariant under the addition of constants, we can assume without loss of generality that $\mathbb{E}_\rho\langle\psi|A|\psi\rangle = \operatorname{tr}(A\rho) = 0$.

Let $(|n\rangle)_{n=1,\dots,D}$ be an orthonormal basis of eigenvectors of ρ corresponding to the eigenvalues p_n. Let $\psi \in \mathbb{S}(\mathcal{H})$, $c_n = \langle n|\psi\rangle$ and $A_{mn} = \langle m|A|n\rangle$. Then we obtain

$$\langle\psi|A|\psi\rangle = \sum_{m,n} c_m^* A_{mn} c_n \tag{1.127}$$

and therefore

$$\begin{aligned}\mathrm{Var}_\rho\langle\psi|A|\psi\rangle &= \sum_{m,n,m',n'} A_{mn}A_{m'n'}\mathbb{E}_\rho(c_n^* c_m c_{m'}^* c_{n'}) &\text{(1.128a)}\\ &= \sum_{m,n} A_{mn}^2 p_m p_n (1+\delta_{mn})K_{mn} + \sum_{m,m'} A_{mm}A_{m'm'}p_m p_{m'}(1+\delta_{mm'})K_{mm'}\\ &\quad - 2\sum_m A_{mm}^2 p_m^2 K_{mm} &\text{(1.128b)}\end{aligned}$$

$$= \sum_{m,n} \left[A_{mm}A_{nn} + A_{mn}^2 \right] p_m p_n K_{mn}, \tag{1.128c}$$

where we used Lemma 1.21 in the second line. Next we have to compute the integrals K_{mn}. For $m, n \in \{1, \ldots, D\}$ we define the functions $g_{mn} : [0, \infty) \to \mathbb{R}$ by

$$g_{mn}(x) := (1 + xp_m)^{-1}(1 + xp_n)^{-1}. \tag{1.129}$$

By Taylor's theorem there exist functions $\theta_{mn}, \chi_{mn} : [0, \infty) \to [0, 1]$ such that

$$g_{mn}(x) = g_{mn}(0) + x g'_{mn}(0) + \frac{x^2}{2} g''_{mn}(x\theta_{mn}(x)) \tag{1.130a}$$

$$= 1 - x(p_m + p_n) + x^2(p_m^2 + p_m p_n + p_n^2)\chi_{mn}(x). \tag{1.130b}$$

This implies

$$K_{mn} = K^{(0)} - (p_m + p_n)K^{(1)} + 2(p_m^2 + p_m p_n + p_n^2)\kappa_{mn} K^{(2)}, \tag{1.131}$$

where $\kappa_{mn} \in [0, 1]$ and

$$K^{(k)} := \frac{1}{k!} \int_0^\infty x^k \prod_{l=1}^{D} (1 + xp_l)^{-1} \, dx, \; k = 0, 1, 2. \tag{1.132}$$

Reimann [106] showed that

$$K^{(k)} \leq \prod_{j=1}^{k+1} \frac{1}{1 - jp_{\max}}, \; k = 0, 1, 2. \tag{1.133}$$

Because of this and our assumption that $\operatorname{tr}(A\rho) = \sum_n A_{nn} p_n = 0$ we obtain for the first sum in (1.128c) that

$$\left| \sum_{m,n} A_{mm} A_{nn} p_m p_n K_{mn} \right| = 2K^{(2)} \left| \sum_{m,n} A_{mm} A_{nn} (p_m^3 p_n + p_m^2 p_n^2 + p_n^3 p_m) \kappa_{mn} \right| \tag{1.134a}$$

$$\leq \frac{2\Delta_A^2 \left(2 \operatorname{tr} \rho^3 + (\operatorname{tr} \rho^2)^2\right)}{(1 - p_{\max})(1 - 2p_{\max})(1 - 3p_{\max})} \tag{1.134b}$$

$$\leq \frac{\Delta_A^2 \operatorname{tr} \rho^2 \left(4\sqrt{\operatorname{tr} \rho^2} + 2 \operatorname{tr} \rho^2\right)}{(1 - p_{\max})(1 - 2p_{\max})(1 - 3p_{\max})}, \tag{1.134c}$$

where the last inequality follows from

$$\operatorname{tr} \rho^3 = \sum_n p_n^3 \le p_{\max} \operatorname{tr} \rho^2 \le \sqrt{\sum_n p_n^2} \operatorname{tr} \rho^2 = (\operatorname{tr} \rho^2)^{3/2}. \tag{1.135}$$

From the definition of the K_{mn} we immediately see that $K_{mn} \le K^{(0)}$. Therefore we find for the second sum in (1.128c) that

$$\sum_{m,n} A_{mn}^2 p_m p_n K_{mn} \le K^{(0)} \sum_{m,n} A_{mn}^2 p_m p_n = K^{(0)} \operatorname{tr}(\rho A \rho A). \tag{1.136}$$

Let $(|\nu\rangle)$ be an orthonormal eigenbasis of A corresponding to the eigenvalues a_ν. Note that $a_\nu \le \Delta_A$ for all ν. By evaluating $\operatorname{tr}(A\rho A\rho)$ in this basis we obtain

$$\sum_{m,n} A_{mn}^2 p_n p_m K_{mn} \le \frac{1}{1-p_{\max}} \sum_{\mu,\nu} a_\nu a_\mu \langle \mu|\rho|\nu\rangle\langle\nu|\rho|\mu\rangle \tag{1.137a}$$

$$\le \frac{\Delta_A^2}{1-p_{\max}} \sum_{\mu,\nu} \langle \mu|\rho|\nu\rangle\langle\nu|\rho|\mu\rangle = \frac{\Delta_A^2 \operatorname{tr}\rho^2}{1-p_{\max}}. \tag{1.137b}$$

The bound for the variance now follows from (1.134c) and (1.137b). □

Reimann's bound on the variance $\operatorname{Var}_\rho\langle\psi|A|\psi\rangle$ together with Chebyshev's inequality shows that as soon as the *purity* $\operatorname{tr}\rho^2$ of ρ is small and Δ_A is not too large, $\langle\psi|A|\psi\rangle \approx \mathbb{E}_\rho\langle\psi|A|\psi\rangle = \operatorname{tr}(\rho A)$ for GAP(ρ)-most $\psi \in \mathbb{S}(\mathcal{H})$.

One can think of the purity $\operatorname{tr}\rho^2 = \sum_n p_n^2$ as the average size of the eigenvalues p_n. Note that $\operatorname{tr}\rho^2 \le 1$ and $\operatorname{tr}\rho^2 = 1$ if and only if ρ is pure, i.e., if it is of the form $\rho = |\psi\rangle\langle\psi|$ for some $\psi \in \mathbb{S}(\mathcal{H})$. The purity can be related to the maximum eigenvalue $p_{\max} = \|\rho\|$ as follows:

$$\operatorname{tr}\rho^2 \le \|\rho\| \le \sqrt{\operatorname{tr}\rho^2}. \tag{1.138}$$

The first inequality follows from $p_n \le \|\rho\|$ for all n and $\sum_n p_n = 1$ while the second inequality is an immediate consequence of $p_{\max}^2 \le \sum_n p_n^2$.

As a last property we mention that if ρ is a canonical density matrix and therefore GAP(ρ) is the thermal equilibrium distribution of wave functions, for relevant Hamiltonians, $\psi \sim \mathrm{GAP}(\rho)$ is with probability one infinitely often differentiable and even analytic, see [142].

1.5.4. Construction in Infinite Dimensions

This section is devoted to the construction of GAP measures on the sphere of separable Hilbert spaces $\mathcal{H}$, i.e., $\mathcal{H}$ has a finite or countably infinite orthonormal basis.

While three of the four definitions of GAP(ρ) for a density matrix ρ presented in Section 1.5.2 make use of the uniform distribution and therefore require that $\mathcal{H}$ is finite-dimensional, the construction of GAP(ρ) to which the acronym GAP refers can be generalized to separable Hilbert spaces. This was done by Tumulka [141] in 2020 and we closely follow this reference for the rest of this section.

Before we come to the actual definition of GAP measures on separable Hilbert spaces, we clarify what the mean and covariance operator of a probability measure μ on $\mathcal{H}$ are. As usual, $\mathcal{H}$ is equipped with its Borel σ-algebra $\mathcal{B}(\mathcal{H})$.

Definition 1.22 (Mean and covariance operator)**.** Let μ be a probability measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$, where $\mathcal{H}$ is a separable Hilbert space. The vector $\psi_0 \in \mathcal{H}$ is called the *mean* of μ if, for every $\phi \in \mathcal{H}$,

$$\langle\phi|\psi_0\rangle = \int_{\mathcal{H}} \langle\phi|\psi\rangle \, \mu(d\psi). \tag{1.139}$$

The operator $C_\mu : \mathcal{H} \to \mathcal{H}$ is called the *covariance operator* of μ if for every $\phi, \chi \in \mathcal{H}$,

$$\langle\phi|C_\mu|\chi\rangle = \int_{\mathcal{H}} \langle\phi|\psi - \psi_0\rangle\langle\psi - \psi_0|\chi\rangle \, \mu(d\psi). \tag{1.140}$$

Note that while the mean and covariance of a probability measure μ on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ need not exist, if they exist, they are unique. Moreover, if C_μ exists, it is a positive operator. However, if μ is a probability measure on $(\mathbb{S}(\mathcal{H}), \mathcal{B}(\mathbb{S}(\mathcal{H}))$, then it can be shown that the mean and covariance operator exist, see [141, Lemma 1]. Additionally, in this case there exists a unique operator $\rho_\mu : \mathcal{H} \to \mathcal{H}$ such that for all $\phi, \chi \in \mathcal{H}$,

$$\langle\phi|\rho_\mu|\chi\rangle = \int_{\mathcal{H}} \langle\phi|\psi\rangle\langle\psi|\chi\rangle \, \mu(d\psi), \tag{1.141}$$

see also Lemma 1 in [141]. The operator ρ_μ is called the *density operator* of μ, i.e., it is a positive trace class operator with trace 1. If the mean of μ is zero, ρ_μ is equal to the covariance operator C_μ of μ.

In the finite-dimensional setting, the starting point for the construction of GAP(ρ) for some density matrix ρ was the Gaussian distribution $G(\rho)$ with mean 0 and covariance ρ. Gaussian distributions can also be defined on infinite-dimensional spaces:

Definition 1.23 (Gaussian measure)**.** A probability measure μ on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ is a *Gaussian measure* if for every $\phi \in \mathcal{H}$ and $w \sim \mu$, the random variable $\langle\phi|w\rangle$ is a complex-valued Gaussian random variable.

Prohorov [102] showed that for every given $\psi_0 \in \mathcal{H}$ and every positive trace-class operator $C : \mathcal{H} \to \mathcal{H}$ there is a unique Gaussian measure μ on $\mathcal{H}$ with mean ψ_0 and covariance operator C (and conversely, every Gaussian measure has a mean and covariance operator). Choosing $\psi_0 = 0$ and $C = \rho$, we therefore obtain the existence and uniqueness of the Gaussian measure $G(\rho)$ with mean 0 and covariance ρ.

It is also possible to explicitly construct $G(\rho)$ as in the finite-dimensional setting from a sequence $Z = (Z_n)_{n\in\mathbb{N}}$ of independent complex Gaussian random variables with mean zero and variances p_n, where the p_n are the eigenvalues of ρ. The existence of such a random sequence in $\mathbb{C}^{\mathbb{N}}$ follows from Kolmogorov's extension theorem [12] and the corresponding probability (product) measure $\tilde{\mu}$ on the σ-algebra generated by the cylinder sets defines, by Lemma 3 in [141], a measure μ' on $\mathcal{B}(\ell^2)$ via restriction, $\mu' = \tilde{\mu}|_{\mathcal{B}(\ell^2)}$, where $\ell^2 \subset \mathbb{C}^{\mathbb{N}}$ is the space of square-summable sequences. The measure μ' is a probability measure because, as in the finite-dimensional case, we find that

$$\mathbb{E}\|Z\|_{\ell^2}^2 = \mathbb{E}\sum_n |Z_n|^2 = \sum_n \mathbb{E}|Z_n|^2 = \sum_n p_n = 1 \tag{1.142}$$

and thus $\|Z\|_{\ell^2} < \infty$, i.e., $Z \in \ell^2$, almost surely. With this we can define the random vector

$$\Psi^G := \sum_n Z_n |n\rangle, \tag{1.143}$$

where $(|n\rangle)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of ρ such that $\rho|n\rangle = p_n|n\rangle$. Because of (1.142), $\mathbb{E}\|\Psi^G\|^2 = 1$. Note that the distribution μ of the random vector Ψ^G is obtained from μ' via the unitary isomorphism $\ell^2 \to \mathcal{H}$.

For every $\phi \in \mathcal{H}$, the random variable

$$\langle\phi|\Psi^G\rangle = \sum_n \langle\phi|n\rangle Z_n \tag{1.144}$$

is Gaussian as it is the limit of linear combinations of complex Gaussian random variables. Therefore, by definition, the distribution μ of the random vector Ψ^G is Gaussian. Moreover, one easily sees that this distribution has mean zero and covariance ρ and we therefore have that $\mu = G(\rho)$.

Next we note that the adjustment and projection procedure from Section 1.5.2 also work for separable Hilbert spaces $\mathcal{H}$. With this we are now able to give a definition of GAP measures on separable Hilbert spaces:

Definition 1.24 (GAP measure)**.** Let $\mathcal{H}$ be a separable Hilbert space. A probability measure ν on $(\mathbb{S}(\mathcal{H}), \mathcal{B}(\mathbb{S}(\mathcal{H})))$ is a *GAP measure* if $\nu = P_* A\mu$ for a suitable Gaussian measure μ with mean zero.

It follows from Lemma 2 in [141] that if μ satisfies $\int \|\psi\|^2 \mu(d\psi) = 1$, then μ possesses a mean and a covariance operator and if the mean is zero, then $\rho_{P_* A\mu} = C_\mu$, i.e., the density operator of $P_* A\mu$ is equal to the covariance operator of μ. With the help of this lemma and Prohorov's existence and uniqueness theorem for Gaussian measures [102], Tumulka showed the existence and uniqueness of GAP measures:

Theorem 1.25 (Tumulka (2020) [141])**.** *For every positive trace-class operator ρ with $\operatorname{tr}\rho = 1$ on a separable Hilbert space $\mathcal{H}$ there exists a unique GAP measure with density operator ρ.*

Finally, we remark that sometimes an important tool for generalizing results from GAP measures on finite-dimensional Hilbert spaces to separable Hilbert spaces is their continuous dependence on the density matrix ρ, see Proposition 1.19 (f), which is also true for separable Hilbert spaces, see Theorem 3 in [141]. More precisely, one can approximate ρ by a sequence of finite-rank density operators (ρ_n), consider $\mathrm{GAP}(\rho_n)$ as measures on a finite-dimensional Hilbert space and then use the results proved for GAP measures in finite dimensions together with the weak convergence of the measures $\mathrm{GAP}(\rho_n)$ to the measure $\mathrm{GAP}(\rho)$.

1.6. Equilibration and Thermalization in Closed Quantum Systems

In this section we discuss several aspects of the equilibration and thermalization in closed quantum systems. For more details, see, e.g., [48, 91] and the references therein.

We start with presenting and discussing several results concerning equilibration in infinite as well as in finite time in Section 1.6.1. A system or quantity *equilibrates* if it approaches a certain state or value and stays close to it for an extended period of time. In contrast to thermalization, this state does not need to be thermal.

Section 1.6.2 is devoted to the concept of *macroscopic thermal equilibrium* (MATE), a definition of thermal equilibrium based on von Neumann's decomposition (1.12) of the system's Hilbert space into macro spaces. In Section 1.6.3 we discuss the notion of *microscopic thermal equilibrium* (MITE), a definition of thermal equilibrium inspired by the phenomenon of canonical typicality. Moreover, in Section 1.6.4, we discuss the approach to MATE and MITE.

As Goldstein, Huse, Tumulka, and Zanghì [53, 54] point out, the distinction between the two notions of thermal equilibrium is particularly interesting in the context of many-body localized (MBL) systems [4, 94, 8]. For such systems, there are at least some eigenstates that are in some way localized such that they are not thermal and thus do not approach thermal equilibrium, i.e., MBL systems fail to thermalize. It can even be argued that for MBL systems most, if not all, eigenstates are not thermal, see, e.g., [7, 8, 96, 118, 66]. We will see in Theorem 1.31 that generally most eigenstates are in MATE and thus this also holds true for MBL systems. The apparent contradiction is resolved by the concept of MITE: Even though for MBL systems most eigenstates are in MATE, the results in [7, 8, 96, 118, 66] in fact show that at most few (or even none) of the eigenstates are in MITE. For more details and a discussion of several examples, see [54].

After having discussed different notions of thermal equilibrium in isolated quantum systems and the approach to it, we come to the study of time scales of equilibration and thermalization in Section 1.6.5. Finally, in Section 1.6.6, we comment on the definition of thermal equilibrium in classical systems.

1.6.1. Equilibration

Roughly speaking, a system or a quantity *equilibrates* if after some amount of time it reaches a certain state or value and stays close to it for an extended period of time. Note that we cannot expect convergence to a particular state or value due to the phenomenon of recurrence: Every system returns arbitrarily close to its initial state after a sufficiently long time. Moreover, note that equilibration does not imply thermalization as the state towards which a system equilibrates need not be thermal.

Reimann [105] and later Short [124] considered the following setting: Let $\mathcal{H}$ be a Hilbert space of dimension $D = \dim \mathcal{H} < \infty$ and H a Hamiltonian on $\mathcal{H}$ with spectral decomposition

$$H = \sum_n E_n \, \Pi_n \tag{1.145}$$

where the E_n are the distinct eigenvalues of H and Π_n is the orthogonal projection onto the eigenspace of H with eigenvalue E_n. We assume that the eigenvalue gaps of H are non-degenerate, see (1.22). Let ρ be a density matrix on $\mathcal{H}$ and $\rho(t) = e^{-iHt}\rho(0)e^{iHt}$ its time evolution with $\rho(0) = \rho$. We compute

$$\overline{\rho(t)} = \sum_{n,m} \overline{e^{-i(E_n - E_m)t}} \, \Pi_m \rho(0) \Pi_n = \sum_n \Pi_n \rho(0) \Pi_n =: w. \tag{1.146}$$

Short [124] showed the following theorem:

Theorem 1.26 (Reimann (2008) [105], Short (2011) [124])**.** *Let H be a Hamiltonian with non-degenerate eigenvalue gaps on a finite-dimensional Hilbert space $\mathcal{H}$ and let ρ be a density matrix on $\mathcal{H}$. For any operator A on $\mathcal{H}$,*

$$\overline{\left|\operatorname{tr}(A\rho(t)) - \operatorname{tr}(Aw)\right|^2} \leq \frac{\Delta(A)^2}{4 d_{\text{eff}}} \leq \frac{\|A\|^2}{d_{\text{eff}}}, \tag{1.147}$$

where $\Delta(A) := 2 \min_{c \in \mathbb{C}} \|A - cI\|$ and

$$d_{\text{eff}} := \left(\sum_n (\operatorname{tr}(\Pi_n \rho))^2 \right)^{-1} \tag{1.148}$$

is the effective dimension of ρ.

The bound in (1.147) shows that as soon as the effective dimension d_{eff} is large and $\|A\|$ is not too large, for most times $t \geq 0$ we have that $\mathrm{tr}(A\rho(t)) \approx \mathrm{tr}(Aw)$, i.e., the quantity $\mathrm{tr}(A\rho(t))$ equilibrates towards $\mathrm{tr}(Aw)$. Note that the effective dimension is a measure for the number of different energies that contribute significantly to the initial state ρ. Therefore the initial state ρ has to be sufficiently spread out over the energy eigenvalues of H to ensure equilibration.

The result above was first proved by Reimann [105] for self-adjoint A and Short [124] generalized the result to arbitrary operators, improved the bound by a factor of 4 and corrected a subtle mistake made in [105].

If ρ is *pure*, i.e., if it is of the form $\rho = |\psi\rangle\langle\psi|$ for some $\psi \in \mathbb{S}(\mathcal{H})$, then the proof of the larger bound in (1.147) is particularly easy [91]: If the eigenvalues of H are degenerate, we choose an eigenbasis of H such that for each distinct energy the initial state ψ has non-zero overlap with only one eigenstate $|n\rangle$. As a consequence, for the computation of the effective dimension the projectors Π_n can without loss of generality be assumed to be of the form $\Pi_n = |n\rangle\langle n|$. We have that

$$|\psi\rangle = \sum_n c_n |n\rangle, \qquad |\psi_t\rangle = \sum_n c_n e^{-iE_n t} |n\rangle, \tag{1.149}$$

where $c_n = \langle n|\psi\rangle$ and the effective dimension of ρ is easily computed to be

$$d_{\mathrm{eff}} = \frac{1}{\sum_n |c_n|^4} \tag{1.150}$$

Moreover, we find that

$$\langle\psi_t|A|\psi_t\rangle - \mathrm{tr}(Aw) = \sum_{n\neq m} c_n^* c_m e^{i(E_n - E_m)t} \langle n|A|m\rangle \tag{1.151}$$

and since the eigenvalue gaps of H are, by assumption, non-degenerate, we obtain

$$\begin{aligned}
\overline{\left|\langle\psi_t|A|\psi_t\rangle - \mathrm{tr}(Aw)\right|^2} &= \sum_{\substack{n\neq m\\ k\neq l}} c_n^* c_m c_k c_l^* \overline{e^{i(E_n - E_m - E_k + E_l)t}} \langle n|A|m\rangle\langle l|A^*|m\rangle && \text{(1.152a)}\\
&= \sum_{n\neq m} |c_n|^2 |c_m|^2 \, |\langle n|A|m\rangle|^2 && \text{(1.152b)}\\
&\leq \frac{\|A\|^2}{2} \sum_{n\neq m} \left(|c_n|^4 + |c_m|^4\right) \leq \frac{\|A\|^2}{d_{\mathrm{eff}}}, && \text{(1.152c)}
\end{aligned}$$

where we used that $|c_n|^2|c_m|^2 \leq (|c_n|^4 + |c_m|^4)/2$.

Short [124] also first proves his bounds for pure states and later extends them to mixed states via *purification*: This is the fact that for every initial state ρ on $\mathcal{H}$ there exists a pure state $|\phi\rangle$ on the tensor product $\mathcal{H} \otimes \mathcal{H}$ such that the reduced state of the first system is given by ρ. After the purification one can apply the results already proved for pure states and in this way often also obtain corresponding results for ρ.

With the help of Short's [124] generalization of the result of Reimann [105] he re-derived a result of Linden, Popescu, Short, and Winter [77] concerning the equilibration of small subsystems. To state their result, let $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ be a Hilbert space of dimension $D = \dim \mathcal{H} < \infty$ with Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ of dimensions d_a and d_b respectively. Let H be a Hamiltonian on $\mathcal{H}$ and for $\psi_t = e^{-itH}\psi_0$ we define $\rho(t) := |\psi_t\rangle\langle\psi_t|$, $\rho_a(t) := \operatorname{tr}_b \rho(t)$, $\rho_b(t) := \operatorname{tr}_a \rho(t)$, $w = \overline{\rho(t)}$, $w_a = \overline{\rho_a(t)}$ and $w_b = \overline{\rho_b(t)}$. Moreover, the effective dimension of a (mixed) state ρ is defined by

$$d_{\text{eff}}(\rho) := \frac{1}{\operatorname{tr} \rho^2}, \tag{1.153}$$

i.e., it is the inverse of the purity of ρ.

Note that if ρ is pure as above, i.e., $\rho = |\psi\rangle\langle\psi|$ for some $\psi \in \mathbb{S}(\mathcal{H})$, then $d_{\text{eff}}(w)$ agrees with the effective dimension defined above in (1.148); this follows from

$$\operatorname{tr} w^2 = \sum_n \operatorname{tr}(\Pi_n \rho(0) \Pi_n \rho(0) \Pi_n) = \sum_n \langle\psi|\Pi_n|\psi\rangle^2 = \sum_n \left(\operatorname{tr}(\Pi_n \rho)\right)^2. \tag{1.154}$$

Theorem 1.27 (Linden, Popescu, Short, Winter (2009) [77])**.** *Let H be a Hamiltonian with non-degenerate energy gaps on a finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ and let $d_a = \dim \mathcal{H}_a$. Then for every $\psi_0 \in \mathbb{S}(\mathcal{H})$,*

$$\overline{\|\rho_a(t) - w_a\|_{\text{tr}}} \leq \sqrt{\frac{d_a}{d_{\text{eff}}(w_a)}} \leq \sqrt{\frac{d_a^2}{d_{\text{eff}}(w)}} \tag{1.155}$$

Theorem 1.27 shows that as soon as $d_{\text{eff}}(w_a)$ or $d_{\text{eff}}(w)$ is large and d_a is not too large, the time average of $\|\rho_a(t) - w_a\|_{\text{tr}}$ is small. Thus, for most times $t \in [0, \infty)$, $\rho_a(t) \approx w_a$.

In a later paper, Linden, Popescu, Short, and Winter [78] studied the speed of fluctuations around thermodynamic equilibrium (in the same setting as in [77]) and showed that they are extremely small for most times provided that the effective dimension $d_{\text{eff}}(w)$ is sufficiently large and d_a is not too large. Moreover, the operator norm of $H_a + H_{\text{int}}$, where H_a is the part of the Hamiltonian acting only on the small system a and H_{int} describes the interaction between a and b, must not be too large.

All the results mentioned in this section so far are concerned with most times $t \in [0, \infty)$. Unfortunately such results do not tell us anything about *equilibration times*; if we only know that a state is close to another one for most $t \in [0, \infty)$, we have no control over the time it takes an initial state far from the state to which it will eventually equilibrate to get close to it.

One result about equilibration in finite time is due to Short and Farrelly [125]. In order to state their result, we first have to introduce some notation. The general setting is almost the same as it was in the work of Short [124] which was presented above; the only difference is that the energy gaps of the Hamiltonian need not be non-degenerate. We denote the maximum degeneracy of an energy gap by D_G.

Let d_E be the number of distinct energies of H and let

$$\mathcal{G} := \{(i,j) : i,j \in \{1,\dots,d_E\}, i \neq j\}\,. \tag{1.156}$$

Each element $\alpha = (i,j) \in \mathcal{G}$ corresponds to an energy gap $G_\alpha = E_i - E_j$ (and vice versa) and the overall number of gaps is given by $d_E(d_E-1)$. Let $\varepsilon > 0$. We define $G(\varepsilon)$ to be the maximum number of energy gaps in an interval of length ε, i.e.,

$$G(\varepsilon) := \max_{E\in\mathbb{R}} \big|\{\alpha \in \mathcal{G} : G_\alpha \in [E, E+\varepsilon)\}\big|. \tag{1.157}$$

Here, $|\{\cdot\}|$ denotes the number of elements in the set $\{\cdot\}$. Note that by taking the limit $\varepsilon \to 0^+$, we obtain the maximum degeneracy of an energy gap D_G, i.e., $\lim_{\varepsilon\to 0^+} G(\varepsilon) = D_G$.

For $T > 0$ and $f : \mathbb{R}_+ \to \mathbb{C}$ we define the time average of f over the interval $[0,T]$ by

$$\langle f(t)\rangle_T := \frac{1}{T}\int_0^T f(t)\,dt. \tag{1.158}$$

As in the case of the infinite time average we find for pure states $\rho = |\psi\rangle\langle\psi|$ and an arbitrary operator A on $\mathcal{H}$ that

$$\left\langle \Big|\langle\psi_t|A|\psi_t\rangle - \mathrm{tr}(Aw)\Big|^2 \right\rangle_T = \sum_{\substack{n\neq m\\ k\neq l}} c_n^* c_m c_k c_l^* \left\langle e^{i(E_n - E_m - E_k + E_l)t}\right\rangle_T \langle n|A|m\rangle\langle l|A^*|m\rangle. \tag{1.159}$$

Short and Farrelly [125] then defined the Hermitian matrix

$$R_{\alpha\beta} := \left\langle e^{i(G_\alpha - G_\beta)t}\right\rangle_T \tag{1.160}$$

and the vector $v_\alpha = v_{(i,j)} = c_j^*\langle j|A|i\rangle c_i$. With this they obtained

$$\left\langle \Big|\langle\psi_t|A|\psi_t\rangle - \mathrm{tr}(Aw)\Big|^2 \right\rangle_T = \sum_{\alpha,\beta\in\mathcal{G}} v_\alpha^* R_{\alpha\beta} v_\beta \leq \|R\| \sum_{\alpha\in\mathcal{G}} |v_\alpha|^2 \tag{1.161a}$$

$$= \|R\| \sum_{n\neq m} |c_n|^2 |c_m|^2 |\langle n|A|m\rangle|^2 \leq \frac{\|R\|\|A\|^2}{d_{\mathrm{eff}}}, \tag{1.161b}$$

where the last inequality follows from the same computations as in the case of the infinite time average, see (1.152b)–(1.152c).

The matrix elements of R can be easily computed and we find that

$$R_{\alpha\beta} = \frac{1}{T}\int_0^T e^{i(G_\alpha - G_\beta)t}\,dt = \begin{cases} 1 & \text{if } G_\alpha = G_\beta, \\ \frac{e^{i(G_\alpha - G_\beta)T}-1}{i(G_\alpha - G_\beta)T} & \text{otherwise.} \end{cases} \tag{1.162}$$

Starting from this and the bound $\|R\| \leq \max_\beta \sum_\alpha |R_{\alpha\beta}|$, Short and Farrelly [125] showed that the operator norm of R can be bounded as

$$\|R\| \leq G(\varepsilon)\left(1 + \frac{8\log_2 d_E}{\varepsilon T}\right). \tag{1.163}$$

Plugging the bound for $\|R\|$ into (1.161b) gives an estimate analogous to (1.152c) but for the finite-time average. Via purification, the result can again be extended to mixed states. Altogether this gives the following theorem:

Theorem 1.28 (Short, Farrelly (2012) [125])**.** *Let $\varepsilon, T > 0$, let $\rho = |\psi\rangle\langle\psi|$ be a state on $\mathcal{H}$ evolving via a (time-independent) Hamiltonian H and let A be an operator on $\mathcal{H}$. Then,*

$$\left\langle \left|\operatorname{tr}(A\rho(t)) - \operatorname{tr}(Aw)\right|^2 \right\rangle_T \leq \frac{G(\varepsilon)\|A\|^2}{d_{\text{eff}}}\left(1 + \frac{8\log_2 d_E}{\varepsilon T}\right). \tag{1.164}$$

Roughly speaking, Theorem 1.28 shows that as soon as the effective dimension d_{eff} is large, the maximum number of gaps in an interval of length ε is not too large and T is sufficiently large, it follows that $\operatorname{tr}(A\rho(t)) \approx \operatorname{tr}(Aw)$ for most times $t \in [0, T]$. Unfortunately, the times needed to ensure that the right-hand side in (1.164) is small are extremely large. As it is for example explained in [136], for a system of N particles we need $T \gg \exp(N)$ to obtain a small error. The reason is that the Hilbert space $\mathcal{H}$ of a system of N particles has dimension of the order $\exp(N)$ and assuming that no eigenvalue of the Hamiltonian is highly degenerate, the number of eigenvalues is of the same order. For $\varepsilon \sim \exp(-N)$, $G(\varepsilon)$ should be at least of order $\exp(N)$ as already the number nearest-neighbor gaps in the interval $[0, \varepsilon)$ should be of order $\exp(N)$. Therefore we need that $\varepsilon \ll \exp(-N)$ and thus $T \gg \exp(N)$ to make the right-hand side in (1.164) small. Note that on the other hand, the results in [50, 51, 52] show that if H is a Hamiltonian with uniformly distributed eigenbasis, the thermalization times are unrealistically small, see Section 1.6.5 for details.

Reimann [108] showed that for a random Hamiltonian whose eigenbasis is Haar-distributed, for any operator A, the quantity $\operatorname{tr}(A\rho(t))$ can be approximated by

$$\operatorname{tr}(A\rho(t)) \approx \frac{\operatorname{tr}(A)}{D} + F(t)\left(\operatorname{tr}(A\rho) - \frac{\operatorname{tr}(A)}{D}\right), \tag{1.165}$$

where D is the dimension of the underlying Hilbert space,

$$F(t) := \frac{D}{D-1}\left(|\phi(t)|^2 - \frac{1}{D}\right), \tag{1.166}$$

and $\phi(t)$ denotes the Fourier transform of the spectral density from [23],

$$\phi(t) := \frac{1}{D}\sum_{n=1}^{D} e^{iE_n t/\hbar}. \tag{1.167}$$

Here, the E_n are the (fixed) eigenvalues of the system's Hamiltonian and $\hbar$ is not set equal to 1. Under reasonable assumptions Reimann [108] argued that $F(t)$ is small already for extremely small times; in fact, the thermalization times are of the same order as in [51, 52]. For more details we refer to Section 1.6.5. So far, it remains an open problem to find systems for which one can prove "more realistic" equilibration/thermalization times.

Note that by using similar methods as Short [124] and Short and Farrelly [125], Reimann and Kastner [112] extended the result to also cover (countably) infinite Hilbert spaces and to relax the condition of a large effective dimension to the case that the initial state has a macroscopic population of at most one energy level.

We end this section with the remark that Short and Farrelly [125] also proved a theorem analogous to Theorem 1.27 for finite times and without the assumption that the energy gaps of H are non-degenerate:

Theorem 1.29 (Short, Farrelly (2012) [125])**.** *Let $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ be a finite-dimensional Hilbert space, let $d_a = \dim \mathcal{H}_a$, let H be a Hamiltonian on $\mathcal{H}$ with d_E distinct eigenvalues and let $\varepsilon, T > 0$. Then for every $\psi_0 \in \mathbb{S}(\mathcal{H})$,*

$$\left\langle \|\rho_a(t) - w_a\|_{\mathrm{tr}} \right\rangle_T \leq \sqrt{\frac{d_a^2 G(\varepsilon)}{d_{\mathrm{eff}}}\left(1 + \frac{8\log_2 d_E}{\varepsilon T}\right)}. \tag{1.168}$$

Thus we see that $\rho_a(t) \approx w_a$ for most $t \in [0, T]$ provided that d_{eff} and T are sufficiently large and $G(\varepsilon)$ and d_a are not too large. As discussed below Theorem 1.28, the times T needed to ensure a small right-hand side in (1.168) are extremely large.

1.6.2. Macroscopic Thermal Equilibrium

In this section we introduce and discuss the notion of *macroscopic thermal equilibrium* (MATE) [55, 53, 54], see also [132] for a similar but slightly different notion of (macroscopic) thermal equilibrium which we briefly discuss at the end of this section. In the following, we closely follow [53, 54].

Let H be a Hamiltonian on a Hilbert space $\mathcal{H}$ and let $\mathcal{H}_{\mathrm{mc}} \subset \mathcal{H}$ be an energy shell. Moreover, we assume that $\mathcal{H}_{\mathrm{mc}}$ is partitioned into macro spaces as in (1.12). Usually there is one macro space in the decomposition that has most of the dimensions of $\mathcal{H}_{\mathrm{mc}}$ and which corresponds to thermal equilibrium; we denote it by $\mathcal{H}_{\mathrm{eq}}$[14]. For any state in $\mathcal{H}_{\mathrm{eq}}$, the macro observables used for the construction of the macro spaces attain their thermal equilibrium value. In the following we assume that

$$\frac{\dim \mathcal{H}_{\mathrm{eq}}}{\dim \mathcal{H}_{\mathrm{mc}}} = 1 - \delta \tag{1.169}$$

for a small $\delta > 0$. As it is pointed out in [53], if for the construction of the macro spaces the volume of the system is partitioned into small cells and macro observables corresponding to the number of particles, total energy, total momentum and/or total magnetization in a cell are considered, realistic values of δ are exponentially small in the number of degrees of freedom per cell.

Definition 1.30 (MATE)**.** Let $\varepsilon > 0$. A system is in *macroscopic thermal equilibrium* (MATE) with precision ε, denoted by $\mathrm{MATE}_\varepsilon$, if and only if its wave function $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ lies in the set

$$\mathrm{MATE}_\varepsilon := \Big\{\phi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \langle\phi|P_{\mathrm{eq}}|\phi\rangle \geq 1 - \varepsilon\Big\}, \tag{1.170}$$

where P_{eq} denotes the projection to $\mathcal{H}_{\mathrm{eq}}$.

Thus, if $\psi \in \mathrm{MATE}_\varepsilon$, then the probability that all macro observables take their thermal equilibrium value is at least $1-\varepsilon$ and therefore, if ε is small, very close to 1. Note that in a similar way we can say that a mixed state ρ on $\mathcal{H}_{\mathrm{mc}}$ is in macroscopic thermal equilibrium with precision ε if

$$\mathrm{tr}(\rho P_{\mathrm{eq}}) \geq 1 - \varepsilon. \tag{1.171}$$

Moreover, we have that most $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ are in $\mathrm{MATE}_\varepsilon$. To see this, let u_{mc} be the uniform distribution over $\mathbb{S}(\mathcal{H}_{\mathrm{mc}})$, let $\mathbb{E}_{\mathrm{mc}}$ denote the expectation with respect to u_{mc}, let $d_{\mathrm{mc}} := \dim \mathcal{H}_{\mathrm{mc}}$, $d_{\mathrm{eq}} := \dim \mathcal{H}_{\mathrm{eq}}$, and let $P_{\mathrm{neq}} := I_{\mathrm{mc}} - P_{\mathrm{eq}}$, where I_{mc} is the identity on $\mathcal{H}_{\mathrm{mc}}$. Then we obtain with Markov's inequality and Lemma 1.2 that

$$\begin{aligned}
u_{\mathrm{mc}}(\mathrm{MATE}_\varepsilon) &= u_{\mathrm{mc}}\Big\{\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \langle\psi|P_{\mathrm{neq}}|\psi\rangle \leq \varepsilon\Big\} && \text{(1.172a)}\\
&= 1 - u_{\mathrm{mc}}\Big\{\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \langle\psi|P_{\mathrm{neq}}|\psi\rangle < \varepsilon\Big\} && \text{(1.172b)}\\
&\geq 1 - \frac{\mathbb{E}_{\mathrm{mc}}\langle\psi|P_{\mathrm{neq}}|\psi\rangle}{\varepsilon} = 1 - \frac{\delta}{\varepsilon}, && \text{(1.172c)}
\end{aligned}$$

[14]Note that there are exceptions for which there is no dominant macro space: For example for the ferromagnetic Ising model with no external magnetic field we can take $\mathcal{H}_{\nu_+}$ to be a subspace of $\mathcal{H}_{\mathrm{mc}}$ (for a suitable energy interval) with the majority of spins up and $\mathcal{H}_{\nu_-}$ as the subspace with the majority of spins down. Then $\mathcal{H}_{\nu_+}$ and $\mathcal{H}_{\nu_-}$ both have almost 50% of the dimension of $\mathcal{H}_{\mathrm{mc}}$.

i.e., the set of the wave functions $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ that are not in $\mathrm{MATE}_\varepsilon$ has measure at most δ/ε.

Similarly, most eigenstates of H are in MATE: Let $\{|n\rangle : n = 1, \ldots, d_{\mathrm{mc}}\}$ be an orthonormal basis of $\mathcal{H}_{\mathrm{mc}}$ (e.g., consisting of eigenfunctions of H). Then,

$$\frac{1}{d_{\mathrm{mc}}} \sum_{n=1}^{d_{\mathrm{mc}}} \langle n|P_{\mathrm{eq}}|n\rangle = \frac{\mathrm{tr}(P_{\mathrm{eq}})}{d_{\mathrm{mc}}} = 1 - \delta \tag{1.173}$$

and thus for no more than $(\delta/\varepsilon)d_{\mathrm{mc}}$ elements of the orthonormal basis it can hold that $\langle n|P_{\mathrm{eq}}|n\rangle < 1 - \varepsilon$, i.e., at least a fraction of $1 - \frac{\delta}{\varepsilon}$ of the elements of the orthonormal basis is in $\mathrm{MATE}_\varepsilon$.

We summarize these two findings in the following theorem:

Theorem 1.31 (Goldstein, Huse, Lebowitz, Tumulka (2015/2016) [53, 54])**.** *Let $\varepsilon > 0$, let $\mathcal{H}_{\mathrm{mc}}$ and $\mathcal{H}_{\mathrm{eq}}$ be two Hilbert spaces of dimensions $d_{\mathrm{mc}} = \dim \mathcal{H}_{\mathrm{mc}}$ and $d_{\mathrm{eq}} = \dim \mathcal{H}_{\mathrm{eq}}$ and assume that $d_{\mathrm{eq}}/d_{\mathrm{mc}} = 1 - \delta$ for some $\delta > 0$. Let u_{mc} be the uniform distribution on $\mathcal{H}_{\mathrm{mc}}$. Then,*

$$u_{\mathrm{mc}}(\mathrm{MATE}_\varepsilon) \geq 1 - \frac{\delta}{\varepsilon}. \tag{1.174}$$

Moreover, let $\{|n\rangle : n = 1, \ldots, d_{\mathrm{mc}}\}$ be an orthonormal basis of $\mathcal{H}_{\mathrm{mc}}$. Then, a fraction of at least $1 - \delta/\varepsilon$ of its elements are in $\mathrm{MATE}_\varepsilon$, i.e., $|n\rangle \in \mathrm{MATE}_\varepsilon$ for at least $(1 - \delta/\varepsilon)d_{\mathrm{mc}}$ basis vectors.

Note that one can also show that most mixed states are in MATE, see [54].

We end this section with providing an alternative and slightly different definition of macroscopic thermal equilibrium due to Tasaki [132]. The starting point is again a set of observables $M_1, \ldots, M_K$, but for his definition, in contrast to the one above (and in Section 1.2.1), they need not be rounded off and coarse-grained. For any $j \in \{1, \ldots, K\}$ let $T_j = \mathrm{tr}(M_j \rho_{\mathrm{mc}})$ be the thermal equilibrium value of M_j, where $\rho_{\mathrm{mc}} = P_{\mathrm{mc}}/d_{\mathrm{mc}}$ is the micro-canonical density matrix, i.e., the normalized projection to $\mathcal{H}_{\mathrm{mc}}$. Moreover, let ΔM_j be the macroscopic resolution of M_j. We define a set of projections P_j via

$$P_j := \mathbb{1}_{[T_j - \Delta M_j, T_j + \Delta M_j]}(M_j), \tag{1.175}$$

i.e., P_j projects to the subspace of $\mathcal{H}_{\mathrm{mc}}$ spanned by the eigenvectors of M_j with eigenvalues in the interval $[T_j - \Delta M_j, T_j + \Delta M_j]$. Then, according to Tasaki [132], a wave function $\phi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ is in macroscopic thermal equilibrium (with precision ε) if it is contained in the set

$$\bigcap_{j=1}^{K} \Big\{ \psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \langle \psi|P_j|\psi\rangle > 1 - \varepsilon \Big\}. \tag{1.176}$$

If we choose the observables M_j to be the rounded and coarse-grained macro observables used for the definition of $\mathcal{H}_{\mathrm{eq}}$ and therefore of MATE above, then Tasaki's version of macroscopic thermal equilibrium basically agrees with MATE in Definition 1.30. However, as Tasaki [132] points out, there is an essential difference between the two notions of macroscopic thermal equilibrium if the time evolution is considered. If we consider a set of observables on the whole Hilbert space $\mathcal{H}$ from which the macro spaces are constructed, usually the energy is among them. This means that by rounding and coarse-graining this set of observables, we also slightly modify the Hamiltonian in terms of which the energy shells are defined. As any state evolves with the unmodified Hamiltonian, the energy shells are not invariant under the time evolution. In contrast, in Tasaki's approach neither the Hamiltonian nor the energy shell is redefined.

1.6.3. Microscopic Thermal Equilibrium

After having introduced the notion of macroscopic thermal equilibrium in the previous section, we now define the stronger notion of *microscopic thermal equilibrium* (MITE) [53, 54]. Similar concepts were also used in, e.g., [77, 115, 93]. We again closely follow [53, 54] throughout this section.

The notion of MITE is inspired by the phenomenon of *canonical typicality*, which states that if the full Hilbert space $\mathcal{H}$ is of the form $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ and the system a is not too large, then for most wave functions $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ we have that $\rho_a^\psi \approx \operatorname{tr}_b \rho_{\mathrm{mc}}$, see Section 1.4 and the references therein. If a is sufficiently small, it can be argued that $\operatorname{tr}_b \rho_{\mathrm{mc}}$ is close to the partial trace of a canonical density matrix $\rho_{\mathrm{can}} = \operatorname{tr}(e^{-\beta H})/Z$ with suitable inverse temperature $\beta > 0$, i.e., $\rho_a^\psi \approx \operatorname{tr}_b \rho_{\mathrm{mc}} \approx \operatorname{tr}_b \rho_{\mathrm{can}}$ for most $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$. Note that if the interaction between a and b is weak, $\operatorname{tr}_b \rho_{\mathrm{can}}$ is itself approximately canonical.

In the following we consider systems a that correspond to spatial regions such that the diameter of a, denoted by $\operatorname{diam}(a)$, is defined. Moreover, we denote the complement of a by a^c.

Definition 1.32 (MITE)**.** Let $\ell \in \mathbb{R}$. A system is in *microscopic thermal equilibrium (MITE) on the length scale* ℓ, denoted by MITE_ℓ, if and only if its wave function $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ lies in the set

$$\mathrm{MITE}_\ell := \bigcap_{a:\, \operatorname{diam}(a) \leq \ell} \Big\{ \phi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \rho_a^\phi \approx \operatorname{tr}_{a^c} \rho_{\mathrm{mc}} \Big\}. \tag{1.177}$$

Let ℓ_0 be the largest ℓ small enough to ensure that $\operatorname{tr}_{a^c} \rho_{\mathrm{mc}} \approx \operatorname{tr}_{a^c} \rho_{\mathrm{can}}$ for all systems a with $\operatorname{diam}(a) \leq \ell_0$. A system is in MITE if and only if its wave function $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ lies in the set MITE_{ℓ_0}.

Thus a system with wave function $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ is in MITE if for every subsystem

a of diameter $\mathrm{diam}(a) \leq \ell_0$ its reduced density matrix ρ_a^ψ is close to $\mathrm{tr}_{a^c}\,\rho_{\mathrm{can}}$ (with suitable inverse temperature $\beta > 0$).

Note that obviously $\mathrm{MITE}_{\ell'} \subset \mathrm{MITE}_\ell$ for all $\ell' \leq \ell$. Moreover, the condition that $\rho_a^\phi \approx \mathrm{tr}_{a^c}\,\rho_{\mathrm{mc}}$ in the definition of MITE can be made precise by requiring that $\|\rho_a^\phi - \mathrm{tr}_{a^c}\,\rho_{\mathrm{mc}}\|_{\mathrm{tr}} < \varepsilon$ for a small $\varepsilon > 0$. Finally, we remark that by replacing ρ_a^ψ by $\mathrm{tr}_{a^c}\,\rho$, where ρ is an arbitrary density matrix, we can also define MITE for mixed states.

With the help of canonical typicality, we can show that most pure states are in MITE:

Theorem 1.33 (Goldstein, Huse, Lebowitz, Tumulka (2016) [54])**.** *Let $\Lambda \subset \mathbb{R}^3$ be the volume of the whole system. Let $\varepsilon > 0$ and let $a_1, \ldots, a_K \subset \Lambda$ be a cover of the whole system such that every system $a' \subset \Lambda$ with $\mathrm{diam}(a') \leq \ell_0$ is contained in one of the a_j. Then,*

$$u_{\mathrm{mc}}\Big\{\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \forall\, j : \big\|\rho_{a_j}^\psi - \mathrm{tr}_{a_j^c}\,\rho_{\mathrm{mc}}\big\|_{\mathrm{tr}} \leq \varepsilon\Big\} \tag{1.178}$$

$$\geq 1 - 8K \max_j d_{a_j}^2 \exp\left(-\frac{d_{\mathrm{mc}}\varepsilon^2}{36\pi^3 \max_j d_{a_j}^2}\right). \tag{1.179}$$

Suppose that Λ is a cube and $\mathrm{diam}(\Lambda) \geq 4\ell_0$. As it is explained in [53, 54], one can choose, e.g., $K = 8$ cubes $a_1 \ldots, a_8 \subset \Lambda$ of nearly half the volume of Λ and it follows from Theorem 1.33 that for most $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$, $\rho_{a_j}^\psi \approx \mathrm{tr}_{a_j^c}\,\rho_{\mathrm{mc}}$ for all $j = 1, \ldots, 8$. As every system $a' \subset \Lambda$ of diameter $\mathrm{diam}(a') \leq \ell_0$ is contained in one of the a_j, it follows that also for most $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$, $\rho_{a'}^\psi \approx \mathrm{tr}_{a'^c}\,\rho_{\mathrm{mc}}$ for every a' with diameter $\mathrm{diam}(a') \leq \ell_0$, i.e., most $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ are in MITE_{ℓ_0} and therefore in MITE.

Proof of Theorem 1.33. It follows from canonical typicality in the form of Theorem 1.16 that for every fixed subsystem a,

$$u_{\mathrm{mc}}\Big\{\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \big\|\rho_a^\psi - \mathrm{tr}_{a^c}\,\rho_{\mathrm{mc}}\big\|_{\mathrm{tr}} > \varepsilon\Big\} \leq 8d_a^2 \exp\left(-\frac{d_{\mathrm{mc}}\varepsilon^2}{36 d_a^2 \pi^3}\right). \tag{1.180}$$

Applying this bound to every a_j gives

$$u_{\mathrm{mc}}\Big\{\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}}) : \exists\, j : \big\|\rho_{a_j}^\psi - \mathrm{tr}_{a_j^c}\,\rho_{\mathrm{mc}}\big\|_{\mathrm{tr}} > \varepsilon\Big\} \leq 8K \max_j d_{a_j}^2 \exp\left(-\frac{d_{\mathrm{mc}}\varepsilon^2}{36\pi^3 \max_j d_{a_j}^2}\right) \tag{1.181}$$

and from this the claim follows immediately. □

As most states $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ lie in MITE and, according to Theorem 1.31, most states $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ lie in MATE, it follows that most states in MITE lie also in

MATE and vice versa. We can even show a stronger statement: For macroscopic systems, all states in MITE lie also in MATE. To see this, we follow [53] and first introduce a general framework of which MITE and MATE are special cases.

Let A be a self-adjoint operator on $\mathcal{H}$. A wave function $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ defines a probability distribution μ_A^ψ on the spectrum of A by

$$\mu_A^\psi(B) := \langle\psi|\mathbb{1}_B(A)|\psi\rangle = \mathrm{tr}\left(\mathbb{1}_B(A)|\psi\rangle\langle\psi|\right), \quad B \subseteq \mathbb{R}, \tag{1.182}$$

and similarly, the micro-canonical density matrix ρ_{mc} defines a probability distribution μ_A^{mc} on the spectrum of A by

$$\mu_A^{\mathrm{mc}}(B) := \mathrm{tr}\left(\mathbb{1}_B(A)\rho_{\mathrm{mc}}\right), \quad B \subseteq \mathbb{R}. \tag{1.183}$$

Note that by (1.88),

$$\mathbb{E}_{\mathrm{mc}}\,\mu_A^\psi(B) = \mathrm{tr}\left(\mathbb{1}_B(A)\,\mathbb{E}_{\mathrm{mc}}|\psi\rangle\langle\psi|\right) = \mathrm{tr}\left(\mathbb{1}_B(A)\rho_{\mathrm{mc}}\right) = \mu_A^{\mathrm{mc}}(B) \tag{1.184}$$

for all $B \subseteq \mathbb{R}$, where $\mathbb{E}_{\mathrm{mc}}$ denotes the expectation with respect to u_{mc}.

Let $\mathcal{A}$ be a set of observables on $\mathcal{H}$. We say that a wave function $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ is in *thermal equilibrium relative to* $\mathcal{A}$ if for all $A \in \mathcal{A}$,

$$\mu_A^\psi \approx \mu_A^{\mathrm{mc}}. \tag{1.185}$$

We obtain MATE by choosing $\mathcal{A}$ to be the set of the macro observables $M_1, \ldots, M_K$ used to define $\mathcal{H}_{\mathrm{eq}}$ in Section 1.6.2; we denote this set by $\mathcal{A}_{\mathrm{MATE}} = \{M_1, \ldots, M_K\}$.

Moreover, MITE$_\ell$ is obtained for the choice $\mathcal{A} = \mathcal{A}_{\mathrm{MITE}} = \cup_a \mathcal{A}_a$, where the union is taken over all spatial regions a with $\mathrm{diam}(a) \leq \ell_0$. Here, $\mathcal{A}_a$ denotes the set of all observables on $\mathcal{H}_a$, i.e.,

$$\mathcal{A}_a := \left\{\tilde{A} \otimes I_{a^c} : \tilde{A} \text{ is self-adjoint on } \mathcal{H}_a\right\}, \tag{1.186}$$

where I_{a^c} is the identity on $\mathcal{H}_{a^c}$.

In summary, MATE is thermal equilibrium relative to macro observables while MITE is thermal equilibrium relative to local ones. With the help of this common framework, we can now argue why MITE implies MATE for macroscopic systems. Suppose that the volume Λ of the system is partitioned into cells and that the macro observables correspond to quantities in these cells, e.g., to the total number of particles in a cell. Let L be the diameter of the largest of these cells. If $L \leq \ell_0$, then obviously $\mathcal{A}_{\mathrm{MITE}} \subset \mathcal{A}_{\mathrm{MATE}}$ and thus MITE implies MATE. As it is argued in [53, 54], the condition $L \leq \ell_0$ is usually satisfied; for example for a cubic meter of gas at room conditions, we can choose $L \approx 10^{-4}$ m and $\ell_0 \approx 10^{-3}$ m.

1.6.4. Approach to Thermal Equilibrium

In the following we discuss the approach to thermal equilibrium in the sense that an initial state approaches MATE or MITE and stays there for an extended period of time.

An important condition which ensures the approach to thermal equilibrium in isolated quantum systems is the so-called *eigenstate thermalization hypothesis* (ETH) which states that all eigenstates of the Hamiltonian H are in thermal equilibrium. The ETH goes back to Deutsch (1991) [25] and Srednicki (1994) [127] and has been investigated and applied in the physics literature ever since. Moreover, only recently it has also been rigorously proved for certain classes of random matrices, see Section 1.8.3. We note in passing that rather recently there has also been established a link between the ETH and (fast) thermalization in *open* quantum systems, see [17].

We first discuss the approach to MATE. In this case, we have the following theorem:

Theorem 1.34 (Approach to MATE). *Let $\varepsilon, \delta > 0$ and let H be a Hamiltonian with non-degenerate eigenvalues.*[15] *Let $\{|n\rangle : n = 1, \dots, D\}$ be an orthonormal eigenbasis of H and assume that $|n\rangle \in \mathrm{MATE}_{\varepsilon\delta}$ for all n. Then for every $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ and $(1-\delta)$-most $t \in [0, \infty)$,*

$$\psi_t = e^{-itH}\psi_0 \in \mathrm{MATE}_\varepsilon. \tag{1.187}$$

Proof. Let $E_1, \dots, E_D$ be the eigenvalues of H corresponding to the eigenstates $|n\rangle$. We compute

$$\begin{aligned} \overline{\langle\psi_t|P_{\mathrm{neq}}|\psi_t\rangle} &= \sum_{m,n} \overline{e^{i(E_m-E_n)t}} \langle\psi_0|m\rangle\langle m|P_{\mathrm{neq}}|n\rangle\langle n|\psi_0\rangle && (1.188\text{a}) \\ &= \sum_n |\langle n|\psi_0\rangle|^2 \langle n|P_{\mathrm{neq}}|n\rangle && (1.188\text{b}) \\ &\leq \varepsilon\delta \sum_n |\langle n|\psi_0\rangle|^2 = \varepsilon\delta. && (1.188\text{c}) \end{aligned}$$

Thus, for every $\eta > 0$ there is a $T_0 > 0$ such that for every $T \geq T_0$,

$$\frac{1}{T}\int_0^T \langle\psi_t|P_{\mathrm{neq}}|\psi_t\rangle\, dt < \varepsilon\delta + \eta. \tag{1.189}$$

An application of Markov's inequality shows that for every $T \geq T_0$,

$$\frac{1}{T}\Big|\Big\{t \in [0,T] : \langle\psi_t|P_{\mathrm{neq}}|\psi_t\rangle > \varepsilon\Big\}\Big| \leq \frac{\varepsilon\delta + \eta}{\varepsilon} = \delta + \frac{\eta}{\varepsilon}. \tag{1.190}$$

[15] The assumption of non-degeneracy can be dropped, see, e.g., Proposition 1 in [117].

Next we take the limes superior as $T \to \infty$ and use that $\eta > 0$ was arbitrary to obtain

$$\limsup_{T\to\infty} \frac{1}{T}\Big|\big\{t \in [0,T] : \langle\psi_t|P_{\mathrm{neq}}|\psi_t\rangle > \varepsilon\big\}\Big| \leq \delta, \tag{1.191}$$

which finishes the proof. □

Next we discuss the approach to MITE. There are several results concerning the approach to MITE and for all of them it is assumed that the Hamiltonian H has non-degenerate eigenvalue gaps and that the MITE-ETH is fulfilled, i.e., that all eigenstates of H are in MITE.

The results in [105, 77] show that if all energy eigenstates of H in $\mathcal{H}_{\mathrm{mc}}$ are in MITE, then most initial wave functions $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ are in MITE for most of the time. More precisely, it is required that the initial state has sufficiently large effective dimension, i.e., that many different energy eigenstates contribute significantly to it. Note that most wave functions in $\mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ have large effective dimension; as it is proved in Theorem 2 in [77], the set of wave functions $\psi \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ such that $d_{\mathrm{eff}} < d_{\mathrm{mc}}/4$ is of the order $\exp(-\sqrt{d_{\mathrm{mc}}})$ and therefore extremely small as soon as d_{mc} is large.

Another result which is due to Rigol, Dunjko, and Olshanii [114] shows that not only most but *all* initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ approach MITE and stay there for most of the time provided that the eigenstates $\{|n\rangle : n = 1, \ldots, d_{\mathrm{mc}}\}$ of H in $\mathcal{H}_{\mathrm{mc}}$ are all in MITE and additionally that

$$\langle n|A|m\rangle \approx 0 \tag{1.192}$$

for all $m \neq n$ and all $A \in \mathcal{A}_{\mathrm{MITE}}$. This is Srednicki's [128, 129] extension of the ETH to off-diagonal elements.

As the authors of [114] do not give explicit error bounds, we quote the result from [140] which gives such bounds and also provides a rigorous proof.

Theorem 1.35 (Approach to MITE [140])**.** *Let $\delta, \eta > 0$, let $H = \sum_n E_n|n\rangle\langle n|$ be a non-degenerate Hamiltonian with non-degenerate energy gaps and let $\mathcal{A} \neq \emptyset$ be a set of self-adjoint operators on $\mathcal{H}$. Set $\varepsilon = \sqrt{\eta}\delta/2$ and suppose that*

$$(\mathcal{A}\text{-}ETH) \qquad \forall\, n\; \forall A \in \mathcal{A} : \Big|\langle n|A|n\rangle - \mathrm{tr}(\rho_{\mathrm{mc}}A)\Big| < \varepsilon \tag{1.193}$$

and

$$\forall\, n \neq m\; \forall A \in \mathcal{A} : \big|\langle n|A|m\rangle\big| < \varepsilon. \tag{1.194}$$

Then, for every $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ and every $A \in \mathcal{A}$,

$$\Big|\langle\psi_t|A|\psi_t\rangle - \mathrm{tr}(\rho_{\mathrm{mc}}A)\Big| < \delta \tag{1.195}$$

for $(1-\delta)$*-most* $t \in [0,\infty)$. *In particular, if every spectral projection of any* $A \in \mathcal{A}$ *is also contained in* $\mathcal{A}$*, then every* $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ *spends most of the time in thermal equilibrium relative to* $\mathcal{A}$. *Moreover, if* $\mathcal{A} = \mathcal{A}_{\mathrm{MITE}}$*, then every* $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ *spends most of the time in MITE.*

For a proof of Theorem 1.35 we refer to [140].

Theorem 1.34 and Theorem 1.35 concerning the approach to MATE/MITE show that for every initial wave function $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$, the time evolved state ψ_t is in MATE/MITE for most times $t \in [0,\infty)$ provided that a suitable version of the ETH if fulfilled. This, however, does not tell us anything about the time scales of thermalization, i.e., about the time it takes an initial state possibly far from thermal equilibrium to reach it; this is the topic of the next section.

1.6.5. Time Scales of Equilibration and Thermalization

In this section we discuss several results concerning the time scales of equilibration and thermalization in closed quantum systems.

We begin with a result by Goldstein, Hara, and Tasaki [50]; they showed that one can construct situations in which the relaxation time to equilibrium is extremely long while one can also choose an equilibrium subspace to which any initial state equilibrates within only a short time:

Theorem 1.36 (Goldstein, Hara, Tasaki (2013) [50]). *Let* H *be a Hamiltonian on a Hilbert space* $\mathcal{H}$ *and let* $\mathcal{H}_{\mathrm{mc}}$ *be a micro-canonical subspace of* $\mathcal{H}$ *corresponding to an energy interval* $[E, E+\Delta E)$.

1.) *Let* $0 < d \leq d_{\mathrm{mc}} = \dim \mathcal{H}_{\mathrm{mc}}$ *and let* $|\eta\rangle \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$. *Moreover, let*

$$0 < T \leq \frac{\pi}{6\Delta E} d. \tag{1.196}$$

Then there exists a d*-dimensional subspace* $\mathcal{H}_1$ *of* $\mathcal{H}_{\mathrm{mc}}$ *with* $|\eta\rangle \in \mathcal{H}_1$ *such that for any initial state* $\psi_0 \in \mathbb{S}(\mathcal{H}_1)$ *one has*

$$\frac{1}{T}\int_0^T \langle \psi_t | P_1 | \psi_t \rangle \, dt \geq \frac{3}{\pi} > 0.95, \tag{1.197}$$

where P_1 *denotes the projection to* $\mathcal{H}_1$.

2.) *Assume that there is a constant* $c > 0$ *independent of the system's volume* V *such that the density of states within the interval* $[E, E+\Delta E)$ *is at least* e^{cV}. *Let* $\varepsilon_0 > 0$ *with* $\varepsilon_0 \ll \Delta E$ *be independent of* V *such that for any interval* $[E', E''] \subset [E, E+\Delta E)$ *with* $E'' - E' \geq \varepsilon_0$ *the number of eigenvalues of the Hamiltonian in* $[E', E'']$ *is not* *less than* $(E'' - E')e^{cV}$. *Then, for any* T_0 *such that*

$$0 < T_0 \Delta E \le (\Delta E/(2\varepsilon_0))^2, \tag{1.198}$$

there exists a subspace $\mathcal{H}_2$ *of* $\mathcal{H}_{\mathrm{mc}}$*, whose dimension is* $d \sim e^{cV}$*, such that for any initial state* $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ *and any* $T > 0$*,*

$$\frac{1}{T}\int_0^T \langle \psi_t | P_2 | \psi_t \rangle \, dt \le \frac{2}{\sqrt{T_0 \Delta E}} \left(1 + \frac{2T_0}{T} \right), \tag{1.199}$$

where P_2 *denotes the projection to* $\mathcal{H}_2$*.*

The subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are constructed by carefully choosing a basis for them. More precisely, the basis vectors of $\mathcal{H}_1$ are chosen to be in a narrow energy interval while the basis states of $\mathcal{H}_2$ sparsely spread over the whole energy range.

The authors of [50] argue that if the dimension of $\mathcal{H}_1$ is of an order which is typical for a non-equilibrium subspace $\mathcal{H}_{\mathrm{neq}} = \mathcal{H}_{\mathrm{eq}}^{\perp}$, then the times which ensure that an initial state from $\mathcal{H}_1$ does not get far away from it, can easily exceed the age of the universe. So there can always be pathological situations in which the equilibration takes an unphysically long time. On the other hand, the second part of Theorem 1.36 shows that there also is a subspace of $\mathcal{H}_{\mathrm{mc}}$ which any initial state leaves within a reasonable amount of time. However, unfortunately the construction of the subspace $\mathcal{H}_2$ is very artificial and therefore it cannot be expected that $\mathcal{H}_2$ resembles a realistic non-equilibrium subspace.

A natural question is how long thermalization takes for a *typical* subspace $\mathcal{H}_{\mathrm{neq}}$. This question was studied by Goldstein, Hara, and Tasaki in [51, 52]. In order to state their result, we first recall the notion of a random subspace. Let $\mathcal{H}$ be a Hilbert space of dimension D and let $d \le D$. We have already constructed the uniform distribution on $\mathrm{ONB}(\mathcal{H})$, the set of orthonormal bases of $\mathcal{H}$, in Section 1.1.3. Let $(\varphi_j)_{j=1,\dots,D}$ be a random orthonormal basis of $\mathcal{H}$. By restricting to the first d elements of this basis, we obtain a random basis of a d-dimensional subspace of $\mathcal{H}$. We call the distribution of the subspace spanned by $\varphi_1, \dots, \varphi_d$ the uniform distribution on the set of d-dimensional subspaces of $\mathcal{H}$.

Let H be a Hamiltonian on $\mathcal{H}$. As above, we restrict our considerations to a micro-canonical subspace $\mathcal{H}_{\mathrm{mc}}$ corresponding to an energy interval $[E - \Delta E, E]$. Let $E_1, \dots, E_{d_{\mathrm{mc}}}$ be the eigenvalues of H restricted to $\mathcal{H}_{\mathrm{mc}}$. For the proof of the main result in [51] it is assumed that the eigenvalues $E_1, \dots, E_{d_{\mathrm{mc}}}$ can be well described by a function $\rho : \mathbb{R} \to \mathbb{R}$, the density of states, in the sense that for every differentiable function $f : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{d_{\mathrm{mc}}} \left| \sum_{j=1}^{d_{\mathrm{mc}}} f(E_j) - \int_{E-\Delta E-\eta}^{E+\eta} \rho(e) f(e) \, de \right| \le \eta \sup_{e \in [E-\Delta E-\eta, E+\eta]} |f'(e)|, \tag{1.200}$$

where $\eta > 0$ is a small constant. As the authors show in [51], this assumption can always be fulfilled.

Let $\tilde{\beta} = (k_B T)^{-1}$ be the inverse temperature which corresponds to the equilibrium state in the energy shell $\mathcal{H}_{\mathrm{mc}}$. Here, T denotes the absolute temperature and $k_B \sim 1.38 \times 10^{-23}$ J/K is the so-called *Boltzmann constant.* In (1.200) we choose

$$\eta = d_{\mathrm{mc}}^{-\kappa} \tilde{\beta}^{-1}, \tag{1.201}$$

where $\kappa \in (0,1)$ is close to 1. Moreover, we define the *Boltzmann time*

$$\tau_B := h\tilde{\beta} = \frac{h}{k_B T}, \tag{1.202}$$

where $h = 2\pi\hbar \sim 6.626 \times 10^{-34}$ Js is the Planck constant.

Theorem 1.37 (Goldstein, Hara, Tasaki (2014/2015) [51, 52])**.** *Let $\mathcal{H}_{\mathrm{mc}}$ be a microcanonical subspace of dimension d_{mc} as above, let $\varepsilon > 0$, let d_{mc} be sufficiently large and let the dimension $d \geq 1$ be sufficiently small compared to d_{mc} such that*

$$\frac{d}{d_{\mathrm{mc}}} \leq 2 \left(\frac{\log\left(2e^3 d_{\mathrm{mc}}^{5/4}\right)}{\log(1+\varepsilon)} + 1 \right)^{-4}. \tag{1.203}$$

Suppose that the density of states satisfies

$$\rho(e) \leq \tilde{\beta} d_{\mathrm{mc}} \tag{1.204}$$

for any $e \in [E - \Delta E - \eta, E + \eta]$.

Let $\mathcal{H}_{\mathrm{rnd}}$ be a random d-dimensional subspace of $\mathcal{H}_{\mathrm{mc}}$ and let P_{rnd} denote the projection to it. For $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ let $\psi_t = e^{-itH/\hbar}\psi_0$ denote its time evolution[16]*. Then, for any $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$, with probability larger than $1 - (d/d_{\mathrm{mc}})$,*

$$\frac{1}{\tau} \int_0^\tau \langle \psi_t | P_{\mathrm{rnd}} | \psi_t \rangle \, dt \leq \frac{\tau_B}{\tau} \left(1 + 2d_{\mathrm{mc}}^{-\kappa/5}\right)(1+\varepsilon) \sim \frac{\tau_B}{\tau}, \tag{1.205}$$

for any $\tau \geq 0$ such that

$$0 \leq \frac{\tau}{\tau_B} \leq \min\left\{ d_{\mathrm{mc}}^{\kappa/5}, \left(\frac{d_{\mathrm{mc}}}{d}\right)^{1/4} \right\}. \tag{1.206}$$

Theorem 1.37 shows that if we think of the random subspace as the subspace corresponding to non-equilibrium, then any initial state $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ escapes the

[16] Note that for this result, in comparison to the rest of the thesis (if not stated otherwise), we do not set $\hbar$ equal to 1.

non-equilibrium subspace and thus approaches thermal equilibrium on a time scale $\tau \gg \tau_B$. For example for $T \approx 300$ K we find that $\tau_B \approx 1.6 \times 10^{-13}$ s and thus already for $\tau \approx 1\,\mu$s, the right-hand side of (1.205) is smaller than 10^{-6}. With the help of Markov's inequality we find that the set of "bad" times in $t \in [0, \tau]$ such that $\langle \psi_t | P_{\mathrm{rnd}} | \psi_t \rangle \geq 10^{-3}$ has (normalized) measure of at most 10^{-3} and we can conclude that the system thermalizes within less than a micro-second.

As the authors of [51, 52] point out, these extremely quick thermalization times are highly unphysical and thus a purely random subspace is not a good model for a realistic non-equilibrium subspace. The realistic ones therefore belong to the exceptions in Theorem 1.37 and are, in this sense, atypical. Another important point to note is that in realistic systems, the relaxation time to thermal equilibrium should increase with the size of the system, whereas the time scales of thermalization observed in Theorem 1.37 only depend on the Boltzmann time which in turn depends only on the system's temperature but not on its size. Recall that while for the random subspace in Theorem 1.37 the thermalization times are extremely short, one can also construct a non-equilibrium subspace such that the relaxation times to thermal equilibrium are unphysically long, see Theorem 1 in [50] and the discussion above. It is expected that realistic systems show thermalization times somewhere in between but the details still remain to be understood.

Another result concerning the thermalization of a closed quantum system is due to Reimann [108] which we already briefly mentioned in Section 1.6.1. He also obtained very short thermalization times when studying systems with random Hamiltonians. More precisely, he considered the following setting: Let $\mathcal{H}$ be a finite-dimensional Hilbert space with $D = \dim \mathcal{H} \gg 1$, let ρ be a density matrix and let A be an observable on $\mathcal{H}$. For $t \geq 0$ let $\rho(t) = \mathcal{U}_t \rho(0) \mathcal{U}_t^*$ with $\rho(0) = \rho$ and $\mathcal{U}_t = e^{-iHt/\hbar}$ (here we also do not set $\hbar$ equal to 1). The system's Hamiltonian is modeled by a random Hamiltonian in the sense that its eigenvalues $E_1, \ldots, E_D$ are arbitrary but fixed and its eigenbasis is uniformly distributed, i.e., with respect to the Haar measure. Let $\{|n\rangle : n = 1, \ldots, D\}$ be the (random) orthonormal eigenbasis of H and let $\mathbb{E}_U$ denote the expectation with respect to the Haar measure. We define $\rho_{\mathrm{av}} := \mathbb{E}_U w$ where w is defined by $\langle m|w|n\rangle := \delta_{mn} \langle n|\rho|n\rangle$. Reimann [108] showed that for all $t \geq 0$,

$$\mathbb{E}_U \operatorname{tr}(A\rho(t)) = \operatorname{tr}(A\rho_{\mathrm{av}}) + F(t) \left(\operatorname{tr}(A\rho) - \operatorname{tr}(A\rho_{\mathrm{av}}) \right), \tag{1.207}$$

where the function F is given by (1.166).

Moreover, Reimann [108] obtains that for every $t \geq 0$, the variance of $\operatorname{tr}(A\rho(t))$ is of the order $\Delta_A^2 \operatorname{tr} \rho^2 / D$ where Δ_A denotes the difference between the largest and smallest eigenvalue of A. Therefore, the variance is small if D is sufficiently large and then the right-hand side of (1.207) describes the quantity $\operatorname{tr}(A\rho(t))$ very well.

Note that Reimann [108] also shows that

$$\mathrm{tr}(A\rho_{\mathrm{av}}) = \frac{\mathrm{tr}(A)}{D} + \frac{\mathrm{tr}(A\rho) - \frac{\mathrm{tr}(A)}{D}}{D+1}, \tag{1.208}$$

where $\mathrm{tr}(A)/D$ is the micro-canonical expectation of A if $\mathcal{H}$ is a micro-canonical energy shell. Therefore, if $F(t)$ is small, $\mathrm{tr}(A\rho(t))$ is well approximated by $\mathrm{tr}(A)/D$.

Reimann [108] argues that if $\mathcal{H}$ is a micro-canonical energy shell corresponding to an energy window $[E - \Delta E, E]$, then $\Delta E \gg k_B T$ is a reasonable assumption and in this case one obtains

$$F(t) \approx \frac{1}{1 + (k_B T t/\hbar)^2} = \frac{1}{1 + (2\pi t/\tau_B)^2}, \tag{1.209}$$

i.e., the time scale of equilibration is of the same order as in the result from [51, 52] discussed above. He further argues that (1.207) describes the very rapid relaxation of the system towards local equilibrium rather than global equilibration which should take significantly longer. Reimann [108] also compared his findings with experimental results of the rapid prethermalization of ultracold atoms of a coherently split Bose gas [62] and the ultrafast relaxation of hot atoms [9, 63, 40] and with some numerical results of the relaxation of an integrable and non-integrable one-dimensional fermionic model [113], of the prethermalization in a one-dimensional electron gas [139], of the thermalization of a spin qubit coupled to a bath [65] and of the thermalization in a random matrix model [6]. He found very good agreement of his theoretical results with the experimental and numerical data.

Equilibration and thermalization of systems with random Hamiltonians (in the sense that the eigenbasis is sampled from the Haar measure) were also studied in, e.g., [23], and system's with random observables in [82], and in both cases extremely short equilibration/thermalization times are observed. For further results concerning the time scales of equilibration and thermalization see, e.g., [89, 90, 43, 27, 74].

1.6.6. Comparison to the Classical Case

In this section, we discuss how thermal equilibrium can be defined in classical systems and where the similarities and differences to the quantum case lie. Again, we closely follow the presentations in [53, 54] and also [61].

As in these references, we adopt the *individualist* rather than the *ensemblist* attitude. For an ensemblist, a system is in thermal equilibrium if its phase point X is random with a probability density (on phase space) that is approximately given by a micro-canonical or canonical density. However, we consider an individual system and we would like to say something about whether this particular system is in thermal equilibrium or not and a system has a phase point X but no probability distribution ρ over phase space. For more details on this topic, see, e.g., [61].

A pure and mixed state in quantum mechanics corresponds to a phase point X in phase space and a probability distribution over phase space in classical mechanics respectively. A phase point X of the system specifies the positions and momenta of all particles in the system. Thus, if we want to describe a subsystem, we just consider only the positions and momenta of the particles in the subsystem and in this way a phase point X of the whole system provides us also with a phase point $\tilde{X}$ of the subsystem. Therefore, in classical mechanics, we always obtain an analogue of a pure state for a subsystem and never an analogue of a mixed state. In contrast, in quantum mechanics, the reduced density matrix of a subsystem is in general mixed even if the state of the whole system is pure. As a consequence, while in quantum mechanics it is possible that the reduced density matrix of a subsystem is close to a marginal[17] of a micro-canonical (or canonical) density matrix, the state of a subsystem in classical mechanics is always "pure" and thus not close to a thermodynamic ensemble. We conclude that there is no classical analogue to the concept of MITE.

However, there is a classical analogue of MATE which we discuss in the following. Let Γ be the system's phase space. Analogous to the quantum mechanical setting we partition the phase space into "macro sets" $\Gamma_\nu \subset \Gamma$ corresponding to macro states ν,

$$\Gamma = \bigcup_\nu \Gamma_\nu. \tag{1.210}$$

The phase points in a macro set Γ_ν should "look macroscopically the same". Of course, as also in the quantum mechanical setting, there is some arbitrariness in the choice of the macro sets Γ_ν, but it can be argued that for reasonable choices and large enough systems this arbitrariness is unproblematic, see [61].

Usually, there is one macro set in the decomposition (1.210) that has by far the largest phase space volume and is associated with thermal equilibrium; we denote this set by Γ_{eq}. Similarly to the quantum case, following Boltzmann [13, 14, 49], the macro sets can be defined with the help of a set of macro variables $M_j : \Gamma \to \mathbb{R}$, $j = 1, \dots, K$. For each macro variable M_j let $\Delta M_j > 0$ denote its macroscopic resolution and we think of the M_j as being suitably coarse-grained and therefore, e.g., taking values only in $\{k\Delta M_j : k \in \mathbb{Z}\}$. With this we can make precise what we mean by "looking macroscopically the same". Let $X_1, X_2 \in \Gamma$ be two phase points. Then we say that they look macroscopically the same if and only if $M_j(X_1) = M_j(X_2)$ for all $j = 1, \dots, K$. We associate a macro state ν with a list of values of the M_j, i.e., $\nu = (\nu_1, \dots, \nu_K)$ with $\nu_j = k_j \Delta M_j$ for some $k_j \in \mathbb{Z}$. The corresponding macro set Γ_ν is then defined as

$$\Gamma_\nu = \Big\{ X \in \Gamma : M_j(X) = \nu_j \ \forall j \Big\} \tag{1.211}$$

and Γ_{eq} is the set of all the phase points for which all macro variables assume their

[17]The marginal is obtained by tracing out the complement of the subsystem.

thermal equilibrium value. Note that typically the size differences between any two macro sets are huge.

Usually, a coarse-grained version of the energy is among the macro variables used for the construction of the macro sets Γ_ν and in this case the partition of Γ also provides us with a partition of any energy shell Γ_{mc} into macro sets. Here, an energy shell consists of all the phase points for which the coarse-grained version of the energy takes on the same value. Also in each energy shell there is a macro set corresponding to thermal equilibrium which we again denote by Γ_{eq}. With this we can say that a system is in (classical) macroscopic thermal equilibrium if and only if its phase point is contained in the set Γ_{eq}.

As it is shown in [54], the macro set Γ_{eq} takes up almost all of the volume of Γ_{mc}; a realistic value of the ratio between the sizes of the two sets is given by

$$\frac{\mathrm{vol}\ \Gamma_{\mathrm{eq}}}{\mathrm{vol}\ \Gamma_{\mathrm{mc}}} \approx 1 - \exp\left(-10^{-15} N\right), \tag{1.212}$$

where N denotes the number of degrees of freedom of the system and vol is the $6N$-dimensional phase space volume, see Section 7.1 in [54] for more details and the derivation of (1.212).

We are also able to say something concerning the approach to (classical) macroscopic thermal equilibrium. First note that by Liouville's theorem, the phase space volume is conserved under the time evolution and therefore at any time the thermal equilibrium subspace takes up almost all of the volume of Γ_{mc}. Moreover, due to the conservation of energy, an initial phase point $X(0) \in \Gamma_{\mathrm{mc}}$ stays in the energy shell Γ_{mc} for all times. From these two observations it is very plausible that most $X(0) \in \Gamma_{\mathrm{mc}} \backslash \Gamma_{\mathrm{eq}}$ will eventually reach the thermal equilibrium macro set Γ_{eq} and stay there for most of the time. Note that due to recurrence any system that starts out in a non-equilibrium macro set will return arbitrarily close to it after a sufficiently long time.

We end this section with remarking that with the help of the decomposition (1.210) we can define the *Boltzmann entropy* S_B of a phase point $X \in \Gamma_\nu$ by

$$S_B(X) = S_B(\nu) := k_B \log \mathrm{vol}\, \Gamma_\nu, \tag{1.213}$$

where k_B denotes the Boltzmann constant. The idea is that if a system starts out in a small macro set which is "far away" from thermal equilibrium, its phase point evolves through larger and larger macro sets until it finally reaches Γ_{eq} where it stays for a very long time and therefore the entropy increases over time.

More precisely, a macro set $\Gamma_\nu \subset \Gamma_{\mathrm{mc}}$ is transported under the Hamiltonian flow into a set $A_t \subset \Gamma_{\mathrm{mc}}$ such that $\mathrm{vol}(A_t) = \mathrm{vol}(\Gamma_\nu)$ for all $t \geq 0$. Typically, the set

$$\Gamma_{<\nu} := \bigcup_{\nu' : S(\nu') < S(\nu)} \Gamma_{\nu'} \tag{1.214}$$

is much smaller than the set Γ_ν, i.e., $\mathrm{vol}(\Gamma_{<\nu}) \ll \mathrm{vol}(\Gamma_\nu)$, which immediately implies

$$\frac{\mathrm{vol}(A_t \cap \Gamma_{<\nu})}{\mathrm{vol}(A_t)} \ll 1. \tag{1.215}$$

We conclude that for most initial phase points $X(0) \in \Gamma_\nu$,

$$S_B(X(t)) \geq S_B(X(0)), \tag{1.216}$$

i.e., the Boltzmann entropy typically increases and valleys in the entropy curve $t \mapsto S_B(X(t))$ are typically infrequent, shallow and short-lived. However, note that if the system starts out in a non-equilibrium macro set then due to recurrence, after very long times there are deeper valleys in the entropy curve.

Analogously, given a decomposition of the system's Hilbert space $\mathcal{H}$ into macro spaces $\mathcal{H}_\nu$ as in (1.12), we can define the *quantum Boltzmann entropy* S_{qB} by

$$S_{\mathrm{qB}}(\psi) = S_{\mathrm{qB}}(\nu) := k_B \log \dim \mathcal{H}_\nu, \qquad \psi \in \mathbb{S}(\mathcal{H}_\nu). \tag{1.217}$$

This definition agrees well with the suggestion already made by Einstein [29] that in quantum mechanics the entropy of a macro state should be proportional to the logarithm of the "number of elementary quantum states" that are compatible with this macro state.

For more details on the Boltzmann entropy in classical and in quantum mechanics as well as a discussion of another prominent concept of entropy, the so-called Gibbs entropy, see [61].

1.7. The Free Fermi Gas in 1D

In the following we discuss the thermalization of a concrete model, namely of the free, non-relativistic Fermi gas in one dimension. This model has only recently been studied by Shiraishi and Tasaki [123] and Tasaki [134, 133] and we closely follow these references throughout this section. We first introduce the model in Section 1.7.1. In Section 1.7.2, we present and discuss the results obtained in [123] and [134] and briefly comment on [133] and the classical case.

1.7.1. The Model

In this section, we introduce the free, non-relativistic Fermi gas of N particles in one dimension and show some of its properties. Let $\Lambda = \{1, \ldots, L\}$ be a chain of length $L \in \mathbb{N}$ and let $\mathcal{H}$ be the N-particle sector of the fermionic Fock space, i.e., $\mathcal{H} \simeq \mathbb{C}^D$

with $D = \binom{L}{N}$. The Hamiltonian H_θ studied in [123] is given by

$$H_\theta = -\sum_{x=1}^{L} \left(e^{i\theta} c_x^\dagger c_{x+1} + e^{-i\theta} c_{x+1}^\dagger c_x \right), \tag{1.218}$$

where $c_x^\dagger$ and c_x denote the creation and annihilation operators of a fermion at site $x \in \Lambda$ and $\theta \in [0, 2\pi)$. Moreover, we assume periodic boundary conditions, i.e., $c_{L+1} = c_1$. Recall that the creation and annihilation operators satisfy the canonical anticommutation relations

$$\{c_x, c_y\} = 0 \qquad \text{and} \qquad \{c_x, c_y^\dagger\} = \delta_{x,y} \tag{1.219}$$

for any $x, y \in \Lambda$ where the anticommutator $\{\cdot, \cdot\}$ is defined by $\{A, B\} := AB + BA$ for any two operators A and B.

Note that Shiraishi and Tasaki [123] introduced the phase factor θ to avoid degeneracy of the Hamiltonian; more precisely, if $|\theta| > 0$ is sufficiently small, it can be shown that the eigenvalues of H_θ are non-degenerate, see Theorem 1.42.

In order to write down the eigenbasis of H_θ as well as the eigenvalues, we have to introduce some notation first. If the length L of the chain is odd, let

$$\mathcal{K} := \left\{ \frac{2\pi}{L}\nu \,\middle|\, \nu \in \left\{ 0, \pm 1, \dots, \pm \frac{L-1}{2} \right\} \right\} \tag{1.220}$$

and if L is even, let

$$\mathcal{K} := \left\{ \frac{2\pi}{L}\nu \,\middle|\, \nu \in \left\{ 0, \pm 1, \dots, \pm \left(\frac{L}{2} - 1 \right), \frac{L}{2} \right\} \right\} \tag{1.221}$$

be the k-space. For $k \in \mathcal{K}$ we define the creation operator

$$a_k^\dagger := \frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{ikx} c_x^\dagger. \tag{1.222}$$

With the help of the canonical anticommutation relations for the c_x we can show that $\{a_k, a_{k'}\} = 0$ and $\{a_k^\dagger, a_{k'}\} = \delta_{kk'}$. The first equation is obvious and for the second one we compute

$$\{a_k^\dagger, a_{k'}\} = \frac{1}{L} \sum_{x,y=1}^{L} e^{ikx} e^{-ik'y} \{c_x^\dagger, c_y\} = \frac{1}{L} \sum_{x=1}^{L} e^{i(k-k')x}. \tag{1.223}$$

If $k = k'$, the right-hand side of (1.223) is obviously equal to 1 and if $k \neq k'$, then

$$\{a_k^\dagger, a_{k'}\} = \frac{1}{L}\frac{e^{i(k-k')} - e^{i(k-k')(L+1)}}{1 - e^{i(k-k')}} = 0, \tag{1.224}$$

where we used that $(k-k')L$ is a multiple of 2π and therefore $e^{i(k-k')(L+1)} = e^{i(k-k')}$.

Let $|\Phi_{\text{vac}}\rangle$ denote the vacuum state, i.e., the state with no particles. For $k = (k_1, \ldots, k_N) \in \mathcal{K}^N$ such that $k_j < k_{j+1}$ for $j = 1, \ldots, N-1$ we define

$$|\Psi_k\rangle := a_{k_1}^\dagger a_{k_2}^\dagger \ldots a_{k_N}^\dagger |\Phi_{\text{vac}}\rangle. \tag{1.225}$$

In this way we obtain $\binom{L}{N}$ (non-zero) states. We now show that the $|\Psi_k\rangle$ form an orthonormal basis of $\mathcal{H}$ consisting of eigenfunctions of H_θ. Let $k, k' \in \mathcal{K}$ with $k \neq k'$ and suppose without loss of generality that k_1' does not appear in k. Then,

$$\begin{aligned}\langle \Psi_{k'}|\Psi_k\rangle &= \langle \Phi_{\text{vac}}|a_{k_N'} \ldots a_{k_1'} a_{k_1}^\dagger \ldots a_{k_N}^\dagger |\Phi_{\text{vac}}\rangle && (1.226\text{a})\\ &= (-1)^N \langle \Phi_{\text{vac}}|a_{k_N'} \ldots a_{k_2'} a_{k_1}^\dagger \ldots a_{k_N}^\dagger a_{k_1} |\Phi_{\text{vac}}\rangle = 0, && (1.226\text{b})\end{aligned}$$

where we used in the last step that $c_x|\Phi_{\text{vac}}\rangle = 0$ for every $x \in \Lambda$. For $k' = k$ we get

$$\|\Psi_k\|^2 = \langle \Phi_{\text{vac}}|a_{k_1} a_{k_1}^\dagger a_{k_2} a_{k_2}^\dagger \ldots a_{k_N} a_{k_N}^\dagger |\Phi_{\text{vac}}\rangle = \langle \Phi_{\text{vac}}|\Phi_{\text{vac}}\rangle = 1, \tag{1.227}$$

where the second step follows from

$$a_{k_j} a_{k_j}^\dagger |\Phi_{\text{vac}}\rangle = \frac{1}{L}\sum_{x,y=1}^{L} e^{ik_j(x-y)} c_y c_x^\dagger |\Phi_{\text{vac}}\rangle = \frac{1}{L}\sum_{x=1}^{L} c_x c_x^\dagger |\Phi_{\text{vac}}\rangle = |\Phi_{\text{vac}}\rangle \tag{1.228}$$

for all $j = 1, \ldots, N$ as $c_x c_x^\dagger |\Phi_{\text{vac}}\rangle = |\Phi_{\text{vac}}\rangle$ for all $x \in \Lambda$. Altogether this shows that the $|\Psi_k\rangle$ form indeed an orthonormal basis of $\mathcal{H}$. Next we prove that the states $|\Psi_k\rangle$ are eigenstates of H_θ. To this end we first compute for $k \in \mathcal{K}$

$$\begin{aligned}[H_\theta, a_k^\dagger] &= -\frac{1}{\sqrt{L}}\sum_{x,y=1}^{L}\left(e^{i(\theta+ky)}\left[c_x^\dagger c_{x+1}, c_y^\dagger\right] + e^{i(-\theta+ky)}\left[c_{x+1}^\dagger c_x, c_y^\dagger\right]\right), && (1.229\text{a})\\ &= -\frac{1}{\sqrt{L}}\sum_{x,y=1}^{L}\Big(e^{i(\theta+ky)}\left[c_x^\dagger\{c_{x+1}, c_y^\dagger\} - \{c_x^\dagger, c_y^\dagger\}c_{x+1}\right]\\ &\qquad + e^{i(-\theta+ky)}\left[c_{x+1}^\dagger\{c_x, c_y^\dagger\} - \{c_{x+1}^\dagger, c_y^\dagger\}c_x\right]\Big), && (1.229\text{b})\end{aligned}$$

where $[\cdot,\cdot]$ denotes the commutator, i.e., $[A,B] = AB - BA$ for any two operators A and B. Note that in the second line we made use of the identity $[AB, C] = A\{B,C\} - \{A,C\}B$ for any three operators A, B and C. With the help of the canonical anticommutation relations we obtain

$$[H_\theta, a_k^\dagger] = -\frac{1}{\sqrt{L}} \sum_{x,y=1}^{L} \left(e^{i(\theta+ky)} c_x^\dagger \delta_{x+1,y} + e^{i(-\theta+ky)} c_{x+1}^\dagger \delta_{x,y} \right) \tag{1.230a}$$

$$= -\frac{1}{\sqrt{L}} \sum_{x=1}^{L} \left(e^{i(\theta+k)} e^{ikx} c_x^\dagger + e^{-i\theta} e^{ikx} c_{x+1}^\dagger \right) \tag{1.230b}$$

$$= -\left(e^{i(\theta+k)} + e^{-i(\theta+k)} \right) a_k^\dagger \tag{1.230c}$$

$$= -2\cos(k+\theta) a_k^\dagger. \tag{1.230d}$$

Therefore we find that

$$H_\theta |\Psi_k\rangle = H_\theta a_{k_1}^\dagger \dots a_{k_N}^\dagger |\Phi_{\text{vac}}\rangle \tag{1.231a}$$

$$= a_{k_1}^\dagger H_\theta a_{k_2}^\dagger \dots a_{k_N}^\dagger |\Phi_{\text{vac}}\rangle - 2\cos(k_1+\theta)|\Psi_k\rangle = \dots \tag{1.231b}$$

$$= -2\sum_{j=1}^{N} \cos(k_j+\theta)|\Psi_k\rangle + a_{k_1}^\dagger \dots a_{k_N}^\dagger H_\theta |\Phi_{\text{vac}}\rangle. \tag{1.231c}$$

Because of $H_\theta|\Phi_{\text{vac}}\rangle = 0$ this shows that $H_\theta|\Psi_k\rangle = E_k|\Psi_k\rangle$ with

$$E_k = -2\sum_{j=1}^{N} \cos(\theta + k_j), \tag{1.232}$$

i.e., $|\Psi_k\rangle$ is an eigenstate of H_θ with eigenvalue E_k. Altogether we have thus shown that the $|\Psi_k\rangle$ form an orthonormal eigenbasis of H_θ.

1.7.2. Thermalization of the Free Fermi Gas in 1D

After having introduced the free, non-relativistic Fermi gas in one dimension in the previous section, we now want to discuss what can be shown concerning the thermalization of this model.

A first rigorous result was obtained by Shiraishi and Tasaki [123] in 2024. They first considered, more generally, a system of N fermions on a lattice Λ with L sites. The dimension of the system's Hilbert space $\mathcal{H}$ is given by $D = \binom{L}{N}$. They fixed the density ρ of the system and chose L and N accordingly, i.e., such that $N/L \simeq \rho$. Note that

$$D = \binom{L}{N} \sim e^{LS(\rho)}, \tag{1.233}$$

where the binomial entropy $S : \mathbb{R} \to \mathbb{R}$ is defined by

$$S(p) = -p \log p - (1-p)\log(1-p). \tag{1.234}$$

Moreover, the relation $a(L) \sim b(L)$ means that

$$\lim_{L\uparrow\infty} \frac{1}{L} \log \frac{a(L)}{b(L)} = 0. \tag{1.235}$$

Suppose that the lattice Λ is decomposed into two parts $\Lambda = \Lambda_1 \dot{\cup} \Lambda_2$, where $\dot{\cup}$ denotes the disjoint union, such that $|\Lambda_1| = |\Lambda_2| = L/2$ if L is even and $|\Lambda_1| = (L-1)/2$ and $|\Lambda_2| = (L+1)/2$ if L is odd. Let H be the system's Hamiltonian and let $\Psi_1, \dots, \Psi_D$ be its eigenstates with corresponding eigenvalues $E_1, \dots, E_D$. Shiraishi and Tasaki [123] made the following two assumptions:

Assumption 1. The energy eigenvalues $E_1, \dots, E_D$ of H are non-degenerate.

Assumption 2. The energy eigenstates $\Psi_1, \dots, \Psi_D$ satisfy

$$\langle \Psi_j | P_1 | \Psi_j \rangle \leq 2^{-N} \quad \forall j = 1, \dots, D, \tag{1.236}$$

where P_1 denotes the projection to the subspace $\mathcal{H}_1 \subset \mathcal{H}$ where all particles are in the sublattice Λ_1.

These two assumptions are natural and it is expected that they are often fulfilled. Assumption 1 is believed to usually be satisfied if there are, e.g., no symmetries which cause degeneracy of the eigenvalues. However, in case of degeneracies, it should be possible to lift them by a small perturbation (in fact, it can be shown that adding a small random perturbation with a continuous distribution lifts such degeneracies with probability 1, see, e.g., [138]). Note also that as it is shown in Appendix A of [123], the results remain valid for models with degenerate energy eigenvalues as long as the degeneracy is not too large.

Assumption 2 is similar to an ETH for P_1, however, we only have here an inequality rather than an equality. The left-hand side in (1.236) is the probability in the state $|\Psi_j\rangle$ that all particles are in the sublattice Λ_1; this probability would be equal to 2^{-N} if each particle would be put in Λ_1 or Λ_2 with probability 1/2. Shiraishi and Tasaki [123] argue that many quantum lattice gases should fulfill this assumption but they can only prove it for a few special examples, namely for a class of non-interacting fermions, including free fermions on a one-dimensional lattice, and systems of interacting fermions or hardcore bosons on a double lattice with a special symmetry.

As we see below, Assumption 2 ensures that for most initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$ the effective dimension d_{eff} is large provided that N (and L) are sufficiently large and the density ρ is small. Recall that

$$d_{\text{eff}} = \left(\sum_{j=1}^{D} |\langle \psi_0 | \Psi_j \rangle|^4 \right)^{-1}. \tag{1.237}$$

In order to state the result concerning the effective dimension, we have to introduce the notion of $(1-\varepsilon)$-most wave function for $\varepsilon > 0$:

Definition 1.38 (Most wave functions)**.** Let u be the uniform distribution on $\mathbb{S}(\mathcal{H})$ and let $\varepsilon > 0$. A statement $s(\psi)$ is true for $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$ if

$$u\Big\{ \psi \in \mathbb{S}(\mathcal{H}) : s(\psi) \text{ holds} \Big\} \geq 1 - \varepsilon. \tag{1.238}$$

Theorem 1.39 (Shiraishi, Tasaki (2024) [123])**.** *Suppose that Assumption 2 is valid and let $\rho \leq 1/5$. Then, for sufficiently large N (and L), $(1 - e^{-(\rho/3)N})$-most initial wave functions $\psi_0 \in \mathbb{S}(P_1\mathcal{H})$ are such that*

$$\frac{D}{d_{\text{eff}}} \leq e^{\rho N}. \tag{1.239}$$

For any subset $\Gamma \subset \Lambda$ we define

$$N_\Gamma := \sum_{x \in \Gamma} c_x^\dagger c_x, \tag{1.240}$$

i.e., N_Γ is the number operator of the particles in the sublattice Γ. With the help of Theorem 1.39, Shiraishi and Tasaki [123] showed the following theorem:

Theorem 1.40 (Shiraishi, Tasaki (2024) [123])**.** *Consider a system of N particles on a lattice Λ with L sites and let $\mathcal{H}$ be its Hilbert space and H its Hamiltonian. Let $\rho = N/L \leq 1/5$ and suppose that Assumption 1 and Assumption 2 are satisfied. Let $\mu := |\Gamma|/L$, $\varepsilon_0(\rho) := \sqrt{6\mu(1-\mu)\rho}$,*

$$P_{\text{eq}} := \mathbb{1}_{[N(\mu - \varepsilon_0(\rho)), N(\mu + \varepsilon_0(\rho))]}(N_\Gamma) \tag{1.241}$$

and $P_{\text{neq}} := 1 - P_{\text{eq}}$. Then, for sufficiently large N (and L), for $(1 - e^{-(\rho/3)N})$-most $\psi_0 \in \mathbb{S}(P_1\mathcal{H})$ there exists a $T > 0$ such that for $(1 - e^{-(\rho/4)N})$-most $t \in [0, T]$,

$$\|P_{\text{neq}} \psi_t\|^2 \leq e^{-(\rho/4)N}. \tag{1.242}$$

Theorem 1.40 shows that if the eigenvalues of the Hamiltonian are non-degenerate, its energy eigenstates fulfill (1.236), N and L are sufficiently large and the density ρ is not too large, then for typical initial states (in which all particles are in the sublattice Λ_1) and typical times $t \in [0, T]$, the measurement result of N_Γ in the state $|\psi_t\rangle$ is close to the equilibrium value $N\mu$. Note that Theorem 1.40 only states the existence of such a time $T > 0$, but no estimates on the time scale of thermalization are given. Moreover, note that the notion of thermal equilibrium used in Theorem 1.40 is a highly simplified one as it only refers to the spatial distribution of the particles.

After having proved Theorem 1.40 for general lattice gases, Shiraishi and Tasaki [123] proceed to show that Assumption 1 and Assumption 2 are fulfilled for the free, non-relativistic Fermi gas in one dimension introduced in Section 1.7.1. The proof of Assumption 2 in this setting is rather simple and uses similar ideas as the ones used in the proof of Lemma 1.44. However, the proof of Assumption 1 is more involved and makes use of the following result from number theory:

Lemma 1.41. *Let L be an odd prime and let $m_1, \dots, m_{L-1} \in \mathbb{Z}$ such that $m_j \neq 0$ for some j. Then,*

$$\left| \sum_{k=1}^{L-1} m_k e^{2\pi i k/L} \right| \geq \left(\sum_{k=1}^{L-1} |m_k| \right)^{-(L-3)/2}. \tag{1.243}$$

The proof of Lemma 1.41 can be found in [123]. With the help of this lemma, Shiraishi and Tasaki [123] prove that for sufficiently small $|\theta|$, the Hamiltonian H_θ of the free, non-relativistic Fermi gas is non-degenerate:

Theorem 1.42 (Shiraishi, Tasaki (2024) [123])**.** *Let L be an odd prime and let $N \in \mathbb{N}$ with $0 < N \leq L$. For any $\theta \in \mathbb{R}$ such that*

$$0 < |\theta| \leq \frac{1}{(4N + 2L)^{(L-1)/2}} \tag{1.244}$$

the Hamiltonian H_θ is non-degenerate.

The idea of proof of Theorem 1.42 is to first show that for $\theta = 0$ there are only *trivial* degeneracies, i.e., all degeneracies of an eigenvalue E_k of H_θ with $k = (k_1, \dots, k_N) \in \mathcal{K}^N$ and $k_j < k_{j+1}$ for all $j = 1, \dots, N-1$ are due to changing signs of the components k_i. Then it is shown that a non-zero $\theta \in [0, 2\pi)$, $\theta \neq \pi$, lifts these trivial degeneracies and by making use of Lemma 1.41 it is proved that if $|\theta|$ is sufficiently small then no additional degeneracies are generated. Note that the maximum degeneracy in the model with $\theta = 0$ is 2^N and therefore too large to, e.g., apply the results from Appendix A in [123] concerning degenerate models.

The authors conclude that Theorem 1.40 can be applied to the free, non-relativistic Fermi gas in one dimension on a chain whose length is an odd prime if $|\theta| > 0$ is sufficiently small. Thus they established the thermalization of this model (in the restricted sense that it only refers to the spatial distribution of particles) for typical non-equilibrium initial states from $P_1\mathcal{H}$. However, it remained open whether thermalization also holds for any initial state, for the model with $\theta = 0$ (i.e., the "standard" Hamiltonian of the free, non-relativistic Fermi gas) and for the model in higher dimensions.

Shortly after the first appearance of the paper by Shiraishi and Tasaki [123], the result concerning the thermalization of the free, non-relativistic Fermi gas in one dimension was improved by Tasaki [134]. His result is valid for any initial state and he formulated it for a partition of the chain into intervals of equal length. If the initial state has a non-uniform density, it shows the thermalization in the sense that at a sufficiently large and typical time the system's particle density is approximately uniform.

More precisely, Tasaki [134] considered the free, non-relativistic Fermi gas of N fermions on a chain whose length L is an odd prime in order to ensure that, by Theorem 1.42, the Hamiltonian H_θ is non-degenerate. Let $m \in \mathbb{N}$, $m \geq 2$, be of order one and let $I_1, \dots, I_m \subset \Lambda$ be disjoint intervals each containing $\ell = \lfloor \frac{L}{m} \rfloor$ sites of the chain. The union $\bigcup_{j=1}^m I_j$ of the intervals I_j almost covers Λ but it is not equal to the whole chain because $L/m \notin \mathbb{N}$ as L is an odd prime. For $j = 1, \dots, m$ we define

$$N_j := N_{I_j} = \sum_{x \in I_j} c_x^\dagger c_x, \tag{1.245}$$

i.e., N_j is the number operator of the particles in the interval I_j. Moreover, we define $\rho_0 := N/L$. Let $\delta \in (0, \frac{1}{2}]$ and

$$P_{\mathrm{eq}} := \prod_{j=1}^m \mathbb{1}_{((1-\delta)\rho_0\ell,(1+\delta)\rho_0\ell)}(N_j), \tag{1.246}$$

i.e., we say that the system is in thermal equilibrium if the measurement of each N_j yields a value close to the equilibrium value $\rho_0\ell$. Put another way, a system is in thermal equilibrium if the coarse-grained particle distribution defined by the N_j is approximately uniform.

Tasaki [134] showed the following theorem:

Theorem 1.43 (Tasaki (2024) [134])**.** *Consider the free, non-relativistic Fermi gas of N particles on a chain $\Lambda = \{1, \dots, L\}$, where L is an odd prime. Let $\delta, m,$ and P_{eq} be as above and let $P_{\mathrm{neq}} := 1 - P_{\mathrm{eq}}$. Then for every initial state $\psi_0 \in \mathbb{S}(\mathcal{H})$ there exists a $T > 0$ such that for $(1 - e^{-\frac{\delta^2}{8(m-1)}N})$-most $t \in [0, T]$,*

$$\|P_{\mathrm{neq}}\psi_t\|^2 \leq e^{-\frac{\delta^2}{8(m-1)}N}. \tag{1.247}$$

Thus, if N is sufficiently large and δ is small but independent of N, every initial state approaches thermal equilibrium and stays there for most of the time. However, as before, the result does not give us any estimate on the time scale of thermalization.

The key result needed for the proof of Theorem 1.43 is an eigenstate thermalization hypothesis (ETH) for the eigenstates $|\Psi_k\rangle$ of H_θ:

Lemma 1.44 (Tasaki (2024) [134])**.** *Consider the free, non-relativistic Fermi gas with N particles on a chain $\Lambda = \{1, \ldots, L\}$ as in Section 1.7.1 and let P_{neq} be as in Theorem 1.43. Then the eigenstates $|\Psi_k\rangle$ of the Hamiltonian H_θ fulfill*

$$\|P_{\mathrm{neq}}\Psi_k\|^2 \leq e^{-\frac{\delta^2}{3(m-1)}N}. \tag{1.248}$$

Adopting the language of Section 1.6, Lemma 1.44 tells us that all eigenstates of the Hamiltonian H_θ are in $\mathrm{MATE}_\varepsilon$ with $\varepsilon = e^{-\frac{\delta^2}{3(m-1)}N}$. Then it follows from Theorem 1.34 that for most times $t \in [0, \infty)$, the time-evolved state ψ_t is in macroscopic thermal equilibrium, $\psi_t \in \mathrm{MATE}_{\sqrt{\varepsilon}}$, and this immediately implies Theorem 1.43. As we will see in Section 3.3, Lemma 1.44 generalizes to higher dimensions and therefore we now sketch its proof closely following [134].

Proof of Lemma 1.44. The key step of the proof is to show that for any eigenstate $|\Psi_k\rangle$ of H_θ with $k = (k_1, \ldots, k_N) \in \mathcal{K}^N$ and $k_i < k_{i+1}$ for $i = 1, \ldots, N-1$, and for any $j = 1, \ldots, m$,

$$\langle\Psi_k|e^{\lambda N_j}|\Psi_k\rangle \leq \left(\mu e^\lambda + (1-\mu)\right)^N, \tag{1.249}$$

where $\lambda \in (0, 1]$ and $\mu = \ell/L \simeq 1/m$. With the help of (1.249) it follows that

$$\langle\Psi_k|\mathbb{1}_{[(1+\delta)\rho_0\ell,\infty)}(N_j)|\Psi_k\rangle \leq \langle\Psi_k|e^{\lambda(N_j-\mu N(1+\delta))}|\Psi_k\rangle \leq \left(g(\lambda,\mu)e^{-\lambda\mu\delta}\right)^N, \tag{1.250}$$

where

$$g(\lambda,\mu) := \left(\mu e^\lambda + (1-\mu)\right)e^{-\lambda\mu}. \tag{1.251}$$

Tasaki [134] then proceeds to estimate $g(\lambda, \mu)$, chooses λ in a suitable way and arrives at

$$\langle\Psi_k|\mathbb{1}_{([(1+\delta)\rho_0\ell,\infty)}(N_j)|\Psi_k\rangle \leq e^{-\frac{\mu\delta^2}{4(e-2)(1-\mu)}N} \tag{1.252}$$

and similarly

$$\langle\Psi_k|\mathbb{1}_{(-\infty,(1-\delta)\rho_0\ell]}(N_j)|\Psi_k\rangle \leq e^{-\frac{\mu\delta^2}{4(e-2)(1-\mu)}N}. \tag{1.253}$$

By making use of

$$P_{\mathrm{neq}} \leq \sum_{j=1}^m \mathbb{1}_{\mathbb{R}\setminus((1-\delta)\rho_0\ell,(1+\delta)\rho_0\ell)}(N_j) \tag{1.254}$$

as well as $4(e-2) \simeq 2.9$ and $\mu \geq \frac{1}{m} - O(\frac{1}{L})$ Tasaki [134] obtains

$$\langle \Psi_k | P_{\mathrm{neq}} | \Psi_k \rangle \leq 2m\, e^{-\frac{\mu\delta^2}{4(e-2)(1-\mu)}N} \leq e^{-\frac{\delta^2}{3(m-1)}N}, \tag{1.255}$$

see [134] for the details of the derivation of (1.255) from (1.249).

We now turn to the derivation of (1.249). To this end let $j \in \{1, \dots, m\}$, $k \in \mathcal{K}$, and define the operators

$$b_k^\dagger := \frac{1}{\sqrt{L}} \left(e^{\lambda/2} \sum_{x \in I_j} e^{ikx} c_x^\dagger + \sum_{x \in \Lambda \backslash I_j} e^{ikx} c_x^\dagger \right). \tag{1.256}$$

Let $|\Psi_k\rangle$ be an eigenstate of H_θ. Then we can write

$$e^{\lambda N_j/2} |\Psi_k\rangle = b_{k_1}^\dagger b_{k_2}^\dagger \dots b_{k_N}^\dagger |\Phi_{\mathrm{vac}}\rangle \tag{1.257}$$

and obtain

$$\langle \Psi_k | e^{\lambda N_j} | \Psi_k \rangle = \langle \Phi_{\mathrm{vac}} | b_{k_N} \dots b_{k_1} b_{k_1}^\dagger \dots b_{k_N}^\dagger | \Phi_{\mathrm{vac}} \rangle \leq \prod_{j=1}^N \| b_{k_j} b_{k_j}^\dagger \|. \tag{1.258}$$

Since the operators $b_{k_j} b_{k_j}^\dagger$ are self-adjoint and positive, $\| b_{k_j} b_{k_j}^\dagger \|$ equals the largest eigenvalue of $b_{k_j} b_{k_j}^\dagger$. We compute

$$\begin{aligned}
\{b_{k_j}, b_{k_j}^\dagger\} = \frac{1}{L} \Bigg(& e^\lambda \sum_{x,y \in I_j} e^{ik_j(x-y)} \{c_y, c_x^\dagger\} + e^{\lambda/2} \sum_{\substack{x \in I_j \\ y \in \Lambda \backslash I_j}} e^{ik_j(x-y)} \{c_y, c_x^\dagger\} \\
& + e^{\lambda/2} \sum_{\substack{x \in \Lambda \backslash I_j \\ y \in I_j}} e^{ik_j(x-y)} \{c_y, c_x^\dagger\} + \sum_{x,y \in \Lambda \backslash I_j} e^{ik_j(x-y)} \{c_y, c_x^\dagger\} \Bigg) && \text{(1.259a)} \\
= \frac{1}{L} & \left(e^\lambda |I_j| + |\Lambda \backslash I_j| \right) = \mu e^\lambda + (1-\mu). && \text{(1.259b)}
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
b_{k_j}^2 = \frac{1}{L} \Bigg(& e^\lambda \sum_{x,y \in I_j} e^{-ik_j(x+y)} c_x c_y + e^{\lambda/2} \sum_{\substack{x \in I_j \\ y \in \Lambda \backslash I_j}} e^{-ik_j(x+y)} c_x c_y \\
& + e^{\lambda/2} \sum_{\substack{x \in \Lambda \backslash I_j \\ y \in I_j}} e^{-ik_j(x+y)} c_x c_y + \sum_{x,y \in \Lambda \backslash I_j} e^{-ik_j(x+y)} c_x c_y \Bigg).
\end{aligned} \tag{1.260}$$

Because of $c_x c_y = -c_y c_x$ we immediately see that the second and third sum cancel each other. In the first and fourth sum the terms with $x = y$ are equal to zero as $c_x^2 = 0$. Moreover, in these sums for every $x \neq y$ the terms $c_x c_y$ and $c_y c_x$ both appear with the same prefactor and therefore cancel each other which results in these two sums being equal to zero. Altogether we thus find that $b_{k_j}^2 = 0$.

Suppose that $|\chi\rangle$ is an eigenstate of $b_{k_j} b_{k_j}^\dagger$. Then,

$$b_{k_j} b_{k_j}^\dagger b_{k_j} b_{k_j}^\dagger |\chi\rangle = \{b_{k_j}, b_{k_j}^\dagger\} b_{k_j} b_{k_j}^\dagger |\chi\rangle - b_{k_j}^2 (b_{k_j}^\dagger)^2 |\chi\rangle = \left(\mu e^\lambda + (1-\mu)\right) b_{k_j} b_{k_j}^\dagger |\chi\rangle \tag{1.261}$$

and therefore we either have that $b_{k_j} b_{k_j}^\dagger |\chi\rangle = 0$, i.e., $|\chi\rangle$ is in the kernel of $b_{k_j} b_{k_j}^\dagger$, or the eigenvalue corresponding to $|\chi\rangle$ is given by $\mu e^\lambda + (1-\mu)$. From this we conclude that

$$\|b_{k_j} b_{k_j}^\dagger\| = \mu e^\lambda + (1-\mu) \tag{1.262}$$

and with this (1.249) is an immediate consequence of (1.258). □

We remark that in a later paper, Tasaki [133] studied the thermalization of a toy model for a weakly heat-conducting solid in one dimension which is equivalent to the free fermion chain. The Hamiltonian $\tilde{H}$ considered in [133] is slightly different than the one used in this section, more precisely,

$$\tilde{H} = \varepsilon_0 \sum_{x=1}^{L} c_x^\dagger c_x + \frac{\eta \varepsilon_0}{2} \left(\sum_{x=1}^{L-1} (c_x^\dagger c_{x+1} + c_{x+1}^\dagger c_x) - c_L^\dagger c_L \right), \tag{1.263}$$

where $\varepsilon_0, \eta > 0$. As it is shown in Lemma 2.3 in [133], the eigenvalues of $\tilde{H}$ are non-degenerate if $2L+1$ is prime and η^{-1} is an integer which together with an ETH in Lemma 2.4 in [133] proves the approach to MATE as before. Note that while the parameter θ of H_θ had to be "fine-tuned" in dependence of L and N to ensure that H_θ is free of degeneracies, no such fine-tuning is needed for $\tilde{H}$; here it suffices that $\eta^{-1} \in \mathbb{N}$.[18] Thus, the results in [133] can be seen as a variation and improvement of [123, 134].

We finally note that analogous questions for the classical free gas of N particles have been studied in the literature, see, e.g., [10, 11, 24]. In contrast to the quantum case, all the results only hold for most initial states and obviously one cannot expect thermalization for all initial states of a classical system.[19] Note that in [11, 24] also thermalization times were investigated and rather brief ones were obtained.

[18] Note that a similar result concerning the non-degeneracy of the eigenvalues of the Hamiltonian of certain spin systems has been obtained in [68]. For the proof the authors also make use of some number-theoretic results.

[19] Take, e.g., an initial phase point in which all particles are concentrated in a small subregion of

1.8. Some Results from Random Matrix Theory

This section is devoted to a few results from random matrix theory that were used in one of our papers discussed in Section 3.2 and attached to this thesis. Inspired by the works of Wigner [146, 147], the research in random matrix theory has been largely growing over the last decades and a variety of properties of different classes of random matrices were studied. For an introduction to random matrix theory, see, e.g., [87, 3, 37].

The two properties of random matrices that we discuss in the following are the *delocalization* of eigenvectors of random matrices and the *eigenstate thermalization hypothesis* (ETH). Roughly speaking, an eigenvector is delocalized if its components are approximately of equal size. In recent years, the delocalization of eigenvectors of different classes of random matrices has been studied intensively in the literature, see, e.g., [35, 36, 32, 31, 38, 72, 119, 1, 33, 16, 148, 149, 30]. Most of these results were concerned with delocalization in the sup norm, i.e., with delocalization in the sense that with high probability no component of an eigenvector is too large. We discuss one result of this kind from Ajanki, Erdős, and Krüger [1] in Section 1.8.1. For our purposes, however, another kind of delocalization of eigenvectors turned out to be extremely useful, namely, the *no-gaps delocalization* discovered by Rudelson and Vershynin [120]. Their result shows that with high probability there are no large "gaps" in the eigenvectors meaning that no significant fraction of the components of an eigenvector can carry only a negligible fraction of its mass. This result is described in Section 1.8.2. Finally, in Section 1.8.3, we discuss the ETH for Wigner matrices due to Cipolloni, Erdős, and Henheik [20] and related results.

1.8.1. Delocalization in the Sup Norm

Ajanki, Erdős, and Krüger [1] proved a result about the delocalization of the eigenvectors of general Wigner-type matrices in the sup norm, i.e., they showed that with high probability the components of the eigenvectors cannot become too large.

We start with describing the matrices for which their theorem holds. Let $H^{(D)} \in \mathbb{C}^{D\times D}$ be a sequence of self-adjoint random matrices with distributions $\mathbb{P}^{(D)}$. In the following we often suppress the dependence of quantities on the dimension D. We say that a random matrix $H = (h_{ij})$ is of *Wigner-type* if its entries h_{ij} are independent for $i \leq j$ and

$$\mathbb{E}h_{ij} = 0 \qquad \forall i, j = 1, \dots, D, \tag{1.264}$$

i.e., the h_{ij} are centered.

the available volume and all velocities are equal to zero. As there are no external forces, the state is invariant under the time evolution and therefore never approaches a state with a uniform particle distribution and a Maxwellian velocity distribution, i.e., this system never thermalizes.

The matrix of variances $S = (\sigma_{ij}^2)$ is defined by

$$\sigma_{ij}^2 := \mathbb{E}|h_{ij}|^2 \qquad \forall i, j = 1, \dots, D. \tag{1.265}$$

Note that the matrix S is symmetric and its entries are non-negative. The corresponding *quadratic vector equation* (QVE) is given by

$$-\frac{1}{m_i(z)} = z + \sum_{j=1}^{D} \sigma_{ij}^2 m_j(z) \qquad \forall i = 1, \dots, D \text{ and } z \in \mathbb{H}, \tag{1.266}$$

where $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the complex upper half plane and $m = (m_1, \dots, m_D) : \mathbb{H} \to \mathbb{H}^D$. It can be shown that the QVE (on the complex upper half-plane) has a unique solution, see Theorem 2.1 in [2]. The QVE now plays a central role in random matrix theory and a detailed study of this equation can be found in [2], see also the references therein.

Let $p, P > 0$ and $L \in \mathbb{N}$ be parameters independent of the dimension D and let $\mu = (\mu_1, \mu_2, \dots)$ be a sequence of non-negative real numbers. For all D we make the following four assumptions:

Assumption 1 : The matrix S is *flat*, i.e.,

$$\sigma_{ij}^2 \leq \frac{1}{D} \qquad \forall i, j = 1, \dots, D. \tag{1.267}$$

Assumption 2 : The matrix S is *uniformly primitive*, i.e.,

$$(S^L)_{ij} \geq \frac{p}{D} \qquad \forall i, j = 1, \dots, D. \tag{1.268}$$

Assumption 3 : The unique solution m of the QVE (1.266) induced by the matrix S is bounded,

$$|m_i(z)| \leq P \qquad \forall i = 1, \dots, D \text{ and } z \in \mathbb{H}. \tag{1.269}$$

Assumption 4 : The entries h_{ij} of H have *bounded moments*,

$$\mathbb{E}|h_{ij}|^k \leq \mu_k \sigma_{ij}^k \qquad \forall k \in \mathbb{N} \text{ and } i, j = 1, \dots, D. \tag{1.270}$$

Note that sufficient conditions for Assumption 3 can be found in Theorem 6.1 in [2]. A simple example for which all assumptions are fulfilled is a suitably rescaled *Gaussian* *unitary ensemble* (GUE) which is defined as follows: Let h_{jk} for $j \leq k$ be independent random variables, let $h_{kj} = h_{jk}^*$, let h_{jk} for $j < k$ be standard complex Gaussian random variables, i.e., $h_{jk} \sim \mathcal{N}(0, 1/2) + i\mathcal{N}(0, 1/2)$ and let $h_{jj} \sim \mathcal{N}(0, 1)$ for all j. Then the matrix $H = (h_{jk})$ is a GUE matrix[20] and $H/\sqrt{D}$ satisfies Assumption 1-4. In this case, Assumption 1, Assumption 2 as well as Assumption 4 are clearly fulfilled. The unique solution of the QVE on the complex upper half plane is given by the semicircle, i.e., $m_i(z) = m_{\mathrm{sc}}(z) = (-z + \sqrt{z^2 - 4})/2$ for all $i = 1, \ldots, D$ and therefore bounded by 1 such that Assumption 3 is also satisfied.

Before we can state the eigenvector delocalization result from [1], we have to introduce the notion of *stochastic domination*:

Definition 1.45 (Stochastic domination)**.** Let $X = (X^{(D)})_{D\in\mathbb{N}}$ and $Y = (Y^{(D)})_{D\in\mathbb{N}}$ be two sequences of non-negative random variables. We say that X is *stochastically dominated* by Y and write $X \prec Y$ if and only if there exists a function $D_0 : (0, \infty)^2 \to \mathbb{N}$ such that for all $\varepsilon > 0$ and $\alpha > 0$,

$$\mathbb{P}^{(D)}\left(X^{(D)} > D^{\varepsilon} Y^{(D)}\right) \leq D^{-\alpha}, \quad D \geq D_0(\varepsilon, \alpha). \tag{1.271}$$

Roughly speaking, a sequence X of random variables is stochastically dominated by a sequence Y if for large D, with high probability, X is not “much” larger than Y.

With these preparations we are now ready to state the delocalization result by Ajanki, Erdős, and Krüger (Corollary 1.14 in [1]):

Theorem 1.46 (Ajanki, Erdős, Krüger (2017) [1])**.** *Let $(H^{(D)})_{D\in\mathbb{N}}$ be a sequence of random Hermitian $D \times D$ matrices with corresponding distributions $(\mathbb{P}^{(D)})_{D\in\mathbb{N}}$, i.e., $H^{(D)} \sim \mathbb{P}^{(D)}$ for every D. Suppose that Assumption 1-4 are satisfied. Let $E_1^{(D)} \leq \cdots \leq E_D^{(D)}$ be the eigenvalues of $H^{(D)}$ and for $n \in \{1, \ldots, D\}$ let $\phi_n^{(D)} \in \mathbb{C}^D$ be a normalized eigenvector of $H^{(D)}$ with eigenvalue $E_n^{(D)}$. Then for every sequence $(b^{(D)})_{D\in\mathbb{N}}$ of vectors $b^{(D)} \in \mathbb{C}^D$ and every sequence $(n^{(D)})_{D\in\mathbb{N}}$ with $n^{(D)} \in \{1, \ldots, D\}$,*

$$\left|\left\langle b^{(D)} \middle| \phi_{n^{(D)}}^{(D)} \right\rangle\right| \prec \frac{1}{\sqrt{D}}. \tag{1.272}$$

In particular, the eigenvectors are completely delocalized, i.e.,

$$\left\|\phi_{n^{(D)}}^{(D)}\right\|_{\infty} \prec \frac{1}{\sqrt{D}}. \tag{1.273}$$

The function $D_0 : (0, \infty)^2 \to \mathbb{N}$ implicit in the symbol $\prec$ depends only on the constants p, P, L and $(\mu_k)_{k\in\mathbb{N}}$ from Assumption 1-4.

[20]Note that the word “unitary” in GUE refers to the fact that the distribution of H is invariant under conjugation with unitaries. Moreover, note that the analogue for real-valued matrices is called the *Gaussian orthogonal ensemble* (GOE).

Theorem 1.46 shows that the eigenvectors of random matrices satisfying Assumption 1-4 are delocalized in the sense that if the dimension D is sufficiently large the absolute value of the largest entry of an eigenvector is, with high probability, not much larger than $1/\sqrt{D}$.

1.8.2. No-Gaps Delocalization

While most results concerning the delocalization of the eigenvectors of random matrices are concerned with the sup norm, Rudelson and Vershynin [120] studied the phenomenon of *no-gaps delocalization*. Rougly speaking, a $D \times D$ random matrix exhibits no-gaps delocalization if with high probability for every eigenvector $\phi \in \mathbb{C}^D$ and any subset $I \subset \{1, \ldots, D\}$ of size at least κD, where $\kappa \in (0, 1)$, we have that

$$\|\phi_I\| := \left(\sum_{j \in I} |\phi(j)|^2 \right)^{1/2} \geq f(\kappa) \|\phi\|, \tag{1.274}$$

where $\phi(j)$ denotes the jth component of ϕ and $f : (0, 1) \to (0, 1)$ is a “nice” function. As Rudelson and Vershynin [120] point out, delocalization in the sup norm does not imply no-gaps delocalization and vice versa. Thus they are different aspects of the delocalization of eigenvectors, one ensures that (with high probability) there are no “peaks” in the components of the eigenvectors while the other one rules out gaps.

We make the following assumptions on the $D \times D$ random matrix $H = (h_{ij})$:

Assumption 1 : For any $i, j \in \{1, \ldots, D\}$, the entry h_{ij} is independent of the other matrix entries except possibly h_{ji}. Moreover, the real part of H is random and the imaginary part is fixed.

Assumption 2 : The real parts of the matrix entries of H have densities which are bounded by a number $K \geq 1$.

Moreover, for any random $D \times D$ matrix H and any number $J > 0$ we define the “boundedness event”

$$\mathcal{B}_{H,J} := \left\{ \|H\| \leq J\sqrt{D} \right\}. \tag{1.275}$$

For example if the entries of H are centered, have bounded fourth moments and D is sufficiently large, then the boundedness event $\mathcal{B}_{H,J}$ occurs with high probability; this is a consequence of a result by Latala [69].

In the original statement by Rudelson and Vershynin [120] some constants were incorrect; they were corrected in [138] and in the following we state this corrected version of the no-gaps delocalization theorem.

Theorem 1.47 (Rudelson, Vershynin (2016) [120])**.** *Let H be a $D \times D$ random matrix satisfying Assumption 1 and Assumption 2. Choose $J \geq 1$ such that the boundedness event $\mathcal{B}_{H,J}$ holds with probability at least $1/2$. Let $\kappa \in (180/D, 1/2)$ and $s > 0$. Then, conditionally on $\mathcal{B}_{H,J}$, the following holds with probability at least $1 - (Cs)^{\kappa D}$: Every eigenvector ϕ of H satisfies*

$$\|\phi_I\| \geq (\kappa s)^9 \|\phi\| \quad \textit{for all } I \subset \{1, \dots, D\} \textit{ with } |I| \geq \kappa D, \tag{1.276}$$

where $C = C(K, J) \geq 1$.

Note that Rudelson and Vershynin [120] also prove a no-gaps delocalization result for random matrices whose entries follow a discrete distribution satisfying certain assumptions. However, the proof of this statement is much more challenging than the one in the continuous case.

Moreover, we remark that the bounds obtained by Rudelson and Versyhnin in [120] have been improved in the case that all matrix entries are independent, see [80, 81]. But as we are interested in random matrices which can model the Hamiltonian of a system and therefore in Hermitian matrices, we cannot apply these improved results in our situation.

1.8.3. The Eigenstate Thermalization Hypothesis

In this section we discuss a version of the eigenstate thermalization hypothesis (ETH) for Wigner matrices due to Cipolloni, Erdős, and Henheik [20]. The ETH comes from the physics literature and goes back to Deutsch (1991) [25] and Srednicki (1994) [127]. A first rigorous proof for Wigner matrices was given by Cipolloni, Erdős, and Schröder [22] in 2021 and the result was later improved by Cipolloni, Erdős, and Henheik in [20].

We assume that $H = (h_{ij})$ is a $D \times D$ Wigner matrix with bounded moments. More precisely, we make the following assumption on the matrix entries of H:

Assumption 1: The matrix elements h_{ij} are centered random variables that are independent up to Hermitian symmetry. The off-diagonal elements h_{ij} with $i < j$ are identically distributed and also the diagonal entries h_{ii} are identically distributed (with a possibly different distribution than the off-diagonal entries) and real. Moreover, we assume that $\mathbb{E}|h_{ij}|^2 = 1$ and $|\mathbb{E}h_{ij}^2| < 1$ for all $i < j$. Additionally, we assume that for any $p \in \mathbb{N}$ there exists a constant $C_p > 0$ such that

$$\mathbb{E}|h_{11}|^p + \mathbb{E}|h_{12}|^p \leq C_p D^{p/2}. \tag{1.277}$$

Theorem 1.48 (Cipolloni, Erdős, Henheik (2023) [20]). *Let H be a Wigner matrix satisfying Assumption 1, let $\phi_1, \dots, \phi_D$ be an orthonormal eigenbasis of H and let $A \in \mathbb{C}^{D \times D}$ be deterministic. Then,*

$$\max_{i,j} \left| \langle \phi_i | A | \phi_j \rangle - \delta_{ij} \langle A \rangle \right| \prec \frac{\langle |\mathring{A}|^2 \rangle^{1/2}}{\sqrt{D}}, \tag{1.278}$$

where $\langle A \rangle := \operatorname{tr}(A)/D$ and $\mathring{A} := A - \langle A \rangle$ denotes the traceless part of A.

It follows from Theorem 1.48 that if D is sufficiently large and $\langle |\mathring{A}|^2 \rangle$ is not too large, then with high probability the expectation value of A in an eigenstate ϕ_j of H is close to the thermal value $\langle A \rangle$. Moreover, the off-diagonal matrix elements $\langle \phi_i | A | \phi_j \rangle$ of A with $i \neq j$ are close to zero.

We remark that an ETH for Wigner-type matrices has recently been obtained by Erdős and Riabov in [34]. Moreover, for two so-called *deformed Wigner matrices* $H_1 = W + D_1$, $H_2 = W + D_2$, where W is a Wigner matrix and D_1 and D_2 are Hermitian deterministic deformations, a generalization of the ETH for quantities of the form $\langle \phi_i^{(1)} | A | \phi_j^{(2)} \rangle$ where $\phi_i^{(1)}$ and $\phi_j^{(2)}$ are eigenvectors of H_1 and H_2 respectively has been proved recently by Cipolloni, Erdős, Henheik, and Kolupaiev in [21].

2. Objectives

Canonical Typicality for Other Ensembles than Micro-canonical

The objective of [137] was to generalize canonical typicality [79, 45, 100, 58, 101] to GAP measures. More precisely, we wanted to replace the uniform distribution on the sphere in the original statement of canonical typicality by the much more general class of GAP measures as they are not only a natural generalization of the uniform distribution but are also physically very relevant as for certain density matrices they arise as the distribution of wave functions in thermal equilibrium [57, 59].

In Section 1.4 we have seen two rigorous proofs of canonical typicality. One made use of an expression for the variance of $\langle\psi|A|\psi\rangle$ with respect to the uniform distribution on the sphere and led to bounds on the measure of the set of "bad" wave functions that are polynomially small in the dimension of the subspace $\mathcal{H}_R$, see also [130]. The other proof due to Popescu, Short, and Winter [100, 101] was based on Lévy's Lemma [73, 99, 88, 70] and resulted in exponential bounds.

Inspired by these results, our aim was to prove a Lévy Lemma for GAP measures (using ideas similar as in the proof of Lévy's Lemma for the uniform distribution in [88]) and apply it to obtain canonical typicality for GAP measures with bounds exponentially small in $\|\rho\|^{-1}$. Moreover, as a simpler proof, we wanted to generalize bounds on the GAP-variance of $\langle\psi|A|\psi\rangle$ for self-adjoint operators A on a finite-dimensional Hilbert space due to Reimann [106] and use them to show canonical typicality for GAP measures with polynomial bounds.

Time Evolution of Typical Pure States

In this project we wanted to generalize normal typicality [144, 56, 60, 107] to more realistic Hamiltonians and to GAP measures. As the assumption in von Neumann's result [144, 60, 56] that the eigenbasis of the Hamiltonian is uniformly distributed is physically unrealistic, see Section 1.2.3, we wanted to prove a similar result for more general Hamiltonians. In particular, we were interested in a result for Hamiltonians with a band structure in a basis that diagonalizes the projections to the macro spaces $\mathcal{H}_\nu$ as we believed (and saw numerically) that they lead not to immediate thermalization of initial states far from equilibrium but rather the states go through larger and larger macro spaces until they finally reach the largest one, the thermal equilibrium macro space.

It turned out that we can obtain a generalization of normal typicality to arbitrary Hamiltonians and most initial states from any macro space $\mathcal{H}_\mu$ as long as we are only concerned with absolute errors [136]. However, the superposition weights $\|P_\nu \psi_t\|^2$ can be very small themselves and then small absolute errors are not meaningful anymore. Therefore, the goal of [138] was to also obtain bounds for the relative errors. The idea for the proof was to model the Hamiltonian by a suitable random (band) matrix and make use of the delocalization of eigenvectors of random matrices in the form of no-gaps delocalization due to Rudelson and Vershynin [120].

Because of the physical relevance and naturalness of GAP measures, the objective of [143] was to generalize (generalized) normal typicality to this class of measures. The main ingredient for the proof was a further improvement of the bounds on the GAP-variance of $\langle\psi|A|\psi\rangle$ from [106, 137].

Macroscopic Thermalization for Highly Degenerate Hamiltonians After Slight Perturbation

The objective of [117] was to better understand the thermalization of systems of free fermions. Only recently, Shiraishi and Tasaki [123] and Tasaki [134, 133] studied the thermalization (with respect to a coarse grained position distribution) of the free, non-relativistic Fermi gas in one dimension. Shiraishi and Tasaki [123] slightly modified the standard Hamiltonian and proved that this modification leads to the non-degeneracy of the Hamiltonian's eigenvalues. Tasaki [134] showed an ETH for the Hamiltonian's eigenstates and concluded that every initial state thermalizes.

After having read these papers, we mainly had two questions:

1. Can we also prove thermalization for the unmodified Hamiltonian of the free, non-relativistic Fermi gas in one dimension? In this case, the energy eigenvalues are highly degenerate and while an ETH still holds for a special eigenbasis, it might not hold for all eigenvectors of the Hamiltonian.

2. Can we prove thermalization of the free, non-relativistic Fermi gas also in higher dimensions?

Our aim was to prove an ETH for all eigenstates of the unmodified Hamiltonian from which the thermalization of every initial state follows. We succeeded in doing so in the one-dimensional setting, in higher dimensions, however, we were only able to show that there is one eigenbasis whose eigenvectors are in MATE. For highly degenerate Hamiltonians (as the one of the free, non-relativistic Fermi gas) this leaves open the possibility that the ETH is violated and not every initial state thermalizes. Therefore we wanted to develop a general strategy to prove thermalization for highly degenerate Hamiltonians that have an eigenbasis whose eigenvectors are in MATE and the main idea was to slightly perturb the Hamiltonian by adding a small random perturbation.

3. Results and Discussion

In this chapter, we present and discuss the results obtained in the papers of which this thesis consists and which can be found in the appendix. Section 3.1 is concerned with project [137] which is joint work with Stefan Teufel and Roderich Tumulka. In Section 3.2 we present and discuss the results of the projects [136, 138, 143] which are partially joint work with Stefan Teufel and Roderich Tumulka. In Section 3.3 we report on the project [117] which is joint work with Barbara Roos, Stefan Teufel, and Roderich Tumulka. For the contributions of the authors to the different projects see the page titled "Personal Contributions" at the beginning of this thesis.

3.1. Canonical Typicality for Other Ensembles than Micro-canonical

3.1.1. Results

This section is about the results obtained in [137]. We start with presenting the strongest form of our main result, namely the exponential bounds, and a version of Lévy's Lemma for GAP measures which is essential for the proof of this form of our main result. Then we state several corollaries and end with a slightly weaker version of our main result, the polynomial bounds, whose proof is rather elementary.

In the following we always assume that the Hilbert space $\mathcal{H}$ is separable, i.e., that it has either a finite or a countably infinite orthonormal basis, and we equip the unit sphere $\mathbb{S}(\mathcal{H})$ with the Borel σ-algebra.

Recall that we denote the average of a function f with respect to a measure μ by

$$\mu(f) := \int f(\psi)\, \mu(d\psi). \tag{3.1}$$

Moreover, the expectation and variance with respect to $\mathrm{GAP}(\rho)$ are denoted by $\mathbb{E}_\rho$ and Var_ρ respectively.

Exponential Bounds

We first state our main result in its strongest form (cf. Theorem 1 and Remark 2 in [137]):

Theorem 3.1 (Generalized canonical typicality, exponential bounds)**.** *Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be Hilbert spaces with $\mathcal{H}_a$ having finite dimension d_a and $\mathcal{H}_b$ being separable. Let ρ be a density matrix on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ and let $\varepsilon \geq 0$. Then*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \big\|\rho_a^\psi - \mathrm{tr}_b\,\rho\big\|_{\mathrm{tr}} > \varepsilon\Big\} \leq 12 d_a^2 \exp\left(-\frac{\tilde{C}\varepsilon^2}{d_a^2\|\rho\|}\right), \tag{3.2}$$

where $\rho_a^\psi = \mathrm{tr}_b\,|\psi\rangle\langle\psi|$ and $\tilde{C} = \frac{1}{2304\pi^2}$.

Theorem 3.1 shows that for GAP(ρ)-typical $\psi \in \mathbb{S}(\mathcal{H})$, the reduced density matrix ρ_a^ψ is approximately given by $\mathrm{tr}_b\,\rho$ provided that $\|\rho\|$ is small and d_a is not too large.

Similarly as the Lévy Lemma for the uniform distribution on the unit sphere was the main tool in the proof of canonical typicality by Popescu, Short, and Winter [100, 101], the key ingredient for the proof of Theorem 3.1 is the following version of Lévy's Lemma for GAP measures (cf. Theorem 2 together with Remark 3 in [137]):

Theorem 3.2 (Lévy's Lemma for GAP measures)**.** *Let $\mathcal{H}$ be a separable Hilbert space, let $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$ be a Lipschitz continuous function with Lipschitz constant[21] η, let ρ be a density matrix on $\mathcal{H}$, and let $\varepsilon \geq 0$. Then*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : |f(\psi) - \mathrm{GAP}(\rho)(f)| > \varepsilon\Big\} \leq 12 \exp\left(-\frac{C\varepsilon^2}{\eta^2\|\rho\|}\right), \tag{3.3}$$

where $C = \frac{1}{576\pi^2}$.

It follows from Theorem 3.2 that if $\|\rho\|$ is small, for every Lipschitz continuous function f on $\mathbb{S}(\mathcal{H})$, GAP(ρ)-most $\psi \in \mathbb{S}(\mathcal{H})$ are such that $f(\psi) \approx \mathrm{GAP}(\rho)(f)$.

Some Corollaries

In the following we give some corollaries of Theorem 3.1 and Theorem 3.2.

First, we apply Theorem 3.2 to the function $f(\psi) = \langle\psi|B|\psi\rangle$, where B is a bounded operator on $\mathcal{H}$. Note that by Lemma 5 in [100], $\eta \leq 2\|B\|$. This immediately gives the following corollary (Corollary 1 in [137]):

Corollary 3.3. *Let $\mathcal{H}$ be a separable Hilbert space and let ρ be a density matrix and B a bounded operator on $\mathcal{H}$, let $\tilde{C}$ be as in Theorem 3.1 and let $\varepsilon \geq 0$. Then,*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : |\langle\psi|B|\psi\rangle - \mathrm{tr}(\rho B)| > \varepsilon\Big\} \leq 12 \exp\left(-\frac{\tilde{C}\varepsilon^2}{\|B\|^2\|\rho\|}\right). \tag{3.4}$$

[21] Note that here (as also in [88, 100, 101]) the Lipschitz constant η refers to the spherical metric. However, as the Euclidean metric d_{Eucl} and the spherical metric d_{sph} are equivalent, more precisely, $d_{\mathrm{Eucl}}(\psi,\phi) \leq d_{\mathrm{sph}}(\psi,\phi) \leq \frac{\pi}{2} d_{\mathrm{Eucl}}(\psi,\phi)$, using the Euclidean metric instead of the spherical metric would at most change the Lipschitz constant by a factor of $\pi/2$.

Thus, if $\|\rho\|$ is small and $\|B\|$ is not too large, GAP(ρ)-most $\psi \in \mathbb{S}(\mathcal{H})$ are such that $\langle\psi|B|\psi\rangle \approx \mathrm{tr}(\rho B)$

As the map $\rho \mapsto \mathrm{GAP}(\rho)$ is covariant, i.e., for any unitary operator U on $\mathcal{H}$, $U_*\mathrm{GAP}(\rho) = \mathrm{GAP}(U\rho U^*)$, see Proposition 1.19 (b), the measure GAP(ρ_0) will evolve to GAP(ρ_t) under the unitary time evolution which raises the question how $t \mapsto \psi_t$ looks like. With the help of Theorem 3.2 we showed the following version of "dynamical typicality" (Corollary 2 in [137]):

Corollary 3.4. *Let $\mathcal{H}$ be a separable Hilbert space, ρ a density matrix and B a bounded operator on $\mathcal{H}$. Moreover, let $t \mapsto U_t$ be a measurable family of unitary operators and let $\varepsilon, t \geq 0$. Then*

$$\mathrm{GAP}(\rho)\Big\{\psi_0 \in \mathbb{S}(\mathcal{H}) : |\langle\psi_t|B|\psi_t\rangle - \mathrm{tr}(\rho_t B)| > \varepsilon\Big\} \leq 12\exp\left(-\frac{\tilde{C}\varepsilon^2}{\|B\|^2\|\rho\|}\right), \tag{3.5}$$

where $\rho_t = U_t\rho U_t^*$, $\psi_t = U_t\psi_0$ and $\tilde{C} = \frac{1}{2304\pi^2}$. Moreover, for every $\varepsilon, T > 0$,

$$\mathrm{GAP}(\rho)\left\{\psi_0 \in \mathbb{S}(\mathcal{H}) : \frac{1}{T}\int_0^T |\langle\psi_t|B|\psi_t\rangle - \mathrm{tr}(\rho_t B)|\, dt > \varepsilon\right\} \leq 9\exp\left(-\frac{\tilde{C}\varepsilon^2}{36\|B\|^2\|\rho\|}\right). \tag{3.6}$$

Corollary 3.4 shows that for GAP(ρ)-most $\psi_0 \in \mathbb{S}(\mathcal{H})$, the whole curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ is close to the deterministic curve $t \mapsto \mathrm{tr}(\rho_t B)$ on any finite time interval $[0,T]$. Of course, the cases of most interest in Corollary 3.4 are that $U_t = \exp(-iHt)$ if the Hamiltonian H does not depend on time and that U_t is the solution of $i\frac{d}{dt}U_t = H_t U_t$ with initial condition $U_0 = I$ if the Hamiltonian H_t is time-dependent.

While in the two corollaries above we consider arbitrary separable Hilbert spaces, in the next two corollaries we again turn to the case of bi-partite systems with Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. The first result is a version of dynamical typicality for reduced density matrices (Corollary 3 in [137]).

Corollary 3.5. *Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be Hilbert spaces with $\mathcal{H}_a$ having finite dimension d_a and $\mathcal{H}_b$ being separable and let ρ be a density matrix on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. Moreover, let $t \mapsto U_t$ be a measurable family of unitary operators and let $\varepsilon, t \geq 0$. Then,*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \left\|\rho_a^{\psi_t} - \mathrm{tr}_b\,\rho_t\right\|_{\mathrm{tr}} > \varepsilon\Big\} \leq 12 d_a^2 \exp\left(-\frac{\tilde{C}\varepsilon^2}{d_a^2\|\rho\|}\right), \tag{3.7}$$

where $\rho_t = U_t\rho U_t^*$, $\psi_t = U_t\psi$ and $\tilde{C} = \frac{1}{2304\pi^2}$. Moreover, for every $\varepsilon, T > 0$,

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \frac{1}{T}\int_0^T \left\|\rho_a^{\psi_t} - \mathrm{tr}_b\,\rho_t\right\|_{\mathrm{tr}} dt > \varepsilon\right\} \leq 9 d_a^2 \exp\left(-\frac{\tilde{C}\varepsilon^2}{36 d_a^2\|\rho\|}\right). \tag{3.8}$$

Thus, for GAP(ρ)-typical $\psi \in \mathbb{S}(\mathcal{H})$, the dynamics of $\rho_a^{\psi_t}$ on any finite time interval is approximately given by $\operatorname{tr}(\rho_t B)$ if $\|\rho\|$ is small and d_a is not too large.

For the next corollary (Corollary 4 in [137]), we assume that both $\mathcal{H}_a$ and $\mathcal{H}_b$ are finite-dimensional. It is concerned with the distribution of the conditional wave function of the subsystem a and shows that for GAP(ρ)-typical $\psi \in \mathbb{S}(\mathcal{H})$ and a typical ONB of $\mathcal{H}_b$ this distribution is close to GAP($\operatorname{tr}_b \rho$).

Corollary 3.6. *Let $\varepsilon, \delta \in (0,1)$, let $\mathcal{H}_a$ be a Hilbert space of finite dimension d_a, let $f : \mathbb{S}(\mathcal{H}_a) \to \mathbb{R}$ be any continuous (test) function, and let $\mathcal{H}_b$ be a Hilbert space of finite dimension $d_b \geq \max\{4, d_a, 32\|f\|_\infty^2/\varepsilon^2\delta\}$. Then there is a $p > 0$ such that for every density matrix ρ on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ with $\|\rho\| \leq p$,*

$$\begin{aligned} \mathrm{GAP}(\rho) \times u_{\mathrm{ONB}} \Big\{ (\psi, B) \in \mathbb{S}(\mathcal{H}) \times \mathrm{ONB}(\mathcal{H}_b) : \\ \Big| \mathrm{Born}_a^{\psi,B}(f) - \mathrm{GAP}(\operatorname{tr}_b \rho)(f) \Big| < \varepsilon \Big\} \geq 1 - \delta\,, \end{aligned} \tag{3.9}$$

where $\mathrm{Born}_a^{\psi,B}$ is the distribution of the conditional wave function ψ_a, $\mathrm{ONB}(\mathcal{H}_b)$ is the set of all orthonormal bases of $\mathcal{H}_b$, and u_{ONB} denotes the uniform distribution over this set.

We remark that in the case that $\rho = \rho_R = P_R/d_R$, i.e., if ρ is the normalized projection to a subspace $\mathcal{H}_R \subset \mathcal{H}$, this result was already obtained in [57, Theorem 3].

Polynomial Bounds

Besides the exponential bounds presented above whose proof is based on Lévy's Lemma for GAP measures, we also gave a proof which led to only polynomial bounds but is much more elementary as it only needs Chebyshev's inequality and a bound on the GAP(ρ)-variance of $\langle\psi|A|\psi\rangle$, where A is an arbitrary operator on $\mathcal{H}$. In this way we obtained the following result (cf. Theorem 3 and Remark 6 in [137]):

Theorem 3.7 (Generalized canonical typicality, polynomial bounds)**.** *Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be Hilbert spaces with $\mathcal{H}_a$ having finite dimension d_a and $\mathcal{H}_b$ being separable. Let ρ be a density matrix on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ with $\|\rho\| < 1/4$ and let $\varepsilon > 0$. Then,*

$$\mathrm{GAP}(\rho) \Big\{ \psi \in \mathbb{S}(\mathcal{H}) : \big\| \rho_a^\psi - \operatorname{tr}_b \rho \big\|_{\mathrm{tr}} > \varepsilon \Big\} \leq \frac{28 d_a^5 \operatorname{tr} \rho^2}{\varepsilon^2}. \tag{3.10}$$

3.1.2. Strategy of Proof

Exponential Bounds

Once Lévy's Lemma for GAP measures is proved, the proof of Theorem 3.1 is almost the same as the one of canonical typicality from [100], see also Theorem 1.16, where we repeat this proof. The idea is again to apply Lévy's Lemma, now in the version for GAP measures, to functions of the form $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$, $f(\psi) = \mathrm{tr}((U_a \otimes I_b)|\psi\rangle\langle\psi|)$, where U_a is a unitary operator on $\mathcal{H}_a$. Similarly as in (1.79) we have that

$$\mathbb{E}_\rho(\mathrm{tr}_a(U_a \rho_a^\psi)) = \mathrm{tr}_a(U_a \mathbb{E}_\rho(\rho_a^\psi)) = \mathrm{tr}_a(U_a \,\mathrm{tr}_b\, \rho), \tag{3.11}$$

where we used in the last step that $\mathbb{E}_\rho|\psi\rangle\langle\psi| = \rho$ which implies

$$\mathbb{E}_\rho(\rho_a^\psi) = \mathbb{E}_\rho(\mathrm{tr}_b\,|\psi\rangle\langle\psi|) = \mathrm{tr}_b\,\mathbb{E}_\rho|\psi\rangle\langle\psi| = \mathrm{tr}_b\,\rho. \tag{3.12}$$

Note that interchanging the partial trace tr_b with the expectation $\mathbb{E}_\rho$ is obviously unproblematic if $\mathcal{H}_b$ is finite-dimensional. If $\mathcal{H}_b$ is infinite-dimensional, (3.12) is still valid which can be proved by showing that for every $\phi \in \mathcal{H}_a$, ${}_a\langle\phi|\mathbb{E}_\rho(\rho_a^\psi)|\phi\rangle_a$ is equal to ${}_a\langle\phi|\,\mathrm{tr}_b\,\rho|\phi\rangle_a$ which in turn can be proved by evaluating the partial trace in an orthonormal basis of $\mathcal{H}_b$ and applying Fubini's theorem. Then (3.12) follows from the fact that a bounded operator A is uniquely determined by the quadratic form $\phi \mapsto \langle\phi|A|\phi\rangle$, see Remark 1 and its proof in [137] for the details.

From here on, the proof of Theorem 3.1 is the same as the one of canonical typicality for the uniform distribution: We expand ρ_a^ψ and $\mathrm{tr}_b\,\rho$ in a basis of the space of operators on $\mathcal{H}_a$ that consists of unitary operators, relate $\|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}}$ to the coefficients in the expansions, use Lévy's Lemma for GAP measures to show that the difference between these coefficients is small with high probability and conclude that also $\|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}}$ is small with high probability.

Next we turn to the proof of Theorem 3.2, i.e., Lévy's Lemma for GAP measures. The idea is to first show a concentration-of-measure-type result for $G(\rho)$, then for $\mathrm{GA}(\rho)$ and finally for $\mathrm{GAP}(\rho)$ (for the definition of these measures see Section 1.5). Note that at first we assume that $\mathcal{H}$ is finite-dimensional and only at the end of the proof of Theorem 3.2 we generalize our result to separable Hilbert spaces.

Let ρ be a density matrix on a D-dimensional Hilbert space $\mathcal{H}$. If $Z \sim G(\rho)$ then we can write $Z = \sqrt{\rho/2}\tilde{Z}$, where the components of $\tilde{Z}$ in the eigenbasis of ρ are D independent complex-valued centered Gaussian random variables such that their real and imaginary parts are standard Gaussian random variables. Moreover, if the function $F : \mathcal{H} \to \mathbb{R}$ is Lipschitz continuous with constant η, then $F \circ \sqrt{\rho/2} : \mathcal{H} \to \mathbb{R}$ is Lipschitz continuous with constant $\eta\sqrt{\|\rho\|/2}$. As $\mathcal{H}$ can be identified with $\mathbb{R}^{2D}$, Lemma 1.15 regarding Gaussian distributions implies the following concentration-of-measure-type result for $G(\rho)$ (Theorem 7 in [137]):

Theorem 3.8. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, let ρ be a density matrix on $\mathcal{H}$ and let Z be a random vector with distribution $G(\rho)$. Moreover, let $F : \mathcal{H} \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant η and let $\varepsilon > 0$. Then,*

$$\mathbb{P}\Big\{|F(Z) - \mathbb{E}F(Z)| > \varepsilon\Big\} \leq 2\exp\left(-\frac{4\varepsilon^2}{\pi^2\eta^2\|\rho\|}\right). \tag{3.13}$$

The next step is to show that a similar result also holds true if $\mathbb{P}$ is replaced by $\mathrm{GA}(\rho)$. Recall that $\mathrm{GA}(\rho)$ is the measure that has density $\|\psi\|^2$ with respect to $G(\rho)$. We obtained the following theorem (Theorem 8 in [137]):

Theorem 3.9. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, let ρ be a density matrix on $\mathcal{H}$, let $F : \mathcal{H} \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant η and let $\varepsilon > 0$. Then,*

$$\mathrm{GA}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : |F(\psi) - \mathrm{GA}(\rho)(F)| > \varepsilon\Big\} \leq 4\exp\left(-\frac{2\varepsilon^2}{\pi^2\eta^2\|\rho\|}\right). \tag{3.14}$$

For the proof of Theorem 3.9 we closely follow the proof of Lemma 1.15 from [88]. Let $D = \dim\mathcal{H}$, let $Z = (Z_1, \dots, Z_D)$ and $\tilde{Z} = (\tilde{Z}_1, \dots, \tilde{Z}_D)$ be two independent $\mathrm{GA}(\rho)$-distributed random vectors and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-negative convex function. The first step is to show with the help of the definition of $\mathrm{GA}(\rho)$ as well as Jensen's and Hölder's inequality that

$$\mathrm{GA}(\rho)\left[\varphi(F(\psi) - \mathrm{GA}(\rho)(F))\right] \leq \sum_{n,m}\left(\mathbb{E}\left(|Z_n|^4|\tilde{Z}_m|^4\right)\mathbb{E}\left(\varphi(F(Z) - F(\tilde{Z}))^2\right)\right)^{1/2}, \tag{3.15}$$

where we use the two notations $F(Z)$ and $F(\psi)$ interchangeably. An explicit computation which makes use of the known formulas for moments of a Gaussian random variable shows that

$$\sum_{n,m}\left(\mathbb{E}\left(|Z_n|^4|\tilde{Z}_m|^4\right)\right)^{1/2} = 2 \tag{3.16}$$

and therefore it remains to bound $\mathbb{E}(\varphi(F(Z) - F(\tilde{Z}))^2)$. As φ is convex and non-negative, also φ^2 is convex. We can identify the random vector Z with $X := (\mathrm{Re}\, Z_1, \mathrm{Im}\, Z_1, \mathrm{Re}\, Z_2, \dots, \mathrm{Re}\, Z_D, \mathrm{Im}\, Z_D)$, i.e., with a vector of $2D$ independent centered Gaussian random variables with variances $\mathbb{E}|\mathrm{Re}\, Z_j|^2 = \mathbb{E}|\mathrm{Im}\, Z_j|^2 = p_j/2$, where the p_j are the eigenvalues of ρ, and similarly we identify $\tilde{Z}$ with the vector $Y = (\mathrm{Re}\, \tilde{Z}_1, \mathrm{Im}\, \tilde{Z}_1, \mathrm{Re}\, \tilde{Z}_2, \dots, \mathrm{Re}\, \tilde{Z}_D, \mathrm{Im}\, \tilde{Z}_D)$. By making use of the invariance of normal distributions under rotations, Jensen's inequality, Fubini's theorem and assuming without loss of generality that F is continuously differentiable, we can show as in (1.65a)–(1.65e) (with φ replaced by φ^2) that

$$\mathbb{E}\varphi(F(Z) - F(\tilde{Z}))^2 \leq \mathbb{E}\varphi\left(\frac{\pi}{2}\langle\nabla F(X), Y\rangle\right)^2. \tag{3.17}$$

Let $\lambda \in \mathbb{R}$. We apply (3.17) to the case that $\varphi(x) = \exp(\lambda x)$. With the help of the formula (1.67) for the moment generating function of a real Gaussian random variable we find as in the proof of Lemma 1.15 that

$$\mathbb{E}\exp[2\lambda(F(X)-F(Y))] \leq \exp\left(\frac{\lambda^2\pi^2\|\rho\|\eta^2}{4}\right) \tag{3.18}$$

and thus altogether we arrive at

$$\mathrm{GA}(\rho)\left[\exp\left(\lambda(F(\psi)-\mathrm{GA}(\rho)(F))\right)\right] \leq 2\exp\left(\frac{\lambda^2\pi^2\|\rho\|\eta^2}{8}\right). \tag{3.19}$$

Now Theorem 3.9 follows again analogously to the proof of Lemma 1.15 by an application of Markov's inequality and an optimization in λ.

Besides Theorem 3.9 we also need the following lemma (Lemma 2 in [137]) for the proof of Theorem 3.2:

Lemma 3.10. *For all $r>0$ it holds that*

$$\mathrm{GA}(\rho)\{\|\psi\|<r\} \leq \sqrt{2}\exp\left(-\frac{1/2-r^2}{2\|\rho\|}\right). \tag{3.20}$$

For the proof of Lemma 3.10 we first use Hölder's inequality and the definition of $\mathrm{GA}(\rho)$ to show that

$$\mathrm{GA}(\rho)\{\|\psi\|<r\} \leq \sqrt{2}\left(\mathbb{P}\{\|\psi\|<r\}\right)^{1/2}. \tag{3.21}$$

The main tool to bound $\mathbb{P}\{\|\psi\|<r\}$ is the so-called *Chernoff bound*. It states that for a random variable Y and any $a\in\mathbb{R}$,

$$\mathbb{P}\{Y\leq a\} \leq \inf_{t<0} M_Y(t)e^{-ta}, \tag{3.22}$$

where $M_Y(t)=\mathbb{E}(e^{tY})$ is the moment generating function of Y. With the help of the Chernoff bound we find that

$$\mathbb{P}\{\|\psi\|<r\} \leq \inf_{t<0} e^{-tr^2}\prod_n M_{2(\mathrm{Re}\,\tilde{Z}_n)^2}\left(\frac{p_n t}{2}\right) M_{2(\mathrm{Im}\,\tilde{Z}_n)^2}\left(\frac{p_n t}{2}\right), \tag{3.23}$$

where the $\tilde{Z}_n$ are independent complex standard Gaussian random variables. Note that here we used that we can write $\|\psi\|^2=\sum_n p_n|\tilde{Z}_n|^2$. As $2(\mathrm{Re}\,\tilde{Z}_n)^2$ and $2(\mathrm{Im}\,\tilde{Z}_n)^2$ are χ_1^2-distributed, their moment generating function is given by

$$M_{2(\mathrm{Re}\,\tilde{Z}_n)^2}(s) = M_{2(\mathrm{Im}\,\tilde{Z}_n)^2}(s) = (1-2s)^{-1/2} \quad \text{for } s<1/2. \tag{3.24}$$

From this and after choosing $t=-\|\rho\|^{-1}$ Lemma 3.10 follows after a short computation.

With the help of Theorem 3.9 and Lemma 3.10 we are now able to prove Theorem 3.2. Our proof is inspired by the proof of Lévy's Lemma in [88]. The main idea is to use f to define a suitable Lipschitz continuous function $\tilde{f}$ on $\mathcal{H}$ to which Theorem 3.9 can be applied.

We first assume that $\mathcal{H}$ is finite-dimensional and that f is real-valued. Moreover, we suppose that without loss of generality $\mathrm{GAP}(\rho)(f) = 0$. For $0 < r < 1$ we define the function $\tilde{f} : \mathcal{H} \to \mathbb{R}$ by

$$\tilde{f}(\psi) = \begin{cases} f\left(\frac{\psi}{\|\psi\|}\right) & \text{if } \|\psi\| \geq r, \\ r^{-1}\|\psi\| \; f\left(\frac{\psi}{\|\psi\|}\right) & \text{if } \|\psi\| \leq r. \end{cases} \tag{3.25}$$

A standard but somewhat lengthy computation shows that $\tilde{f}$ is indeed Lipschitz continuous with Lipschitz constant $6\eta/r$. We then show that

$$\begin{aligned} &\mathrm{GAP}(\rho)\left\{|f(\psi)| > \varepsilon\right\} \\ &\quad \leq \mathrm{GA}(\rho)\left\{\left|\tilde{f} - \mathrm{GA}(\rho)(\tilde{f})\right| > \varepsilon - \left|\mathrm{GA}(\rho)(\tilde{f})\right|\right\} + \mathrm{GA}(\rho)\{\|\psi\| < r\}. \end{aligned} \tag{3.26}$$

The second term can be bounded by Lemma 3.10 while Theorem 3.9 provides an upper bound for the first term. However, before we can apply Theorem 3.9 to the first term, we have to find an upper bound for $|\mathrm{GA}(\rho)(\tilde{f})|$. With the help of the assumption that $\mathrm{GAP}(\rho)(f) = 0$, the boundedness of f (the function f is bounded by $\pi\eta$) and Lemma 3.10 we can show that

$$\left|\mathrm{GA}(\rho)(\tilde{f})\right| \leq 5\eta \exp\left(-\frac{1/2 - r^2}{2\|\rho\|}\right). \tag{3.27}$$

Putting all estimates together we finally arrive at

$$\mathrm{GAP}(\rho)\{|f(\psi)| > \varepsilon\} \leq 6 \exp\left(-\frac{\varepsilon^2}{288\pi^2\eta^2\|\rho\|}\right). \tag{3.28}$$

If f is complex-valued, we consider its real and imaginary part separately and find that we have to replace the factor 6 in (3.28) by 12 and 288 by 576. This finishes the proof in the finite-dimensional setting.

If $\mathcal{H}$ has a countably infinite orthonormal basis, we approximate ρ by finite-rank density matrices ρ_n, $n \in \mathbb{N}$, defined by

$$\rho_n := \sum_{m=1}^{n-1} p_m |m\rangle\langle m| + \left(\sum_{m=n}^{\infty} p_m\right) |n\rangle\langle n|, \tag{3.29}$$

where the states $|n\rangle$ form an eigen-ONB of ρ with corresponding eigenvalues p_n which are ordered decreasing in size, i.e., $p_1 \geq p_2 \geq \dots$. Because of $\|\rho_n - \rho\|_{\mathrm{tr}} \to 0$ as $n \to \infty$, the measures $\mathrm{GAP}(\rho_n)$ converge weakly to $\mathrm{GAP}(\rho)$, see Proposition 1.19. As the set

$$A_\varepsilon := \{\psi \in \mathbb{S}(\mathcal{H}) : |f(\psi)| > \varepsilon\} \tag{3.30}$$

is open in $\mathbb{S}(\mathcal{H})$, the weak convergence of $\mathrm{GAP}(\rho_n)$ implies by the *Portmanteau Theorem* [12, Theorem 2.1] that for every $\varepsilon' > 0$,

$$\mathrm{GAP}(\rho)(A_\varepsilon) \leq \liminf_{n\to\infty} \mathrm{GAP}(\rho_n)(A_\varepsilon) \leq \mathrm{GAP}(\rho_N)(A_\varepsilon) + \varepsilon' \tag{3.31}$$

provided that $N \in \mathbb{N}$ is large enough. As the density matrices ρ_N live on a finite-dimensional subspace of $\mathcal{H}$, we can apply the result we already proved in the finite-dimensional setting. Because of $\|\rho_N\| = \|\rho\|$ for sufficiently large N and since $\varepsilon' > 0$ was arbitrary, we obtain that the bound (3.28) as well as its generalization for complex-valued functions remain valid in the infinite-dimensional case and this finishes the proof of Theorem 3.2.

Some Corollaries

The first part of Corollary 3.4 is obtained by inserting $U_t^* B U_t$ for B in Corollary 3.3. For the second part we define

$$Y_t := |\langle\psi_t|B|\psi_t\rangle - \mathrm{tr}(\rho_t B)| \tag{3.32}$$

and apply Theorem 3.2 in order to show that for every $\delta, s > 0$,

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : e^{sY_t} > \delta\Big\} \leq 12 \exp\left(-\frac{\tilde{C}}{\|B\|^2\|\rho\|}\frac{\ln(\delta)^2}{s^2}\right). \tag{3.33}$$

Next note that we have that

$$\mathrm{GAP}(\rho)(e^{sY_t}) \leq \sum_{n=0}^{\infty}(n+1)\,\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : e^{sY_t} \in (n, n+1]\Big\}. \tag{3.34}$$

With (3.33) we obtain after a short computation that $\mathrm{GAP}(\rho)(e^{sY_t}) \leq 9e^{5/a}$, where $a := \frac{\tilde{C}}{\|B\|^2\|\rho\|s^2}$, provided that $a \leq 1$ (however, as the final bound becomes trivial for $a > 1$, we can drop this assumption). Using Jensen's inequality and Fubini's theorem we find that

$$\mathrm{GAP}(\rho)\left(\exp\left(\frac{1}{T}\int_0^T Y_t\,dt\; s\right)\right) \leq \frac{1}{T}\int_0^T \mathrm{GAP}(\rho)(e^{Y_t s})\,dt \leq 9e^{5/a}. \tag{3.35}$$

By choosing $s := \frac{\varepsilon \tilde{C}}{6\|B\|^2\|\rho\|}$ the claim follows from an application of Markov's inequality.

The first part of Corollary 3.5 is an immediate consequence of Theorem 3.1 and the covariance of the map $\rho \mapsto \mathrm{GAP}(\rho)$, see Proposition 1.19 (b). The second part can be proved in the same way as the second part of Corollary 3.4.

The proof of Corollary 3.6 is based on two results from [57] as well as Theorem 3.1. We choose $\psi \sim \mathrm{GAP}(\rho)$ and $B \sim u_{\mathrm{ONB}}$ independently. Roughly speaking, Theorem 2 of [57] tells us that $\mathrm{Born}_a^{\psi,B}(f) \approx \mathrm{GAP}(\rho_a^\psi)(f)$ with high probability provided that d_b is sufficiently large. Moreover, it follows from Lemma 5 of [57] that $\mathrm{GAP}(\rho_a^\psi)(f) \approx \mathrm{GAP}(\mathrm{tr}_b\,\rho)(f)$ if $\|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}}$ is sufficiently small, a condition that is fulfilled with high probability by Theorem 3.1. Putting everything carefully together yields the claim.

Polynomial Bounds

The main ingredient for the proof of the polynomial bounds in Theorem 3.7 is a generalization of Reimann's [106] upper bound on the $\mathrm{GAP}(\rho)$-variance of $\langle\psi|A|\psi\rangle$. Reimann proved a bound in the case that A is self-adjoint and $\mathcal{H}$ is finite-dimensional. What we show is that Reimann's bound essentially remains valid also in the case that A is an arbitrary bounded operator on a separable Hilbert space. More precisely, we proved the following proposition (Proposition 1 in [137]):

Proposition 3.11. *Let ρ be a density matrix on a separable Hilbert space $\mathcal{H}$ with eigenvalues $p_n > 0$ such that $p_{\max} = \|\rho\| < 1/4$ and let $\dim\mathcal{H} \geq 4$. Then for any bounded operator A on $\mathcal{H}$,*

$$\mathbb{E}_\rho\langle\psi|A|\psi\rangle = \mathrm{tr}(A\rho), \tag{3.36}$$

$$\mathrm{Var}_\rho\langle\psi|A|\psi\rangle \leq \frac{\|A\|^2\,\mathrm{tr}\,\rho^2}{1-p_{\max}}\left(1 + \frac{4\sqrt{\mathrm{tr}\,\rho^2} + 2\,\mathrm{tr}\,\rho^2}{(1-2p_{\max})(1-3p_{\max}))}\right). \tag{3.37}$$

We remark that the only difference to the bound on the variance that was obtained by Reimann [106], see also Lemma 1.20, is that in Proposition 3.11 the distance between the largest and smallest eigenvalue of A is replaced by its operator norm.

The formula for the expectation follows basically from the fact that $\mathbb{E}_\rho|\psi\rangle\langle\psi| = \rho$ which is also true in infinite dimensions.

For the proof of the bound on the variance we first assume that $D = \dim\mathcal{H} < \infty$. Moreover, we assume without loss of generality that $\mathbb{E}_\rho\langle\psi|A|\psi\rangle = 0$. Similarly as in (1.128c) we find that

$$\mathrm{Var}_\rho\langle\psi|A|\psi\rangle = \sum_{m,n}\left[A_{mm}^* A_{nn} + |A_{mn}|^2\right] p_m p_n K_{mn}, \tag{3.38}$$

where the K_{mn} are defined in (1.125). Since $|A_{mm}| \leq \|A\|$ and $K_{mn} \leq \frac{1}{1-p_{\max}}$ (which is shown in [106]) the same computation as in the proof of Lemma 1.20 due to Reimann [106] shows that

$$\sum_{m,n} A^*_{mm} A_{nn} p_m p_n K_{mn} \leq \frac{2\|A\|^2 \operatorname{tr}\rho^2 \left(2\sqrt{\operatorname{tr}\rho^2} + \operatorname{tr}\rho^2\right)}{(1-p_{\max})(1-2p_{\max})(1-3p_{\max})}, \tag{3.39}$$

$$\sum_{m,n} |A_{mn}|^2 p_m p_n K_{mn} \leq \frac{1}{1-p_{\max}} \operatorname{tr}(A^*\rho A\rho). \tag{3.40}$$

While Reimann [106] proceeds to evaluate the trace on the right-hand side of (3.40) in an orthonormal eigenbasis of A which is possible as he considers only self-adjoint operators, we have to proceed in a different way. To this end, we make use of two inequalities for the trace,

$$\operatorname{tr}(B^*C) \leq \sqrt{\operatorname{tr}(B^*B)\operatorname{tr}(C^*C)}, \tag{3.41}$$

$$|\operatorname{tr}(BC)| \leq \|B\| \operatorname{tr}(|C|), \tag{3.42}$$

where B and C are arbitrary operators. Note that (3.41) is the Cauchy-Schwarz inequality and a proof of (3.42) can be found in [126, Theorem 3.7.6].

With the help of these inequalities we find that

$$\operatorname{tr}(A^*\rho A\rho) \leq \sqrt{\operatorname{tr}(A^*\rho^2 A)\operatorname{tr}(\rho A^* A\rho)} = \sqrt{\operatorname{tr}(AA^*\rho^2)\operatorname{tr}(A^*A\rho^2)} \leq \|A\|^2 \operatorname{tr}\rho^2. \tag{3.43}$$

By combining this with (3.40) and plugging the result and (3.39) into (3.38) we obtain the claim in the finite-dimensional case.

If $\mathcal{H}$ has a countably infinite orthonormal basis, we consider as in the proof of Theorem 3.2 the sequence of finite-rank density matrices $(\rho_n)_{n\in\mathbb{N}}$ defined in (3.29) which converges in trace norm to ρ. For every $n \in \mathbb{N}$ we find that

$$\operatorname{Var}_{\rho_n}\langle\psi|A|\psi\rangle \leq \frac{\|A\|^2 \operatorname{tr}\rho_n^2}{1-p_{\max,n}} \left(1 + \frac{4\sqrt{\operatorname{tr}\rho_n^2} + 2\operatorname{tr}\rho_n^2}{(1-2p_{\max,n})(1-3p_{\max,n})}\right), \tag{3.44}$$

where $p_{\max,n}$ denotes the largest eigenvalue of ρ_n. As $p_{\max,n} = p_{\max}$ for n large enough and $\operatorname{tr}\rho_n^2 \to \operatorname{tr}\rho^2$, the right-hand side of (3.44) converges to the desired bound as $n \to \infty$. Due to the weak convergence of the measures $\mathrm{GAP}(\rho_n)$ to $\mathrm{GAP}(\rho)$ we have that $\mathrm{GAP}(\rho_n)(g) \to \mathrm{GAP}(\rho)(g)$ for every bounded continuous function g. With the help of this fact we can show that $\operatorname{Var}_{\rho_n}\langle\psi|A|\psi\rangle \to \operatorname{Var}_{\rho}\langle\psi|A|\psi\rangle$ which concludes the proof of Proposition 3.11.

We now turn to the proof of the polynomial bounds in Theorem 3.7. Let $\varepsilon > 0$.

First we note that it follows from Chebyshev's inequality and Proposition 3.11 that

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : |\langle\psi|A|\psi\rangle - \mathrm{tr}(A\rho)| > \varepsilon\Big\} \leq \frac{28\|A\|^2 \operatorname{tr}\rho^2}{\varepsilon^2}. \tag{3.45}$$

The rest of the proof closely follows the one of Theorem 1.13 in the case of the uniform distribution. Let $\{|l\rangle_a : l = 1, \ldots, d_a\}$ be an orthonormal basis of $\mathcal{H}_a$ and consider again the operators $A_{lm} := (|l\rangle_{aa}\langle m|) \otimes I_b$. With the help of the relation between the trace norm and Hilbert-Schmidt norm in (1.57) and by applying (3.45) to the operators A_{lm} we can use similar steps as in the proof of Theorem 1.13 to obtain the claim.

3.1.3. Discussion

The goal of [137] was to generalize canonical typicality to a much broader class of measures, the so-called GAP measures. These measures form a natural generalization of the uniform distribution on the sphere and they are physically relevant since for certain density matrices ρ, $\mathrm{GAP}(\rho)$ arises as the thermal equilibrium distribution of wave functions [59, 57], see also Section 1.5 for details.

Our main result in [137] shows that for $\mathrm{GAP}(\rho)$-typical wave functions $\psi \in \mathbb{S}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ with $\mathcal{H}_a$ being finite-dimensional and $\mathcal{H}_b$ being separable, the reduced density matrix ρ_a^ψ is close (in trace norm) to $\operatorname{tr}_b \rho$ (and thus nearly independent of ψ) provided that $\|\rho\|$ is sufficiently small and d_a is not too large. Put another way, the set of wave functions $\psi \in \mathbb{S}(\mathcal{H})$ for which the distance $\|\rho_a^\psi - \operatorname{tr}_b \rho\|_{\mathrm{tr}}$ is not smaller than a given $\varepsilon > 0$ has very small $\mathrm{GAP}(\rho)$ measure if no eigenvalue of ρ is too large and the system a is not too big. We gave bounds on the measure of this set of "bad" wave functions that are polynomially (cf. Theorem 3.7) and exponentially (cf. Theorem 3.1) small in $\|\rho\|$ resp. $\|\rho\|^{-1}$. Our main motivation for writing down both proofs of the generalization of canonical typicality were the different strategies of proof: While the proof of the polynomial bounds is based on Chebyshev's inequality and is rather elementary, the proof of the exponential bounds which required proving a Lévy Lemma for GAP measures is much more involved. Moreover, contrary to what one might think at a first glance, the exponential bounds are not always smaller than the polynomial ones. As an example we consider in Remark 7 in [137] the case that the largest eigenvalue of ρ is equal to $\frac{1}{\sqrt{D}}$ while all the other ones are equal. In this case we find that the polynomial bound is smaller than the exponential one for $D \sim 10^{13} - 10^{31}$.

With the help of our bounds, we can also say something about how large the subsystem a can be (cf. Remark 8 in [137]). Suppose that $\mathcal{H}$ is finite-dimensional with $D = \dim \mathcal{H}$ and that $\|\rho\|^{-1}$ and $1/\operatorname{tr}\rho^2$ are comparable to D (which is exponential in the number of degrees of freedom of the system). Then it follows from the polynomial bounds in Theorem 3.7 that the subsystem a must comprise fewer than 20% of the degrees of freedom of the full system, whereas the exponential bounds in Theorem 3.1 show that the subsystem a can even have up to 50% of the degrees of freedom, see also [53, 54].

Recall that if $\rho = \rho_R = P_R/d_R$ is a normalized projection to a subspace $\mathcal{H}_R$, then the corresponding GAP measure GAP(ρ_R) is just the uniform distribution over the sphere $\mathbb{S}(\mathcal{H}_R)$, see Proposition 1.19 (e). Therefore, our results in [137] for GAP measures include the case of the uniform distribution as a special case. We compare the bounds we obtain for canonical typicality and Lévy's Lemma to the original theorems in Remark 12 in [137]. For the polynomial bounds we obtain almost the same bound except for a constant factor and d_a^5 instead of d_a^4, see also Theorem 1.13. The reason for this different exponent is the following: The bound in (1.58b) on the measure w.r.t. u_R of the set of wave functions $\psi \in \mathbb{S}(\mathcal{H}_R)$ such that $\langle\psi|A|\psi\rangle$ differs from $\mathrm{tr}(A\rho_R)$ by more than a given $\varepsilon > 0$ does not involve the operator norm of A but the trace $\mathrm{tr}(A^*\rho_R A)$. When applied to the operators $A_{lm} = (|l\rangle_{aa}\langle m|) \otimes I_b$, the upper bound does depend on l (but not on m). Summing over m gives a factor d_a and the sum over l equals one. In the analogous bound (3.45) for the GAP measure we have $\|A\|^2$ instead of a trace involving A and for $A = A_{lm}$ this norm equals one. Therefore the summation over l and m gives a factor d_a^2 instead of d_a.

Setting $\rho = \rho_R$ in the exponential bounds in Theorem 3.1 gives the same bound as in the original result, Theorem 1.16, up to worse constants, and the same holds true for Lévy's Lemma for GAP measures in Theorem 3.2 and the original Lévy Lemma (Theorem 1.14). Thus, altogether, we obtain for a general density matrix ρ more or less the same bounds in the theorems concerning GAP measures as for the uniform distribution on $\mathbb{S}(\mathcal{H}_R)$ with d_R replaced by $\|\rho\|^{-1}$. Note, however, that our theorems about GAP measures allow for a separable Hilbert space; they therefore also cover situations in which the Hilbert space is infinite-dimensional and the uniform distribution on the sphere does not exist.

One case of particular interest is that ρ is a canonical density matrix.[22] The reason is that in this case the GAP measures describe the thermal equilibrium distribution of wave functions [59, 57] and they can be viewed as a quantum analogue of the canonical ensemble from classical statistical physics. As the uniform distribution on a micro-canonical energy shell can be understood as a quantum analogue of the micro-canonical ensemble, generalizing results from the uniform distribution to GAP measures as we did in [137] expresses a kind of equivalence of ensembles.[23] In particular, from canonical typicality and our generalization thereof, the following picture emerges: First note that if the subsystem a and its environment b are only weakly interacting and b is sufficiently large, any micro-canonical density matrix ρ_{mc} as well

[22]It can be argued that canonical density matrices should usually fulfill the property that $\|\rho\|$ is small, see Remark 9 in [137] for details.

[23]Recall that equivalence of ensembles means that it does not matter much which equilibrium ensemble we use, in particular, it makes no huge difference whether we use the micro-canonical ensemble or the canonical one (with suitable inverse temperature β).

as any canonical density matrix ρ_{can} on the Hilbert space $\mathcal{H}_S = \mathcal{H}_a \otimes \mathcal{H}_b$ lead to reduced density matrices for the subsystem a that are approximately canonical. If the inverse temperature β in ρ_{can} is suitably chosen for the energy corresponding to a given $\mathcal{H}_{\text{mc}}$, we obtain

$$\operatorname{tr}_b \rho_{\text{mc}} \approx \rho_{a,\text{can}} \approx \operatorname{tr}_b \rho_{\text{can}}. \tag{3.46}$$

The original statement of canonical typicality asserts that for u_{mc}-typical wave functions $\psi \in \mathbb{S}(\mathcal{H})$, the reduced density matrix ρ_a^ψ is close to $\operatorname{tr}_b \rho_{\text{mc}} \approx \rho_{a,\text{can}}$. Moreover, generalized canonical typicality shows that for $\mathrm{GAP}(\rho_{\text{can}})$-typical $\psi \in \mathbb{S}(\mathcal{H})$, the reduced density matrix ρ_a^ψ is close to $\operatorname{tr}_b \rho_{\text{can}} \approx \rho_{a,\text{can}}$. Therefore it does not matter much whether we start from u_{mc} or $\mathrm{GAP}(\rho_{\text{can}})$, for both ensembles of ψ we get that the reduced density matrix ρ_a^ψ is nearly constant and nearly canonical.[24]

Generally speaking, our theorems show a kind of robustness of canonical typicality towards changes of the measure of typicality. If we expand a wave function $\psi \sim u_{\text{mc}}$ in an energy eigenbasis of the Hamiltonian, the coefficients of eigenfunctions corresponding to eigenvalues outside the micro-canonical energy interval $[E - \Delta E, E]$ are equal to zero. Our results imply that, as one would expect, this sharp energy cut-off is not essential for canonical typicality to be valid.

Besides our main results Theorem 3.1, Theorem 3.2 and Theorem 3.7 we proved several corollaries thereof. The first one, Corollary 3.3, shows that for any bounded operator B and $\mathrm{GAP}(\rho)$-typical $\psi \in \mathbb{S}(\mathcal{H})$, the quantity $\langle\psi|B|\psi\rangle$ is close to $\operatorname{tr}(\rho B)$. If B is self-adjoint, its spectrum is real and we can coarse grain it by covering it with a not too large number of intervals. By inserting the corresponding spectral projections for B in Corollary 3.3, we then find that, on a coarse-grained level, the probability distribution that a wave function ψ defines over the (coarse-grained) spectrum of B is the same for $\mathrm{GAP}(\rho)$-most ψ. We also call this distribution the coarse-grained Born distribution of B.

Corollary 3.4 is a generalization of dynamical typicality (see Section 1.3) from the uniform distribution to GAP measures. It shows that the time evolution "looks" the same for typical initial wave functions $\psi_0 \in \mathbb{S}(\mathcal{H})$. Here, the look refers to the distribution we discussed in the discussion of Corollary 3.3 above, namely the (coarse-grained) Born distribution for the observable under consideration. Note that in Corollary 3.4 we do not only show that for every $t \geq 0$ and $\mathrm{GAP}(\rho)$-typical ψ_0, the quantity $\langle\psi_t|B|\psi_t\rangle$ is close to $\operatorname{tr}(\rho_t B)$, but also that for $\mathrm{GAP}(\rho)$-typical ψ_0 this holds jointly for most times $t \in [0, T]$. Moreover, we remark that we allow an arbitrary unitary

[24] Note, however, that a $\mathrm{GAP}(\rho_{\text{can}})$-typical wave function $\psi \in \mathbb{S}(\mathcal{H})$ does usually not lie in any micro-canonical subspace $\mathcal{H}_{\text{mc}}$. And even if it does, then it is not typical w.r.t. the measure u_{mc}. The reason is that u_{mc}-typical wave functions are superpositions of many energy eigenfunctions and the superposition weights are approximately of equal size, while $\mathrm{GAP}(\rho_{\text{can}})$-typical wave functions (which are also superpositions of many energy eigenfunctions) have non-equal superposition weights.

time evolution U_t which in particular allows for a time-dependent Hamiltonian.

Corollary 3.5 is a version of dynamical typicality for the reduced density matrix $\rho_a^{\psi_t}$. It shows that for every $t \geq 0$, GAP(ρ)-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that $\rho_a^{\psi_t}$ is close (in trace norm) to $\mathrm{tr}_b\, \rho_t$ provided that $\|\rho\|$ is sufficiently small and the subsystem a is not too large. Moreover, for GAP(ρ)-typical ψ_0 this holds again jointly for most times $t \in [0, T]$. Note that as in Corollary 3.4 we consider an arbitrary unitary time evolution U_t which allows for a time-dependent Hamiltonian. However, even if H does not depend on time, the density matrix ρ can evolve in a non-trivial way; this is different in the original statement of canonical typicality if we consider the uniform distribution over a micro-canonical energy shell $\mathcal{H}_{\mathrm{mc}}$ as the density matrix ρ_{mc} is invariant under the time evolution induced by H.

The fourth and last corollary, Corollary 3.6, is concerned with the conditional wave function ψ_a of the subsystem a. It is shown in [57, Theorem 3] that for most wave functions ψ (most w.r.t. the uniform distribution u_R on the sphere $\mathbb{S}(\mathcal{H}_R)$) and most orthonormal bases of $\mathcal{H}_b$ the Born distribution of the conditional wave function ψ_a is approximately given by GAP($\mathrm{tr}_b\, \rho_R$) provided that d_R is sufficiently large. Corollary 3.6 is a generalization of this result to GAP measures and shows that for GAP(ρ)-most ψ and most orthonormal bases of $\mathcal{H}_b$, the Born distribution of ψ_a is close to GAP($\mathrm{tr}_b\, \rho$) provided that no eigenvalue of ρ is too large and d_b is sufficiently large. More precisely, the corollary shows that for every continuous test function $f : \mathbb{S}(\mathcal{H}_a) \to \mathbb{R}$, GAP($\rho$)-most ψ and most orthonormal bases B of $\mathcal{H}_b$ are such that $\mathrm{Born}_a^{\psi,B}(f)$ is close to GAP($\mathrm{tr}_b\, \rho$)(f). We conjecture that this closeness holds even for bounded measurable functions f uniformly in f with given $\|f\|_\infty$, see Remark 5 in [137] for details.

Counterexamples

In this section we give a couple of counterexamples to generalized canonical typicality and Lévy's Lemma. These counterexamples show in particular that the generalization of the results for the uniform distribution to GAP measures is not straightforward.

As Lévy's Lemma holds for the uniform distribution as well as for GAP measures, one might think at a first glance that it is true for all rather spread out distributions on the sphere. This, however, is *not* the case as Remark 13 in [137] shows: Consider a sequence $(\mu_D)_{D\in\mathbb{N}}$ on $\mathbb{S}(\mathbb{R}^D)$ with corresponding densities $(g_D)_{D\in\mathbb{N}}$ relative to the uniform measure u_D on $\mathbb{S}(\mathbb{R}^D)$ and suppose that g_D is bounded uniformly in D, is Lipschitz continuous with Lipschitz constant bounded uniformly in D, but deviates significantly from 1 on a not-too-small subset of $\mathbb{S}(\mathbb{R}^D)$. Let $F_D : \mathbb{S}(\mathbb{R}^D) \to \mathbb{R}$ be a Lipschitz continuous function. Since $F_D g_D$ is Lipschitz continuous, it follows from Lévy's Lemma that the set of $x \in \mathbb{S}(\mathbb{R}^D)$ for which $F_D(x) g_D(x)$ differs significantly from $u_D(F_D g_D) = \mu_D(F_D)$ has small u_D-measure if D is sufficiently large. Moreover, as g_D is bounded uniformly in D, it follows that this set of "bad" $x \in \mathbb{S}(\mathbb{R}^D)$ also has small μ_D-measure. Thus, for μ_D-most $x \in \mathbb{S}(\mathbb{R}^D)$, $F_D(x) \approx \mu_D(F_D)/g_D(x)$, which is, due to our assumption that g_D deviates significantly from 1 on a non-negligible set, not constant at all. Therefore, a Lévy Lemma for μ_D does not hold even though μ_D might well be very spread out over the sphere. A concrete example is given by the von Mises-Fisher distribution which has parameters $\kappa \in \mathbb{R}_+$ and $\mu \in \mathbb{S}(\mathbb{R}^D)$ and can be obtained by taking a Gaussian distribution in $\mathbb{R}^D$ with mean μ and covariance matrix $\kappa^{-1}I$ and conditioning it on the sphere $\mathbb{S}(\mathbb{R}^D)$. It has density

$$g(x) = C(D,\kappa)\exp\left(\kappa\langle\mu, x\rangle\right), \tag{3.47}$$

with respect to the uniform distribution on $\mathbb{S}(\mathbb{R}^D)$ and it fulfills all assumptions on the sequence $(\mu_D)_{D\in\mathbb{N}}$ above. The von Mises-Fisher distribution is spread out over the sphere but, by our discussion above, does not fulfill a Lévy Lemma.

We remark that the situation is different for GAP measures. As one can see from (1.108), the density of $\mathrm{GAP}(\rho)$ with respect to the uniform distribution on the sphere and its Lipschitz constant are in general not uniformly bounded in D.

Our generalization of canonical typicality does not hold for every measure with density matrix ρ (cf. Remark 15 in [137]). To see this we write $\rho = \sum_{n=1}^{D} p_n|n\rangle\langle n|$, where the $|n\rangle$ form an orthonormal basis of eigenvectors of ρ with corresponding eigenvalues p_n. We then consider the measure

$$\mu = \sum_{n=1}^{D} p_n \delta_{|n\rangle}, \tag{3.48}$$

where $\delta_{|n\rangle}$ denotes the dirac measure concentrated in $|n\rangle$. The measure μ has density matrix ρ and is the most concentrated measure on the sphere for which this holds. If $\psi \sim \mu$, then $\psi = |n\rangle$ with probability p_n. Suppose that each $|n\rangle \in \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ is of the form $|n\rangle = |\ell\rangle_a \otimes |m\rangle_b$ and thus $\rho_a^{|n\rangle} = |\ell\rangle_{aa}\langle\ell|$, i.e., ρ_a^ψ is always a pure state. Writing the spectral decomposition of ρ as $\rho = \sum_{\ell,m} p_{\ell m}|\ell\rangle_{aa}\langle\ell| \otimes |m\rangle_{bb}\langle m|$ we find that $\mathrm{tr}_b\,\rho = \sum_{\ell,m} p_{\ell m}|\ell\rangle_{aa}\langle\ell|$ which is in general highly mixed. We conclude that ρ_a^ψ is far away from $\mathrm{tr}_b\,\rho$ and thus generalized canonical typicality does not hold.

In our results concerning generalized canonical typicality we need that $\|\rho\|$ is small, i.e., that no eigenvalue of ρ is too large. As we argue in Remark 16 in [137], this assumption is necessary: Consider the case that ρ has one large eigenvalue p with corresponding eigenvector $|0\rangle$ and all other eigenvalues are equal, i.e.,

$$\rho = p|0\rangle\langle 0| + \frac{1-p}{D-1}(I - |0\rangle\langle 0|). \tag{3.49}$$

Let $\psi \sim \mathrm{GAP}(\rho)$ and suppose that $|0\rangle = |0\rangle_a \otimes |0\rangle_b$. One can show that

$$\rho_a^\psi = \cos^2\theta|0\rangle_{aa}\langle 0| + \sin^2\theta(I_a/d_a) + O(1/\sqrt{d_b}), \tag{3.50}$$

$$\mathrm{tr}_b\,\rho = p|0\rangle_{aa}\langle 0| + (1-p)(I_a/d_a) + O(1/d_b), \tag{3.51}$$

where $O(1/\sqrt{d_b})$ and $O(1/d_b)$ refer to the trace norm in the limit $d_b \to \infty$ and I_a denotes the identity on $\mathcal{H}_a$. Moreover, $\theta \in [0, \pi/2]$ is random and has density

$$\frac{2(1-p)^2}{p} \frac{\cos\theta}{\sin^5\theta} \exp\left(\left(1 - \frac{1}{p}\right)\cot^2\theta\right). \tag{3.52}$$

Thus ρ_a^ψ depends on θ and has a non-trivial distribution while $\operatorname{tr}_b \rho$ is deterministic. It follows that $\rho_a^\psi \not\approx \operatorname{tr}_b \rho$, i.e., generalized canonical typicality does not hold. Note that an example of a system for which this situation occurs is an N-body quantum system at very low temperature for which the ground state $|0\rangle$ is gapped.

3.2. Time Evolution of Typical Pure States

3.2.1. Results

In the following we assume that the system's Hilbert space $\mathcal{H}$ is finite-dimensional except for the result concerning the generalization of normal typicality to GAP measures where we only assume that it is separable. Moreover, we again assume that $\mathcal{H}$ is decomposed into (mutually orthogonal) macro spaces $\mathcal{H}_\nu$ of dimension d_ν as in (1.12).

Generalized Normal Typicality – Absolute Errors

In this section we consider Hamiltonians H with spectral decomposition

$$H = \sum_{e\in\mathcal{E}} e\, \Pi_e. \tag{3.53}$$

Here, $\mathcal{E}$ denotes the set of distinct eigenvalues of H and Π_e is the projection onto the eigenspace of H corresponding to the eigenvalue e. Let $D_E := \max_{e\in\mathcal{E}} \operatorname{tr}(\Pi_e)$ be the maximum degeneracy of an eigenvalue and D_G the maximum degeneracy of an eigenvalue gap. Moreover, let $d_E := \#\mathcal{E}$ and for any $\kappa > 0$ let $G(\kappa)$ be the maximal number of gaps in an energy interval of length κ, i.e.,

$$G(\kappa) = \max_{E\in\mathbb{R}} \#\left\{(e, e') \in \mathcal{E}\times\mathcal{E} : e \neq e' \text{ and } e - e' \in [E, E+\kappa)\right\}. \tag{3.54}$$

Note that $D_G = \lim_{\kappa\to 0^+} G(\kappa)$.

We obtained the following theorem (Theorem 4 in [136]):

Theorem 3.12. *Let B be an operator on $\mathcal{H}$, let $\varepsilon, \delta, \kappa, T > 0$, let μ be an arbitrary macro state, and define*

$$M_{\mu B} := \frac{1}{d_\mu} \sum_{e\in\mathcal{E}} \operatorname{tr}(P_\mu \Pi_e B \Pi_e). \tag{3.55}$$

Then $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ *are such that for* $(1-\delta)$*-most* $t \in [0,T]$

$$\begin{aligned}\Big|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}\Big| \\ \leq 4\sqrt{\frac{D_E\, G(\kappa)\|B\|}{\delta\varepsilon d_\mu}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\mathrm{tr}(|B|)}{d_\mu}\right\}}. \qquad (3.56)\end{aligned}$$

Theorem 3.12 shows that as soon as $d_\mu \gg D_E G(\kappa)\|B\|^2$ and T is large enough, the right-hand side of (3.56) is small. This implies that $\langle\psi_t|B|\psi_t\rangle$ is close to $M_{\mu B}$ for most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ for most times $t \in [0,T]$.

Setting $B = P_\nu$ in Theorem 3.12, choosing $\kappa > 0$ small enough such that $G(\kappa) = D_G$ and then taking the limit $T \to \infty$ immediately gives the following corollary (Theorem 2 in [136]):

Corollary 3.13 (Generalized normal typicality)**.** *Let* μ, ν *be any macro states and define*

$$M_{\mu\nu} := \frac{1}{d_\mu}\sum_{e\in\mathcal{E}} \mathrm{tr}(P_\mu \Pi_e P_\nu \Pi_e). \qquad (3.57)$$

Let $\varepsilon, \delta > 0$*. Then* $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ *are such that for* $(1-\delta)$*-most* $t \in [0,\infty)$

$$\Big|\|P_\nu\psi_t\|^2 - M_{\mu\nu}\Big| \leq 4\sqrt{\frac{D_E D_G}{\delta\varepsilon d_\mu}\min\left\{1, \frac{d_\nu}{d_\mu}\right\}}. \qquad (3.58)$$

It follows from Corollary 3.13 that as soon as $d_\mu \gg D_E D_G$, i.e., as soon as d_μ is large and no eigenvalue and no gap of H is macroscopically degenerate, the right-hand side of (3.58) is small. Therefore Corollary 3.13 shows that most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ from a macro space $\mathcal{H}_\mu$ are such that for most times in the long run the superposition weight $\|P_\nu\psi_t\|^2$ is close to the value $M_{\mu\nu}$.

Next we present another corollary (Corollary 1 in [136]) of Theorem 3.12 (or, more precisely, of Corollary 3.13). A system of N particles or N degrees of freedom has dimension $D \sim \exp(N)$. More precisely, we expect that $D \approx \exp(s_{\mathrm{eq}}N/k_B)$, where s_{eq} denotes the entropy per particle in the thermal equilibrium macro state. Similarly we define for all macro states μ the entropy per particle s_μ by

$$d_\mu = \exp(s_\mu N/k_B). \qquad (3.59)$$

Corollary 3.14. *Suppose* (3.59)*. Let* μ, ν_-, ν_+ *be macro states such that*

$$s_{\nu_-} \leq s_\mu \leq s_{\nu_+}. \qquad (3.60)$$

Then $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$ most $t \in [0,\infty)$,

$$\left| \|P_{\nu_+}\psi_t\|^2 - M_{\mu\nu_+} \right| \leq \frac{4\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{s_\mu N}{2k_\mathrm{B}}\right), \tag{3.61}$$

$$\left| \|P_{\nu_-}\psi_t\|^2 - M_{\mu\nu_-} \right| \leq \frac{4\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{(s_\mu - \frac{s_{\nu_-}}{2})N}{k_\mathrm{B}}\right). \tag{3.62}$$

Thus if s_μ and $s_{\nu_\pm}$ are fixed and D_E and D_G are of order 1, the error bounds in Corollary 3.14 are exponentially small in N.

Generalized Normal Typicality – Relative Errors

If the quantities $M_{\mu B}$ are small, the absolute errors obtained in [136] and presented above are not meaningful anymore. Therefore we studied the relative errors in [138], where we modeled the Hamiltonian by a random matrix and made use of results concerning the delocalization of eigenvectors of random matrices. For the results regarding the relative errors we always assume that $D_E = D_G = 1$. The reason for this is that for the random matrices that we consider in [138] the joint distribution of their entries is absolutely continuous with respect to the Lebesgue measure which implies that $D_E = D_G = 1$ with probability 1 (cf. Proposition 4 in [138]):

Proposition 3.15. *Let H be a random $D \times D$ matrix with eigenvalues $E_1, \ldots, E_D$ such that the joint distribution of its entries is absolutely continuous with respect to the Lebesgue measure. Then,*

$$\mathbb{P}\Big(E_i - E_j = E_k - E_l \text{ for some } (i \neq j \text{ or } k \neq l) \text{ and } (i \neq k \text{ or } j \neq l)\Big) = 0. \tag{3.63}$$

Regarding the system's Hamiltonian we make the following assumption:

Assumption 1. The Hamiltonian H is of the form $H = H_0 + V$ where H_0 is a deterministic Hermitian $D \times D$ matrix and V is a Hermitian random Gaussian perturbation, i.e., $V = \frac{1}{\sqrt{2}}(A + A^*)$ where $A = (a_{ij})$ is a random $D \times D$ matrix whose entries are independent complex Gaussian random variables with mean zero. More precisely, all random variables $\operatorname{Re} a_{ij}$, $\operatorname{Im} a_{ij}$ are independent and there are $\sigma_{ij} > 0$ with $\sigma_{ij} = \sigma_{ji}$ such that $\operatorname{Re} a_{ij}, \operatorname{Im} a_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2/2)$.

We remark that due to Assumption 1 the entries of the random matrix $V = (v_{ij})$ satisfy $\operatorname{Re} v_{ij}, \operatorname{Im} v_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2)$ for $i \neq j$ and $v_{ii} = \operatorname{Re} v_{ii} \sim \mathcal{N}(0, \sigma_{ii}^2)$ and V is a Hermitian Gaussian Wigner-type matrix.

For Hamiltonians $H = H_0 + V$ of the form as in Assumption 1 we define

$$C_{H_0} := D^{-1/2} \max\{\|\operatorname{Re} H_0\|, \|\operatorname{Im} H_0\|\}. \tag{3.64}$$

For example for many-body Hamiltonians we expect that $C_{H_0} \sim \log(D)/\sqrt{D}$ and thus that C_{H_0} is very small for large D.

For Hamiltonians satisfying Assumption 1 we were able to find a lower bound for $M_{\mu\nu}$ (cf. Theorem 4 in [138]):

Theorem 3.16 (Lower bounds for $M_{\mu\nu}$)**.** *Let $\varepsilon' \in (0, 1/2)$ and let μ and ν be arbitrary macro states such that $d_\mu, d_\nu > \max\{166, 4|\log_2(\varepsilon'/\sqrt{2})|\}$. Let H satisfy Assumption 1 and let $\sigma_- := \min_{i,j} \sigma_{ij}$ and $\sigma_+ := \max_{i,j} \sigma_{ij}$. Then with probability at least $1 - \varepsilon'$,*

$$M_{\mu\nu} \geq \left(\sqrt{\varepsilon' C_\sigma} \frac{\max\{d_\mu, d_\nu\}}{D}\right)^{16} \min\left\{1, \frac{d_\nu}{d_\mu}\right\}, \tag{3.65}$$

where

$$C_\sigma := \frac{c_- \sigma_-}{c_+ \sigma_+ + C_{H_0}} \tag{3.66}$$

with absolute constants $c_-, c_+ > 0$.

A combination of Theorem 3.16 and Corollary 3.13 gives an analogue of Corollary 3.14 for the relative errors (Corollary 2 in [138]):

Corollary 3.17 (Generalized normal typicality – relative errors)**.** *Let $\varepsilon, \delta > 0$ and let ε', μ, ν, C_σ and H be as in Theorem 3.16. Then with probability at least $1 - \varepsilon'$, $(1 - \varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1 - \delta)$-most $t \in [0, \infty)$,*

$$\begin{aligned} &\frac{\left|\|P_\nu \psi_t\|^2 - M_{\mu\nu}\right|}{M_{\mu\nu}} \\ &\quad \leq \frac{4}{\sqrt{\varepsilon\delta}} (C_\sigma \varepsilon')^{-8} \exp\left(-\frac{N}{2k_B}\left(\min\{s_\mu, s_\nu\} - 32(s_{\mathrm{mc}} - \max\{s_\mu, s_\nu\})\right)\right), \end{aligned} \tag{3.67}$$

where s_{mc} is defined by $D = \exp(s_{\mathrm{mc}} N/k_B)$.

Corollary 3.17 shows that if the entropies s_μ, s_ν and s_{mc} satisfy $s_{\mathrm{mc}} < \max\{s_\mu, s_\nu\} + \min\{s_\mu, s_\nu\}/32$, then the relative errors are exponentially small in N. As $s_{\mathrm{mc}} \approx s_{\mathrm{eq}}$, at least one of the macro spaces $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$ has to be very large in order to have small relative errors. This might seem rather restrictive at first, however, even in the case of normal typicality we have that

$$M_{\mu\nu} \approx \frac{d_\nu}{D} \approx \exp\left(-\frac{s_{\mathrm{mc}} - s_\nu}{k_B} N\right) \tag{3.68}$$

which implies that even in this case the relative errors are only small if $s_{\mathrm{mc}} < \max\{s_\nu, s_\mu\} + \min\{s_\nu, s_\mu\}/2$. Therefore also in this situation at least one of the macro spaces has to have a relative entropy close to s_{eq}.

We now turn to a more general version of Theorem 3.16 which we also prove in [138]. For this result, Assumption 1 is replaced by the following weaker assumption on the Hamiltonian $H = (h_{ij})$:

Assumption 2. The random variables $(\operatorname{Re} h_{ij})_{i\leq j}$ and $(\operatorname{Im} h_{ij})_{i<j}$ are mutually independent and continuously distributed. Moreover, the densities ρ_{ij}^{Re} of $\operatorname{Re} h_{ij}$ are bounded.

Under this assumption we proved the following theorem (Theorem 10 in [138]):

Theorem 3.18. *Let $\varepsilon' \in (0, 1/2)$ and let μ be an arbitrary macro state such that $d_\mu > \max\{166, 2|\log_2 \varepsilon'|\}$. Let B be a Hermitian $D \times D$ matrix and let $H = (h_{ij})$ be a random Hermitian $D \times D$ matrix satisfying Assumption 2. Let $K > 0$ be the least upper bound for the densities ρ_{ij}^{Re}, i.e.,*

$$K := \sup \bigcup_{i\leq j} \left\{\rho_{ij}^{\mathrm{Re}}(x) : x \in \mathbb{R}\right\} < \infty. \tag{3.69}$$

Let $\eta \in (0, 1/2)$ be the unique solution of

$$1 - \varepsilon' = \left(1 - 2^{-d_\mu/2}\right)(1 - \eta)^4 \tag{3.70}$$

and let $J^{\mathrm{Re}}, J^{\mathrm{Im}} \in (0, \infty)$ be the unique numbers such that

$$\mathbb{P}\left(\|\operatorname{Re} H\| \leq J^{\mathrm{Re}} \sqrt{D}\right) = 1 - \eta, \tag{3.71}$$

$$\mathbb{P}\left(\|\operatorname{Im} H\| \leq J^{\mathrm{Im}} \sqrt{D}\right) = 1 - \eta. \tag{3.72}$$

Set $J := \max\{K^{-1}, J^{\mathrm{Re}}, J^{\mathrm{Im}}\}$. Then with probability at least $1 - \varepsilon'$,

$$|M_{\mu B}| \geq \max\left\{b_{\min}^{+}, \left(\frac{d_\mu}{4C\sqrt{KJD}}\right)^{16} \frac{\operatorname{tr}(B^+)}{d_\mu}\right\} - \min\left\{b_{\max}^{-}, \frac{\operatorname{tr}(B^-)}{d_\mu}\right\}, \tag{3.73}$$

where B^+ and B^- are the positive and negative part of B, $b_{\min}^{+}$ and $b_{\max}^{-}$ denote the smallest and largest eigenvalue of B^+ and B^- respectively and $C \geq 1$ is a constant. In particular, if $B = P_\nu$ for some macro state ν, then

$$M_{\mu\nu} \geq \frac{d_\nu}{d_\mu}\left(\frac{d_\mu}{4C\sqrt{KJD}}\right)^{16}. \tag{3.74}$$

Moreover, if $d_\nu \geq 166$, then for any $\eta \in (0,1/2)$ and J chosen as above it holds with probability at least $\left(1-2^{-d_\nu/2}\right)(1-\eta)^4$ that

$$M_{\mu\nu} \geq \left(\frac{d_\nu}{4C\sqrt{KJD}}\right)^{16}. \tag{3.75}$$

Theorem 3.18 gives us a non-trivial lower bound on $|M_{\mu B}|$ if the spectrum of B satisfies certain assumptions, e.g., if B is a positive operator or $b^+_{\min} > b^-_{\max}$.

From Theorem 3.18 together with Theorem 3.12 we immediately obtain the following corollary concerning the relative errors (Corollary 3 in [138]):

Corollary 3.19. *Let $\varepsilon, \delta, \kappa, T > 0$, $\varepsilon' \in (0,1/2)$ and let μ be an arbitrary macro state such that $d_\mu > \max\{166, 2|\log_2 \varepsilon'|\}$. Let B be a Hermitian $D \times D$ matrix and let H be a random Hermitian $D \times D$ matrix satisfying Assumption 2. Let $K, J > 0$, $C \geq 1$ and $\eta \in (0,1/2)$ be as in Theorem 3.18. Then with probability at least $1-\varepsilon'$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,T]$*

$$\frac{\left|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}\right|}{|M_{\mu B}|} \leq \frac{4\left(\frac{G(\kappa)\|B\|}{\delta\varepsilon d_\mu}\left(1+\frac{8\log_2 D}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2}}{\max\left\{b^+_{\min}, \left(\frac{d_\mu}{4c_C\sqrt{KJD}}\right)^{16}\frac{\operatorname{tr}(B^+)}{d_\mu}\right\} - \min\left\{b^-_{\max}, \frac{\operatorname{tr}(B^-)}{d_\mu}\right\}}, \tag{3.76}$$

whenever the denominator of the right-hand side is positive.

In the case that $B = P_\nu$ for some macro state ν a combination of Theorem 3.13 together with Theorem 3.18 yields the following result for the relative errors (Corollary 4 in [138]):

Corollary 3.20 (Generalized normal typicality – relative errors)**.** *Let $\varepsilon, \delta > 0$, $\varepsilon' \in (0,1/2)$ and let μ and ν be two macro states with $d_\mu, d_\nu > \max\{166, 2|\log_2(\varepsilon'/\sqrt{2})|\}$. Let H be a random Hermitian $D \times D$ matrix satisfying Assumption 2 and let $K > 0$ be defined as in Theorem 3.18. Moreover, let $\eta \in (0,1/2)$ be the unique solution of*

$$1-\varepsilon' = \left(1-2^{-d_\mu/2}-2^{-d_\nu/2}\right)(1-\eta)^4 \tag{3.77}$$

and let $J > 0$ be defined as in Theorem 3.18. Then with probability at least $1-\varepsilon'$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,\infty)$,

$$\frac{\left|\|P_\nu\psi_t\|^2 - M_{\mu\nu}\right|}{M_{\mu\nu}} \leq \frac{4}{\sqrt{\varepsilon\delta\min\{d_\mu, d_\nu\}}}\left(\frac{4C\sqrt{KJD}}{\max\{d_\mu, d_\nu\}}\right)^{16}, \tag{3.78}$$

where $C \geq 1$ is a universal constant.

Dynamical Typicality

In the following we present results on dynamical typicality which were obtained in [136, 138]. The first result is Theorem 3 from [136] (with corrected constants) and the corollary afterwards is Theorem 1 in [136].

Theorem 3.21 (Dynamical typicality)**.** *Let μ, ν be arbitrary macro states and let B be any operator on $\mathcal{H}$. Then there exists a function $w_{\mu B} : \mathbb{R} \to [0,1]$ such that for every $t \in \mathbb{R}$ and every $\varepsilon > 0$, for $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,*

$$\Big|\langle\psi_t|B|\psi_t\rangle - w_{\mu B}(t)\Big| \leq \min\left\{\frac{\|B\|}{\sqrt{\varepsilon d_\mu}}, \sqrt{\frac{\|B\|\,\mathrm{tr}(|B|)}{\varepsilon d_\mu^2}}, \sqrt{\frac{36\pi^3\log(8/\varepsilon)}{d_\mu}}\|B\|\right\}. \quad (3.79)$$

Moreover, for every μ and B, every $T > 0$, and $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,

$$\frac{1}{T}\int_0^T |\langle\psi_t|B|\psi_t\rangle - w_{\mu B}(t)|^2 \, dt \leq \frac{\|B\|^2}{\varepsilon d_\mu}. \quad (3.80)$$

Theorem 3.21 shows that if $d_\mu \gg 1/\varepsilon$ and $\|B\|$ is not too large, then for every $t \in \mathbb{R}$ and uniformly distributed $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$, the random variable $\langle\psi_t|B|\psi_t\rangle$ is close to $w_{\mu B}(t)$ with very high probability. Note that the quantity $w_{\mu B}$ is given by

$$w_{\mu B}(t) = \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle = \frac{1}{d_\mu}\,\mathrm{tr}\left[P_\mu \exp(iHt) B \exp(-iHt)\right], \quad (3.81)$$

where $\mathbb{E}_\mu$ denotes the expectation with respect to the uniform distribution u_μ on $\mathbb{S}(\mathcal{H}_\mu)$. Moreover, the curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ is close to $t \mapsto w_{\mu B}(t)$ in L^2-norm on the interval $[0,T]$.

Corollary 3.22 (Dynamical typicality for P_ν)**.** *Let μ, ν be two arbitrary macro states. Then there is a function $w_{\mu\nu} : \mathbb{R} \to [0,1]$ such that for every $t \in \mathbb{R}$ and every $\varepsilon > 0$, for $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,*

$$\Big|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)\Big| \leq \frac{1}{\sqrt{\varepsilon d_\mu}}. \quad (3.82)$$

Moreover, for every μ, ν, every $T > 0$, and $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,

$$\frac{1}{T}\int_0^T \big|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)\big|^2 \, dt \leq \frac{1}{\varepsilon d_\mu}. \quad (3.83)$$

In [138] we also wanted to study the relative errors for dynamical typicality. We considered the same setting as for the results regarding the relative errors of generalized normal typicality. While we were not able to bound the relative errors for all times[25] (except for a very special case, see Proposition 3.24), we could give upper bounds for the *comparative error*

$$\frac{\left|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)\right|}{\overline{\|P_\nu\psi_t\|^2}}. \tag{3.84}$$

Thus we compare the absolute error to the infinite time average of $\|P_\nu\psi_t\|^2$ which, roughly speaking, expresses the magnitude that we expect of $\|P_\nu\psi_t\|^2$. We obtained the following result (Theorem 5 in [138]):

Theorem 3.23 (Dynamical typicality – comparative errors)**.** *Let $\varepsilon > 0$, $\varepsilon' \in (0, 1/2)$ and let μ and ν be macro states such that $d_\nu > \max\{166, 4|\log_2 \varepsilon'|\}$. Let H be a random Hermitian $D \times D$ matrix as in Theorem 3.16. Then with probability at least $1-\varepsilon'$, for every $t \in \mathbb{R}$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that*

$$\frac{\left|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)\right|}{\overline{\|P_\nu\psi_t\|^2}} \leq \frac{1}{\sqrt{\varepsilon}}(C_\sigma\varepsilon')^{-8} \exp\left(-\frac{N}{2k_B}\left(2s_\mu - \min\{s_\mu, s_\nu\} - 32(s_{\mathrm{mc}} - s_\nu)\right)\right), \tag{3.85}$$

where C_σ is defined in (3.66)*. Moreover, with probability at least $1-\varepsilon'$, for every $T > 0$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that*

$$\frac{1}{T}\int_0^T \frac{\left|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)\right|^2}{\overline{\|P_\nu\psi_t\|^2}^2}\, dt \leq \frac{1}{\varepsilon}(C_\sigma\varepsilon')^{-16} \exp\left(-\frac{N}{k_B}\left(s_\mu - 32(s_{\mathrm{mc}} - s_\nu)\right)\right). \tag{3.86}$$

Theorem 3.23 shows that as soon as $s_{\mathrm{mc}} < s_\nu + s_\mu/16 - \min\{s_\mu, s_\nu\}/32$ resp. $s_{\mathrm{mc}} < s_\nu + s_\mu/32$, the comparative errors are exponentially small in N.

Next we state two more general results concerning dynamical typicality. For the relative errors we could show with the help of Theorem 3.21 and without any further assumptions on the Hamiltonian the following proposition (Proposition 2 in [138]):

Proposition 3.24. *Let B be a Hermitian operator on $\mathcal{H}$ such that $b := \max\{b^+_{\min} - b^-_{\max}, b^-_{\min} - b^+_{\max}\} > 0$, where $b^\pm_{\min}$ and $b^\pm_{\max}$ are the smallest and largest eigenvalues of $B^\pm$. Let $\varepsilon > 0$, $t \in [0, \infty)$, and let μ be an arbitrary macro state. Then*

$$|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle| \geq b, \tag{3.87}$$

[25] We obtained a result for most times but for most times already generalized normal typicality tells us that $\|P_\nu\psi_t\|^2$ is close to $M_{\mu\nu}$.

which implies that $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{\left|\langle\psi_t|B|\psi_t\rangle - \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|}{\left|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|} \leq b^{-1}\min\left\{\frac{\|B\|}{\sqrt{\varepsilon d_\mu}}, \sqrt{\frac{\|B\|\operatorname{tr}(|B|)}{\varepsilon d_\mu^2}}, \sqrt{\frac{36\pi^3\log(8/\varepsilon)}{d_\mu}}\|B\|\right\}. \tag{3.88}$$

Moreover, for every μ, B and $T>0$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{1}{T}\int_0^T \frac{\left|\langle\psi_t|B|\psi_t\rangle - \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|^2}{\left|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|^2}\,dt \leq \frac{\|B\|^2}{b^2\varepsilon d_\mu}. \tag{3.89}$$

We remark that the condition that $b>0$ in Proposition 3.24 is fulfilled if B is, e.g., a (strictly) positive or (strictly) negative operator. However, if B is neither positive nor negative, its spectrum has to be rather special to satisfy the condition $b>0$. In particular, in the case that $B = P_\nu$ for some macro state ν we have that $b=0$.

We obtained more useful bounds for the comparative error instead of the relative error by assuming that the Hamiltonian is a random matrix which satisfies Assumption 2. More precisely, we showed the following result (Theorem 11 in [138]):

Theorem 3.25. *Let $\varepsilon>0$, $\varepsilon' \in (0,1/2)$ and let μ and ν be two macro states such that $d_\mu, d_\nu > \max\{166, 4|\log_2\varepsilon'|\}$. Let B be a Hermitian $D\times D$ matrix and let H be a random Hermitian $D\times D$ matrix satisfying Assumption 2. Then with probability at least $1-\varepsilon'$, for $t\in\mathbb{R}$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that*

$$\begin{aligned}&\frac{\left|\langle\psi_t|B|\psi_t\rangle - \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|}{\left|\overline{\langle\psi_t|B|\psi_t\rangle}\right|}\\ &\qquad\leq \mathrm{LB(B,\psi)}^{-1}\min\left\{\frac{\sqrt{2}\|B\|}{\sqrt{\varepsilon d_\mu}}, \sqrt{\frac{2\|B\|\operatorname{tr}(|B|)}{\varepsilon d_\mu^2}}, \sqrt{\frac{36\pi^3\log(16/\varepsilon)}{d_\mu}}\|B\|\right\},\end{aligned} \tag{3.90}$$

whenever

$$\begin{aligned}\mathrm{LB}(B,\psi) := \max&\left\{b^+_{\min}, \left(\frac{d_\mu}{4C\sqrt{KJD}}\right)^{16}\frac{\operatorname{tr}(B^+)}{d_\mu}\right\} - \min\left\{b^-_{\max}, \frac{\operatorname{tr}(B^-)}{d_\mu}\right\}\\ &- \sqrt{2}\left(\frac{\|B\|}{\varepsilon d_\mu}\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2} > 0.\end{aligned} \tag{3.91}$$

Moreover, for every μ, B, and $T > 0$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{1}{T}\int_0^T \frac{\left|\langle\psi_t|B|\psi_t\rangle - \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|^2}{\left|\overline{\langle\psi_t|B|\psi_t\rangle}\right|^2}\,dt \leq \frac{2\|B\|^2}{\mathrm{LB}(B,\psi)^2\varepsilon d_\mu}. \tag{3.92}$$

We remark that for large d_μ the lower bound $\mathrm{LB}(B,\psi)$ on $|\overline{\langle\psi_t|B|\psi_t\rangle}|$ is almost the same as the lower bound for $|M_{\mu B}|$ that we obtained in Theorem 3.18.

Generalized Normal Typicality for GAP Measures

The results concerning generalized normal typicality presented so far used the uniform distribution on the sphere of a macro space as the measure of typicality. The goal of [143] was to replace the uniform distribution by a much more general class of natural and physically distributions, the GAP measures, see Section 1.5 for a definition and relevant properties of these measures.

The setting is the same as for the results concerning the absolute errors from [136] except for the Hilbert space $\mathcal{H}$ which need not be finite-dimensional but only separable. In the case that $\mathcal{H}$ is infinite-dimensional, the quantities introduced before stating the results regarding the absolute errors (d_E, D_E, D_G and $G(\kappa)$) are not necessarily finite anymore. To overcome this problem, we define these quantities relative to an operator B on $\mathcal{H}$. This is motivated by the observation that only those eigenvalues e and e' of H with $\Pi_e B \Pi_{e'} \neq 0$ give non-vanishing contributions to the quantities we have to compute such that if we assume that there are only finitely many such e and e', all sums over the eigenvalues of H become effectively finite. We define

$$\begin{aligned}
\mathcal{E}_B &:= \{e \in \mathcal{E} : \exists e' \in \mathcal{E} \text{ such that } \Pi_e B \Pi_{e'} \neq 0 \text{ or } \Pi_{e'} B \Pi_e \neq 0\}, && (3.93)\\
d_{E,B} &:= |\mathcal{E}_B|, && (3.94)\\
D_{E,B} &:= \max_{e\in\mathcal{E}_B} \mathrm{tr}(\Pi_e), && (3.95)\\
D_{G,B} &:= \max_{E\in\mathbb{R}} |\{(e,e') \in \mathcal{E}_B \times \mathcal{E}_B : e \neq e' \text{ and } e - e' = E\}| && (3.96)\\
G_B(\kappa) &:= \max_{E\in\mathbb{R}} |\{(e,e') \in \mathcal{E}_B \times \mathcal{E}_B : e \neq e' \text{ and } e - e' \in [E, E+\kappa)\}|. && (3.97)
\end{aligned}$$

In [143, Theorem 1] we obtained the following result:

Theorem 3.26 (Normal typicality for GAP(ρ))**.** *Let $\mathcal{H}$ be a separable Hilbert space with $\dim\mathcal{H} \geq 4$, let B be an operator on $\mathcal{H}$ with $d_{E,B} < \infty$ and let ρ be a density matrix on $\mathcal{H}$ with $\|\rho\| < 1/4$. Let $\varepsilon, \delta, \kappa, T > 0$ and define*

$$M_{\rho B} := \sum_{e\in\mathcal{E}} \mathrm{tr}(\rho \Pi_e B \Pi_e). \tag{3.98}$$

Then, w.r.t. GAP(ρ), $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H})$ *are such that for* $(1-\delta)$*-most* $t \in [0,T]$,

$$\Big|\langle\psi_t|B|\psi_t\rangle - M_{\rho B}\Big| \leq \left(\frac{188}{\varepsilon\delta}\|B\|^2\|\rho\|D_{E,B}G_B(\kappa)\left(1+\frac{8\log_2 d_{E,B}}{\kappa T}\right)\right)^{1/2}. \tag{3.99}$$

Moreover, w.r.t. GAP(ρ), $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H})$ *are such that for* $(1-\delta)$*-most* $t \in [0,\infty)$,

$$\Big|\langle\psi_t|B|\psi_t\rangle - M_{\rho B}\Big| \leq \left(\frac{188}{\varepsilon\delta}\|B\|^2\|\rho\|D_{E,B}D_{G,B}\right)^{1/2}. \tag{3.100}$$

Theorem 3.26 shows that as soon as $\|\rho\|$ is small, $\|B\|$ is not too large and no eigenvalue and eigenvalue gap of H is hugely degenerate, the expectation $\langle\psi_t|B|\psi_t\rangle$ is close to $M_{\rho B}$ for GAP(ρ)-most initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$ and most times $t \geq 0$. For the finite-time result we need that $T > 0$ is sufficiently large and unfortunately T has to be extremely large to ensure a small error in (3.99), see also Section 1.6.1 and Section 1.6.5. Note that the projections P_ν onto the macro spaces $\mathcal{H}_\nu$ satisfy the assumptions in Theorem 3.26 since usually a (coarse-grained) version of the energy is among the macro observables which are used for the construction of the macro spaces and therefore $\mathcal{H}_\nu$ is a subset of a (finite-dimensional) energy shell $\mathcal{H}_{\mathrm{mc}}$.

3.2.2. Strategy of Proof

Generalized Normal Typicality – Absolute Errors

In the proof of Theorem 3.12 we made use of a formula for the Hilbert space covariance of two operators. Note that the covariance of two complex-valued random variables X, Y is defined as

$$\mathrm{Cov}[X,Y] := \mathbb{E}\left[(X-\mathbb{E}X)^*(Y-\mathbb{E}Y)\right] = \mathbb{E}[X^*Y] - (\mathbb{E}X)^*\mathbb{E}Y. \tag{3.101}$$

We first showed the following lemma (Lemma 1 in [136]):

Lemma 3.27 (Hilbert Space Covariance)**.** *Let* $\mathcal{H}$ *be a Hilbert space of dimension* $D = \dim\mathcal{H}$, *let* $\psi \in \mathbb{S}(\mathcal{H})$ *be uniformly distributed and let* B *and* C *be two operators on* $\mathcal{H}$. *Then,*

$$\mathrm{Cov}\left[\langle\psi|B|\psi\rangle, \langle\psi|C|\psi\rangle\right] = \frac{\mathrm{tr}(B^*C)}{D(D+1)} - \frac{\mathrm{tr}(B^*)\,\mathrm{tr}(C)}{D^2(D+1)}. \tag{3.102}$$

Moreover,

$$\mathbb{E}\left[\langle\psi|B|\psi\rangle^*\langle\psi|C|\psi\rangle\right] = \frac{\mathrm{tr}(B^*)\,\mathrm{tr}(C) + \mathrm{tr}(B^*C)}{D(D+1)}. \tag{3.103}$$

Note that for uniformly distributed $\psi \in \mathbb{S}(\mathcal{H}_\mu)$, Lemma 3.27 implies that

$$\mathbb{E}_\mu\left[\langle\psi|B^*|\psi\rangle\langle\psi|C|\psi\rangle\right] = \frac{1}{d_\mu(d_\mu+1)}\left(\operatorname{tr}(P_\mu B^*)\operatorname{tr}(P_\mu C) + \operatorname{tr}(P_\mu B^* P_\mu C)\right). \tag{3.104}$$

Lemma 3.27 can be proved by a direct computation with the help of the formulas in Lemma 1.2 for the fourth moments of components of a uniformly distributed random vector $\psi \in \mathbb{S}(\mathcal{H})$.

We remark that a close inspection of the proof of Theorem 3.12 that we gave in [136] reveals that we actually do not need a formula for the covariance (the step where we used it was unnecessary) and that the formula for the variance in Lemma 1.2 suffices.

The next step in the proof of Theorem 3.12 is to compute and estimate some averages over $\mathbb{S}(\mathcal{H}_\mu)$. This was done in Proposition 1 in [136] which we quote below. Before doing so, we first note that for every operator B on $\mathcal{H}$ and every $\psi_0 \in \mathbb{S}(\mathcal{H})$ we have that

$$M_{\psi_0 B} = \overline{\langle\psi_t|B|\psi_t\rangle} = \sum_{e\in\mathcal{E}}\langle\psi_0|\Pi_e B \Pi_e|\psi_0\rangle \tag{3.105}$$

and $\mathbb{E}_\mu M_{\psi_0 B} = M_{\mu B}$.

Proposition 3.28. *Let $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ be uniformly distributed and let B be any operator on $\mathcal{H}$. Then for every $\kappa, T > 0$,*

$$\mathbb{E}_\mu\left(\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T\right) \leq \frac{2D_E G(\kappa)}{d_\mu+1}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|^2, \frac{\operatorname{tr}(B^*B)}{d_\mu}\right\}, \tag{3.106}$$

$$\operatorname{Var}_\mu \overline{\langle\psi_t|B|\psi_t\rangle} \leq \frac{\|B\|}{d_\mu+1}\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}, \tag{3.107}$$

where Var_μ denotes the variance with respect to the uniform distribution over $\mathbb{S}(\mathcal{H}_\mu)$.

For the proof of Proposition (3.28) we start similarly as in the proof of Theorem 1 in [125], see also Theorem 1.28 and the paragraphs before. We find that

$$\begin{aligned}\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T &= \sum_{\substack{e\neq e'\\ e''\neq e'''}}\left\langle e^{i(e-e'-e''+e''')t}\right\rangle_T \langle\psi_0|\Pi_e B\Pi_{e'}|\psi_0\rangle\langle\psi_0|\Pi_{e'''}B^*\Pi_{e''}|\psi_0\rangle && (3.108\text{a})\\ &\leq \|R\|\sum_{e\neq e'}\left|\langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle\right|^2, && (3.108\text{b})\end{aligned}$$

where R is the Hermitian matrix defined as in (1.160). With the help of Lemma 3.27 (or alternatively Lemma 1.2) and the bound on $\|R\|$ from (1.163) (which is due to Short and Farrelly [125]) we obtain

$$\begin{aligned}\mathbb{E}_\mu & \left(\left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \right) \\ & \leq \frac{G(\kappa)}{d_\mu(d_\mu+1)} \left(1 + \frac{8\log_2 d_E}{\kappa T}\right) \sum_{e,e'} \left[|\mathrm{tr}(P_\mu \Pi_e B \Pi_{e'})|^2 + \mathrm{tr}(P_\mu \Pi_e B \Pi_{e'} P_\mu \Pi_{e'} B^* \Pi_e) \right].\end{aligned} \tag{3.109}$$

By making use of the trace inequalities (3.41) and (3.42) we can estimate

$$\sum_{e,e'} |\,\mathrm{tr}(P_\mu \Pi_e B \Pi_{e'})|^2 \leq \sum_{e,e'} \mathrm{tr}(\Pi_{e'} P_\mu \Pi_e P_\mu)\,\mathrm{tr}(\Pi_{e'} B^* \Pi_e B) \tag{3.110a}$$

$$\leq D_E \min\{\|B\|^2 d_\mu, \mathrm{tr}(B^*B)\}. \tag{3.110b}$$

The second line follows from applying the (second) trace inequality either to the first or the second factor on the right-hand side of (3.110a). An application of this trace inequality as well as $\|\sum_{e'} \Pi_{e'} P_\mu \Pi_{e'}\| \leq 1$ shows that

$$\sum_{e,e'} \mathrm{tr}(P_\mu \Pi_e B \Pi_{e'} P_\mu \Pi_{e'} B^* \Pi_{e'}) \leq \min\{\|B\|^2 d_\mu, \mathrm{tr}(B^*B)\}. \tag{3.111}$$

Putting everything together proves (3.106).

Next we turn to the bound for the variance $\mathrm{Var}_\mu\, \overline{\langle\psi_t|B|\psi_t\rangle}$. Lemma 1.2 immediately shows that

$$\mathrm{Var}_\mu\, \overline{\langle\psi_t|B|\psi_t\rangle} \leq \frac{1}{d_\mu(d_\mu+1)} \sum_{e,e'} \mathrm{tr}(P_\mu \Pi_e B \Pi_e P_\mu \Pi_{e'} B^* \Pi_{e'}). \tag{3.112}$$

With the help of similar methods as above we can estimate the sum over e and e' and obtain

$$\sum_{e,e'} \mathrm{tr}(P_\mu \Pi_e B \Pi_e P_\mu \Pi_{e'} B^* \Pi_{e'}) \leq \|B\| \min\{d_\mu \|B\|, \mathrm{tr}(|B|)\} \tag{3.113}$$

which proves (3.107).

After having proved Proposition 3.28, the proof of Theorem 3.12 basically reduces to a careful application of Markov's and Chebychev's inequality. Moreover, we remark that Corollary 3.13 and Corollary 3.14 are almost immediate consequences of Theorem 3.12.

Generalized Normal Typicality – Relative Errors

The main tool for the proof of the lower bounds for $M_{\mu\nu}$ in Theorem 3.16 and for $|M_{\mu B}|$ in Theorem 3.18 is an extension of the no-gaps delocalization result of Rudelson and Vershynin [120], see Theorem 1.47 for the statement with corrected constants.

The constants in the original result of Rudelson and Vershynin [120] were slightly different, more precisely, the lower bound on κ was $8/D$ instead of $180/D$ and the exponent of κs in (1.276) was 6 instead of 9. As for our purposes the dependence of constants on certain parameters is important, we went through the proof in [120], followed the exponents and constants and arrived at these slightly different values, see [138] for a more detailed discussion of the steps in the proof where we obtained different constants and exponents than in [120]. Moreover, a close inspection of the proof of Theorem 1.47 revealed that $C \sim \sqrt{K} J^{0.7}$, see Remark 3 in [138] for more details and in particular for a list of all the constants that appeared in the proof of Rudelson and Vershynin [120].

In [138] we extended Theorem 1.47 to random matrices whose imaginary part is not fixed but random as well, we also allowed $0 < J \leq 1$ and we improved the exponent of κs in (1.276) from 9 to 8. More precisely, we showed the following theorem (Theorem 7 in [138]):

Theorem 3.29. *Let $H = (h_{ij})$ be a random $D \times D$ matrix such that $\operatorname{Re} H$ and $\operatorname{Im} H$ are independent, for $i, j \in \{1, \ldots, D\}$ the (continuous) random variable $\operatorname{Re} h_{ij}$ is independent of the other entries of $\operatorname{Re} H$ except possibly $\operatorname{Re} h_{ji}$, for $i, j \in \{1, \ldots, D\}, i \neq j$ the (continuous) random variable $\operatorname{Im} h_{ij}$ is independent of the other entries of $\operatorname{Im} H$ except possibly $\operatorname{Im} h_{ji}$, and the densities ρ_{ij}^{Re} of $\operatorname{Re} h_{ij}$ are bounded by some number $K > 0$. Choose $J > 0$ such that $JK \geq 1$ and such that the boundedness events $\mathcal{B}_{\operatorname{Re} H, J}$ and $\mathcal{B}_{\operatorname{Im} H, J}$ hold with probability at least $1 - \eta$ for some $0 < \eta \leq 1/2$. Let $\kappa \in (83/D, 1/2)$ and $0 < s \leq 1$. Then the following holds with probability at least $\left(1 - (c\sqrt{KJ}s)^{\kappa D}\right)(1-\eta)^4$, where $c \geq 1$ is a universal constant: Every eigenvector ϕ of H satisfies*

$$\|\phi_I\| \geq (\kappa s)^8 \|\phi\| \quad \text{for all } I \subset \{1, \ldots, D\} \text{ with } |I| \geq \kappa D. \tag{3.114}$$

The proof of Theorem 3.29 consists of three steps. The first step is to extend Theorem 1.47 to the case that the imaginary part of H is not fixed but also random (and satisfying the assumptions in Theorem 3.29). To this end we first assume that $J, K \geq 1$ and $\kappa \in (180/D, 1/2)$ and we define

$$\begin{aligned} S := \{&\text{Every eigenvector } \phi \text{ of } H \text{ satisfies } \|\phi_I\| \geq (\kappa s)^9 \|\phi\| \\ &\text{for all } I \subset [D] \text{ with } |I| \geq \kappa D\}, \end{aligned} \tag{3.115}$$

$$E := \left\{ Q \in \mathbb{R}^{D \times D} : \mathbb{P}\left(\mathcal{B}_{\operatorname{Re} H + iQ, 2J}\right) \geq \frac{1}{2} \right\}. \tag{3.116}$$

With the help of Theorem 1.47, the law of total expectation and the monotonicity of the conditional expectation we find that

$$\mathbb{P}\left(S|\mathcal{B}_{H,2J}\right) \geq \left(1-(Cs)^{\kappa D}\right)\mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{H,2J}\right), \tag{3.117}$$

where $C = C(K, 2J) \geq 1$ is the constant from Theorem 1.47.

Because of $\mathcal{B}_{\operatorname{Re} H,J} \cap \mathcal{B}_{\operatorname{Im} H,J} \subset \mathcal{B}_{H,2J}$ and the independence of $\operatorname{Re} H$ and $\operatorname{Im} H$ we can further estimate

$$\mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{H,2J}\right) \geq (1-\eta)\mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{\operatorname{Im} H,J}\right) \geq (1-\eta)^2, \tag{3.118}$$

where we used that $\mathcal{B}_{\operatorname{Im} H,J} \subset \{\operatorname{Im} H \in E\}$. Putting everything together we obtain

$$\mathbb{P}(S) \geq \mathbb{P}(S|\mathcal{B}_{H,2J})\mathbb{P}(\mathcal{B}_{H,2J}) \geq \left(1-(Cs)^{\kappa D}\right)(1-\eta)^4. \tag{3.119}$$

This proves no-gaps delocalization also for random matrices whose imaginary part is not fixed but random.

The second step in the proof of Theorem 3.29 is to relax the condition that $J \geq 1$ to $J > 0$ and $JK \geq 1$. This follows from applying the generalization of Theorem 1.47 that we just proved to the random matrix $\tilde{H} := J^{-1}H$.

The third step is to improve the interval of admissible κ as well as the exponent of κs in the lower bound on $\|\phi_I\|$. This can be done by observing that an estimate in the proof of the Invertibility Theorem 5 at the end of Section 5 in [120] was not optimal. More precisely, the improved bound is obtained by choosing $\tau = t^{9/19}$ instead of $\tau = \sqrt{t}$ at the end of the proof of Theorem 5 in [120], see [138] for the details.

We remark that the proof of Theorem 3.29 still goes through if some entries of $\operatorname{Im} H$ are fixed which is important as we want to consider Hermitian H and for such H the diagonal entries of $\operatorname{Im} H$ are fixed to zero. Moreover, we note that it is also possible to prove Theorem 3.29 without making use of a scaling argument. For this alternative proof one goes through the proof of Rudelson and Vershynin [120] and identifies the steps where the assumption that $J \geq 1$ is actually used and modifies them accordingly to also allow for $0 < J < 1$. This leads to lower bounds on $M_{\mu\nu}$ and more generally on $|M_{\mu B}|$ that are sometimes slightly better and sometimes slightly worse than the ones obtained in Theorem 3.29. However, as the bounds proved in this way are more complicated we do not go into detail here and refer to Remark 5 in [138].

Before coming to the proof of Theorem 3.18, we remark that by Proposition 3.15, the Hamiltonian H has, with probability 1, non-degenerate eigenvalues and eigenvalue gaps. Since we could not find a proof of this well known fact, we gave a short proof using standard results about smooth manifolds [71] in Appendix A in [138].

With the help of Theorem 3.29 we are now able to prove Theorem 3.18. To this end, let (ϕ_n) be an orthonormal basis of eigenvectors of H. An application of the reverse triangle inequality shows that

$$|M_{\mu B}| \geq \frac{1}{d_\mu}\left(\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|B^+|\phi_n\rangle - \sum_m \langle\phi_m|P_\mu|\phi_m\rangle\langle\phi_m|B^-|\phi_m\rangle\right). \quad (3.120)$$

We then set $\kappa := d_\mu/(2D)$ and $s := 1/(2c\sqrt{KJ})$ where $c \geq 1$ is the constant from Theorem 3.29. With $I = I_\mu = \{d_1 + \cdots + d_{\mu-1} + 1, \ldots, d_1 + \cdots + d_\mu\}$ we obtain with probability at least $1 - \varepsilon'$ that

$$\langle\phi_n|P_\mu|\phi_n\rangle = \|(\phi_n)_{I_\mu}\|^2 \geq (\kappa s)^{16}. \quad (3.121)$$

With this we find the following bounds:

$$\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|B^+|\phi_n\rangle \geq \max\left\{b^+_{\min} d_\mu, (\kappa s)^{16}\operatorname{tr}(B^+)\right\}, \quad (3.122)$$

$$\sum_m \langle\phi_m|P_\mu|\phi_m\rangle\langle\phi_m|B^-|\phi_m\rangle \leq \min\left\{b^-_{\max} d_\mu, \operatorname{tr}(B^-)\right\}. \quad (3.123)$$

Combining these bounds gives the lower bound on $|M_{\mu B}|$ and the first lower bound on $M_{\mu\nu}$ as a special case.

If $d_\nu \geq 166$ we set $\kappa := d_\nu/(2D)$ and then Theorem 3.29 shows that with probability at least $\left(1 - 2^{-d_\nu/2}\right)(1-\eta)^4$,

$$\langle\phi_n|P_\nu|\phi_n\rangle \geq (\kappa s)^{16} = \left(\frac{d_\nu}{4c\sqrt{KJ}D}\right)^{16}, \quad (3.124)$$

from which the second lower bound on $M_{\mu\nu}$ follows almost immediately. This finishes the proof of Theorem 3.18.

For the proof of Theorem 3.16 we have to apply Theorem 3.18 in a special case. To this end we first consider the case that H is a Gaussian random matrix with variances bounded by constants (Theorem 8 in [138]):

Theorem 3.30 (Gaussian matrices, variances bounded by constants)**.** *Let $A = (a_{ij})$ be a random $D \times D$ matrix with independent complex Gaussian entries with mean zero, i.e., all random variables $\operatorname{Re} a_{ij}, \operatorname{Im} a_{ij}$, $i, j \in \{1, \ldots, D\}$ are independent and $\operatorname{Re} a_{ij}, \operatorname{Im} a_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2/2)$ for some $\sigma_{ij} > 0$. Suppose that $\sigma_{ij} = \sigma_{ji}$ for all $i, j \in \{1, \ldots, D\}$ and set $\sigma_- := \min_{i,j}\sigma_{ij}$ and $\sigma_+ := \max_{i,j}\sigma_{ij}$. Then the Hermitian matrix $V := \frac{1}{\sqrt{2}}(A + A^*)$ is Gaussian, more precisely, its entries v_{ij} satisfy $\operatorname{Re} v_{ij}, \operatorname{Im} v_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2/2)$ for $i \neq j$ and $v_{ii} = \operatorname{Re} v_{ii} \sim \mathcal{N}(0, \sigma_{ii}^2)$, and V fulfills the assumptions in Theorem 3.29 with parameters $K = \frac{1}{\sqrt{2\pi}\sigma_-}$ and $J = \frac{4}{\eta}\hat{C}\sigma_+$ for arbitrary $\eta \in (0, 1/2)$, where $\hat{C} \geq 1$ is a universal constant.*

For the proof of Theorem 3.30 we first note that the claims concerning the entries of V immediately follow from its construction and the properties of A.

An important ingredient of the proof is a result from Latala [69] which states that for any finite matrix (x_{ij}) of independent mean zero random variables x_{ij},

$$\mathbb{E}\|(x_{ij})\| \leq \hat{C}\left(\max_i \sqrt{\sum_j \mathbb{E}x_{ij}^2} + \max_j \sqrt{\sum_i \mathbb{E}x_{ij}^2} + \sqrt[4]{\sum_{i,j} \mathbb{E}x_{ij}^4}\right), \tag{3.125}$$

where $\hat{C} > 0$ is a universal constant and we assume $\hat{C} \geq 1$ without loss of generality. With this we find that

$$\mathbb{E}\|\operatorname{Re} V\| \leq 4\hat{C}\sigma_+\sqrt{D} \tag{3.126}$$

and with our choice of J it follows that the boundedness event $\mathcal{B}_{\operatorname{Re} V,J}$ holds with probability at least $1-\eta$ (and a similarly for $\operatorname{Im} V$ instead of $\operatorname{Re} V$).

The densities of the entries of $\operatorname{Re} V$ are clearly bounded by K and we have that $JK \geq 1$, so V fulfills the assumptions of Theorem 3.18 for our choice of parameters.

The next step is to show a generalization of Theorem 3.30 to Gaussian matrices with non-zero mean. We obtained the following result (cf. Theorem 9 in [138]):

Theorem 3.31 (Gaussian matrices with non-zero mean). *Let $H_0 = (h_{ij}^0)$ be a (deterministic) Hermitian matrix with C_{H_0} as in* (3.64)*, let V be the random $D \times D$ matrix defined in Theorem 3.30 and let $H := H_0 + V$. Then H is Hermitian with (non-centered) Gaussian independent (up to conjugate symmetry) entries; more precisely, its entries h_{ij} satisfy $\operatorname{Re} h_{ij} \sim \mathcal{N}(\operatorname{Re} h_{ij}^0, \sigma_{ij}^2/2)$, $\operatorname{Im} h_{ij} \sim \mathcal{N}(\operatorname{Im} h_{ij}^0, \sigma_{ij}^2/2)$ for $i \neq j$ and $h_{ii} = \operatorname{Re} h_{ii} \sim \mathcal{N}(h_{ii}^0, \sigma_{ii}^2)$. Moreover, H satisfies the assumptions in Theorem 3.18 with $K = \frac{1}{\sqrt{2\pi}\sigma_-}$ and $J = \frac{1}{\eta}(4\hat{C}\sigma_+ + C_{H_0})$ with $\hat{C}$ as in Theorem 3.30.*

The proof of Theorem 3.31 is analogous to the one of Theorem 3.30. We remark that Theorem 3.31 covers the case that H_0 is a band matrix (in a basis that diagonalizes the projections P_ν to the macro spaces) and V a small Gaussian perturbation thereof.

Now we are ready to prove Theorem 3.16. Similarly as in the proof of Theorem 3.18 we find that with probability at least $1-\varepsilon'$,

$$M_{\mu\nu} \geq \left(\frac{\max\{d_\nu, d_\mu\}}{4C\sqrt{KJ}D}\right)^{16} \min\left\{1, \frac{d_\nu}{d_\mu}\right\}. \tag{3.127}$$

With the help of Theorem 3.31, the factor KJ can be written as

$$KJ = \frac{1}{\sqrt{2\pi}\sigma_-\eta}\left(4\hat{C}\sigma_+ + C_{H_0}\right) \tag{3.128}$$

and one can show that $\eta \geq \varepsilon'/6$. This implies

$$C^2 KJ \leq \frac{6C^2}{\sqrt{2\pi}\sigma_- \varepsilon'} \left(4\hat{C}\sigma_+ + C_{H_0}\right) =: \frac{1}{16\varepsilon'} \frac{c_+\sigma_+ + C_{H_0}}{c_-\sigma_-} = \frac{1}{16\varepsilon' C_\sigma} \tag{3.129}$$

with $c_- := \frac{\sqrt{2\pi}}{96C^2}$ and $c_+ := 4\hat{C}$. Inserting this estimate into (3.127) finishes the proof of Theorem 3.16.

Dynamical Typicality

In order to prove Theorem 3.21 we have to show three bounds. For the first two bounds we note that it follows from Lemma 1.2 that

$$\mathrm{Var}_\mu \langle \psi_t | B | \psi_t \rangle \leq \frac{1}{d_\mu^2} \operatorname{tr}\left(P_\mu \exp(-iHt) B^* \exp(iHt) P_\mu \exp(iHt) B \exp(-iHt) P_\mu\right). \tag{3.130}$$

By applying the trace inequality (3.42) in two different ways, we find that

$$\mathrm{Var}_\mu \langle \psi_t | B | \psi_t \rangle \leq \min\left\{ \frac{\|B\|^2}{d_\mu}, \frac{\|B\| \operatorname{tr}(|B|)}{d_\mu^2} \right\}. \tag{3.131}$$

The first two bounds then follow from Chebyshev's inequality.

For the third bound we make use of Lévy's Lemma, see Theorem 1.14. This theorem can be reformulated as follows: For any Hilbert space $\mathcal{H}$ of dimension D, any Lipschitz continuous function $f : \mathbb{S}(\mathcal{H}) \to \mathbb{R}$ with Lipschitz constant η, and any $\varepsilon > 0$, $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$ are such that

$$|f(\psi) - \mathbb{E}f| \leq \sqrt{\frac{9\pi^3 \log(4/\varepsilon)}{2D}}\eta. \tag{3.132}$$

In [136] we applied this with $\mathcal{H}_\mu$ instead of $\mathcal{H}$ to the function $f(\psi) = \langle \psi_t | B | \psi_t \rangle$ which has Lipschitz constant $\eta = 2\|B\|$. The problem with this reasoning is that the function f is not necessarily real-valued. Lévy's Lemma for complex-valued functions can be obtained from Theorem 1.14 by replacing 4 by 8 and 2/9 by 1/9. This shows that if f is complex-valued, then $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$ are such that

$$|f(\psi) - \mathbb{E}f| \leq \sqrt{\frac{9\pi^3 \log(8/\varepsilon)}{D}}\eta. \tag{3.133}$$

From this result the third bound follows as sketched below (3.132).

For the second claim in Theorem 3.21 we use Fubini's theorem to obtain

$$\mathbb{E}_\mu \left[\int_0^T \left| \langle \psi_t | B | \psi_t \rangle - w_{\mu B}(t) \right|^2 \, dt \right] = \int_0^T \mathrm{Var}_\mu \left[\langle \psi_t | B | \psi_t \rangle \right] \, dt \leq \frac{\|B\|^2 T}{d_\mu}. \tag{3.134}$$

Now the second claim in Theorem 3.21 follows from Markov's inequality. Corollary 3.22 is immediately obtained from Theorem 3.21 be setting $B = P_\nu$.

Next we turn to the proofs of the relative and comparative errors. Proposition 3.24 does not need any further assumptions on the Hamiltonian but the result is rather limited. For its proof we take an orthonormal basis (φ_k) of $\mathcal{H}_\mu$, define $\varphi_{k,t} := e^{-itH}\varphi_k$ and estimate

$$|\mathbb{E}_\mu \langle \psi_t | B | \psi_t \rangle| = \frac{1}{d_\mu} \left| \sum_k \langle \varphi_{k,t} | B | \varphi_{k,t} \rangle \right| \tag{3.135a}$$

$$\geq \frac{1}{d_\mu} \max \left\{ \sum_k \langle \varphi_{k,t} | B^+ - B^- | \varphi_{k,t} \rangle, \sum_k \langle \varphi_{k,t} | B^- - B^+ | \varphi_{k,t} \rangle \right\} \tag{3.135b}$$

and the last expression can clearly be lower bounded by b. With this the remaining claims follow from Theorem 3.21.

The setting of Theorem 3.25 concerning the comparative errors is the same as the one for the results regarding the relative errors of generalized normal typicality. As in [136] we find that $(1-\varepsilon/2)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\left| \overline{\langle \psi_t | B | \psi_t \rangle} - M_{\mu B} \right| \leq \sqrt{2} \left(\frac{\|B\|}{\varepsilon d_\mu} \min \left\{ \|B\|, \frac{\mathrm{tr}(|B|)}{d_\mu} \right\} \right)^{1/2}. \tag{3.136}$$

Then the triangle inequality together with Theorem 3.31 shows that with probability at least $1-\varepsilon'$, $(1-\varepsilon/2)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that $|\overline{\langle \psi_t | B | \psi_t \rangle}| \geq \mathrm{LB}(B, \psi)$. Now the claim follows from this together with Theorem 3.21.

We finally turn to the proof of Theorem 3.23. By writing ψ_0 in an orthonormal basis of eigenvectors of H and applying Theorem 3.29 we obtain that with probability at least $1-\varepsilon'$,

$$\overline{\langle \psi_t | P_\nu | \psi_t \rangle} \geq \left(\frac{\sqrt{\varepsilon' C_\sigma} d_\nu}{D} \right)^{16}. \tag{3.137}$$

Together with Theorem 3.21 this implies that with probability at least $1-\varepsilon'$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{\left| \|P_\nu \psi_t\|^2 - \mathbb{E}_\mu \|P_\nu \psi_t\|^2 \right|}{\overline{\|P_\nu \psi_t\|^2}} \leq \frac{1}{\sqrt{\varepsilon}} (C_\sigma \varepsilon')^{-8} \frac{1}{d_\mu} \sqrt{\min\{d_\mu, d_\nu\}} \left(\frac{D}{d_\nu} \right)^{16}. \tag{3.138}$$

From this the first claim follows by inserting the definitions of s_μ, s_ν and s_{mc}. Moreover, the second claim is a consequence of the lower bound (3.137) on $\overline{\langle\psi_t|P_\nu|\psi_t\rangle}$ and Corollary 3.22.

Generalized Normal Typicality for GAP Measures

The proof of Theorem 3.26 is inspired by the proof of Theorem 3.12 from [136]. An important ingredient for the proof is an upper bound on the GAP(ρ)-variance $\mathrm{Var}_\rho\langle\psi|A|\psi\rangle$. Such a bound was already obtained by Reimann [106] in the case that A is self-adjoint and $\mathcal{H}$ is finite-dimensional, see also Lemma 1.20. In [137] we generalized Reimann's proof to arbitrary operators and separable Hilbert spaces, see also Proposition 3.11. Unfortunately, these bounds are not sufficiently sharp as (ignoring degeneracies of the Hamiltonian) they would lead to an upper bound of the order

$$\operatorname{tr}\rho^2 \sum_{e\neq e'} \|\Pi_e B \Pi_{e'}\|^2 \leq \|\rho\| \sum_{e,e'} \operatorname{tr}(\Pi_{e'} B^* \Pi_e B \Pi_{e'}) = \|\rho\| \operatorname{tr}(B^*B). \tag{3.139}$$

In the case that $\rho = P_\mu/d_\mu$ and $B = P_\nu$ we would obtain an upper bound of the order d_ν/d_μ which need not be small. Therefore, the first step in the proof of Theorem 3.26 is to prove sharper bounds on $\mathrm{Var}_\rho\langle\psi|A|\psi\rangle$. We obtained the following lemma (Lemma 1 in [143]):

Lemma 3.32. *Let ρ be a density matrix on a separable Hilbert space $\mathcal{H}$ with eigenvalues $p_n > 0$ such that $p_{\max} = \|\rho\| < 1/4$ and let $\dim\mathcal{H} \geq 4$. Then for any bounded operator A on $\mathcal{H}$,*

$$\begin{aligned}
\mathrm{Var}_\rho\langle\psi|A|\psi\rangle \leq{} & \frac{1}{1-p_{\max}} \Bigg(\operatorname{tr}(A\rho A^*\rho) + \frac{\operatorname{tr}(A\rho^2A^*\rho) + \operatorname{tr}(A\rho A^*\rho^2)}{1-2p_{\max}} \\
& + \frac{2}{(1-2p_{\max})(1-3p_{\max})} \Bigg[\operatorname{tr}(A\rho^3A^*\rho) + \operatorname{tr}(A\rho^2A^*\rho^2) + \operatorname{tr}(A\rho A^*\rho^3) \\
& + \sum_{m,n} \Big(|\operatorname{tr}(A\rho^3P_m)\operatorname{tr}(A^*\rho P_n)| + |\operatorname{tr}(A\rho^2P_m)\operatorname{tr}(A^*\rho^2P_n)| \\
& + |\operatorname{tr}(A\rho P_m)\operatorname{tr}(A^*\rho^3P_n)| \Big) \Bigg] \Bigg),
\end{aligned} \tag{3.140}$$

where $P_n = |n\rangle\langle n|$ and $\{|n\rangle\}$ is an orthonormal eigenbasis of ρ.

The proof of Lemma 3.32 closely follows the one of Reimann [106] and the one we gave in [137]. We first assume that $D = \dim\mathcal{H} < \infty$ and without loss of generality that $\mathbb{E}_\rho\langle\psi|A|\psi\rangle = \operatorname{tr}(A\rho) = 0$. As in (3.38) and similarly as in (1.128c) we find that

$$\operatorname{Var}_\rho\langle\psi|A|\psi\rangle = \mathbb{E}_\rho\left(\langle\psi|A|\psi\rangle\langle\psi|A^*|\psi\rangle\right) = \sum_{m,n}\left[A_{mm}A_{nn}^* + |A_{mn}|^2\right]p_n p_m K_{mn}, \quad (3.141)$$

where the K_{mn} are defined in (1.125). With the help of a Taylor expansion of $g_{mn}(x) := (1 + xp_m)^{-1}(1 + xp_n)^{-1}$ up to second order we obtain an expansion of K_{mn}, see (1.131), and with this and using that $\operatorname{tr}(A\rho) = 0$ we can show that

$$\left|\sum_{m,n} A_{mm}A_{nn}^* p_n p_m K_{mn}\right| \le 2K^{(2)}\sum_{m,n}|A_{mm}A_{nn}^*|(p_m^3 p_n + p_m^2 p_n^2 + p_m p_n^3), \qquad (3.142)$$

where $K^{(2)}$ is defined in (1.132). With the help of computations such as

$$A_{mm}A_{nn}^* p_m^3 p_n = \langle m|A\rho^3|m\rangle\langle n|A^*\rho|n\rangle = \operatorname{tr}(A\rho^3 P_m)\operatorname{tr}(A^*\rho P_n) \qquad (3.143)$$

we obtain

$$\begin{aligned}&\left|\sum_{m,n} A_{mm}A_{nn}^* p_n p_m K_{mn}\right| \\ &\qquad \le 2K^{(2)}\sum_{m,n}\Big(\left|\operatorname{tr}(A\rho^3 P_m)\operatorname{tr}(A^*\rho P_n)\right| + \left|\operatorname{tr}(A\rho^2 P_m)\operatorname{tr}(A^*\rho^2 P_n)\right| \\ &\qquad + \left|\operatorname{tr}(A\rho P_m)\operatorname{tr}(A^*\rho^3 P_n)\right|\Big). \end{aligned} \qquad (3.144)$$

Moreover, by similar arguments we find that

$$\begin{aligned}\sum_{m,n}|A_{mn}|^2 p_n p_m K_{mn} &= K^{(0)}\operatorname{tr}(A\rho A^*\rho) - K^{(1)}\left(\operatorname{tr}(A\rho^2 A^*\rho) + \operatorname{tr}(A\rho A^*\rho^2)\right) \\ &\qquad + 2K^{(2)}\sum_{m,n}|A_{mn}|^2\kappa_{mn}\left(p_n p_m^3 + p_m^2 p_n^2 + p_n^3 p_m\right), \end{aligned} \qquad (3.145)$$

where $\kappa_{mn} \in [0, 1]$. An upper bound for (3.145) is given by

$$\begin{aligned}&K^{(0)}\operatorname{tr}(A\rho A^*\rho) + K^{(1)}\left(\operatorname{tr}(A\rho^2 A^*\rho) + \operatorname{tr}(A\rho A^*\rho^2)\right) \\ &\qquad + 2K^{(2)}\left(\operatorname{tr}(A\rho^3 A^*\rho) + \operatorname{tr}(A\rho^2 A^*\rho^2) + \operatorname{tr}(A\rho A^*\rho^3)\right). \end{aligned} \qquad (3.146)$$

By combining (3.144) and (3.146) and using the bounds for $K^{(k)}$ from (1.133) we get the desired upper bound for $\operatorname{Var}_\rho\langle\psi|A|\psi\rangle$ in the finite-dimensional case. By similar arguments as in the proof of Proposition 3.11 it can be shown that the bound remains valid in the case that $\dim\mathcal{H} = \infty$ which concludes the proof of Lemma 3.32.

The next step in the proof of Theorem 3.26 is to compute upper bounds for some variances and averaged deviations, more precisely, to prove an analogue of Proposition 3.28 for GAP measures. Due to our assumptions on B, all sums over eigenvalues of the Hamiltonian that occur in our computations are effectively finite and we thus immediately see that

$$M_{\psi_0 B} = \overline{\langle\psi_t|B|\psi_t\rangle} = \sum_{e\in\mathcal{E}} \langle\psi_0|\Pi_e B \Pi_e|\psi_0\rangle \tag{3.147}$$

and $\mathbb{E}_\rho M_{\psi_0 B} = M_{\rho B}$ as interchanging the finite sums with the time average and expectation with respect to GAP(ρ) is unproblematic.

In [143, Proposition 1] we showed the following result:

Proposition 3.33. *Let $\mathcal{H}$ be a separable Hilbert space with $\dim\mathcal{H}\geq 4$, let ρ be a density matrix on $\mathcal{H}$ with eigenvalues $p_n>0$ and $\|\rho\|<1/4$ and let B be an operator on $\mathcal{H}$ such that $d_{E,B}<\infty$. Then for every $\kappa, T>0$,*

$$\mathbb{E}_\rho\left(\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T\right) \leq 24\|B\|^2\|\rho\| D_{E,B} G_B(\kappa)\left(1+\frac{8\log_2 d_{E,B}}{\kappa T}\right), \tag{3.148}$$

$$\mathrm{Var}_\rho\, \overline{\langle\psi_t|B|\psi_t\rangle} \leq 23\|B\|^2\|\rho\|. \tag{3.149}$$

Moreover,

$$\mathbb{E}_\rho\left(\overline{\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2}\right) \leq 24\|B\|^2\|\rho\| D_{E,B} D_{G,B}. \tag{3.150}$$

For the proof of Proposition 3.33 we first assume that the Hilbert space is finite-dimensional. As in the proof of Proposition 3.28 we find that

$$\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T \leq \|R\| \sum_{e,e'} |\langle\psi_0|\Pi_e B\Pi_{e'}|\psi_0\rangle|^2 \tag{3.151}$$

where $\|R\|$ can be bounded as in (1.163). By taking the expectation with respect to GAP(ρ) we obtain

$$\mathbb{E}_\rho\left(\sum_{e,e'} |\langle\psi_0|\Pi_e B\Pi_{e'}|\psi_0\rangle|^2\right) = \sum_{e,e'} \mathrm{Var}_\rho \langle\psi_0|\Pi_e B\Pi_{e'}|\psi_0\rangle + |\mathrm{tr}(\rho\Pi_e B\Pi_{e'})|^2. \tag{3.152}$$

An application of Lemma 3.32 shows that

$$\sum_{e,e'} \mathrm{Var}_\rho\langle\psi_0|\Pi_e B\Pi_{e'}|\psi_0\rangle \leq \frac{1}{1-p_{\max}} \sum_{e,e'} \Big(\mathrm{tr}(\Pi_e B\Pi_{e'}\rho\Pi_{e'}B^*\Pi_e\rho)$$

$$+\frac{\operatorname{tr}(\Pi_e B\Pi_{e'}\rho^2\Pi_{e'}B^*\Pi_e\rho)+\operatorname{tr}(\Pi_e B\Pi_{e'}\rho\Pi_{e'}B^*\Pi_e\rho^2)}{1-2p_{\max}}$$
$$+\frac{2}{(1-2p_{\max})(1-3p_{\max})}\Big[\operatorname{tr}(\Pi_e B\Pi_{e'}\rho^3\Pi_{e'}B^*\Pi_e\rho)+\operatorname{tr}(\Pi_e B\Pi_{e'}\rho^2\Pi_{e'}B^*\Pi_e\rho^2)$$
$$+\operatorname{tr}(\Pi_e B\Pi_{e'}\rho\Pi_{e'}B^*\Pi_e\rho^3)+\sum_{m,n}\Big(|\operatorname{tr}(\Pi_e B\Pi_{e'}\rho^3P_m)\operatorname{tr}(\Pi_{e'}B^*\Pi_e\rho P_n)|$$
$$+|\operatorname{tr}(\Pi_e B\Pi_{e'}\rho^2P_m)\operatorname{tr}(\Pi_{e'}B^*\Pi_e\rho^2P_n)|+|\operatorname{tr}(\Pi_e B\Pi_{e'}\rho P_m)\operatorname{tr}(\Pi_{e'}B^*\Pi_e\rho^3P_n)|\Big)\Big]\Big). \tag{3.153}$$

With the help of the trace inequality (3.42) we obtain, e.g.,

$$\sum_{e,e'}\operatorname{tr}(\Pi_e B\Pi_{e'}\rho\Pi_{e'}B^*\Pi_e\rho)=\sum_e\operatorname{tr}\left(B\left(\sum_{e'}\Pi_{e'}\rho\Pi_{e'}\right)B^*\Pi_e\rho\Pi_e\right) \tag{3.154a}$$
$$\leq\sum_e\|B\|^2\left\|\sum_{e'}\Pi_{e'}\rho\Pi_{e'}\right\|\operatorname{tr}(\Pi_e\rho\Pi_e)\leq\|B\|^2\|\rho\|. \tag{3.154b}$$

All the other terms in (3.153) can be bounded in a similar way.

Next we have to bound sums over products of traces. By applying the Cauchy-Schwarz inequality (3.41) for the trace we find that

$$\sum_{e,e'}\sum_{m,n}\left|\operatorname{tr}(\Pi_e B\Pi_{e'}\rho^3P_m)\operatorname{tr}(\Pi_{e'}B^*\Pi_e\rho P_n)\right|$$
$$=\sum_{m,n}\sum_{e,e'}|\operatorname{tr}(\Pi_e B\Pi_{e'}\Pi_{e'}P_m\Pi_e)\operatorname{tr}(\Pi_{e'}B^*\Pi_e\Pi_eP_n\Pi_{e'})|p_m^3p_n \tag{3.155a}$$
$$\leq\sum_{m,n}p_m^3p_n\sum_{e,e'}\operatorname{tr}(\Pi_eB\Pi_{e'}B^*)\left(\langle m|\Pi_e|m\rangle\langle m|\Pi_{e'}|m\rangle\langle n|\Pi_e|n\rangle\langle n|\Pi_{e'}|n\rangle\right)^{1/2} \tag{3.155b}$$
$$\leq\|B\|^2D_E\sum_{m,n}p_m^3p_n\left(\sum_e\left(\langle m|\Pi_e|m\rangle\langle n|\Pi_e|n\rangle\right)^{1/2}\right)^2 \tag{3.155c}$$
$$\leq\|B\|^2D_E\sum_{m,n}p_m^3p_n\leq\|B\|^2D_E\|\rho\|^2. \tag{3.155d}$$

The other sums over products of traces in (3.153) can be bounded in a similar way. Moreover, by similar methods we get

$$\sum_{e,e'}|\operatorname{tr}(\rho\Pi_eB\Pi_{e'})|^2\leq\|B\|^2D_E\|\rho\|. \tag{3.156}$$

Putting everything together and using that $\|\rho\| < 1/4$ by assumption we obtain (3.148) in the finite-dimensional case.

For the GAP-variance of $\overline{\langle\psi_t|B|\psi_t\rangle}$ we simply apply Lemma 3.32 with $A = \sum_e \Pi_e B \Pi_e$ and use the same trace inequalities as above to bound the occurring terms.

Moreover, for the proof of (3.150) we choose κ small enough such that $G(\kappa) = D_G$ and then take the limit $T \to \infty$ in (3.148). Note that the time limit and $\mathbb{E}_\rho$ can be interchanged by dominated convergence.

In the case that $\mathcal{H}$ is infinite-dimensional, we see that all sums over $e \in \mathcal{E}$ are effectively sums over $e \in \mathcal{E}_B$ and therefore, by assumption, finite. Furthermore, we find that d_E, D_E, D_G and $G(\kappa)$ can be replaced by the corresponding quantities relative to B and that then all steps of the proof remain valid which proves the bounds in Proposition 3.33 also in the infinite-dimensional case.

After having proved Proposition 3.33, the proof of Theorem 3.26 basically reduces to a careful application of Markov's and Chebychev's inequality.

3.2.3. Discussion

Absolute Errors

The first paper of this project was concerned with a generalization of normal typicality to more realistic Hamiltonians and with dynamical typicality and we proved error bounds for *absolute errors* [136].

Theorem 3.12 shows that for any operator B on a finite-dimensional Hilbert space $\mathcal{H}$, most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for most times $t \in [0,T]$, $\langle\psi_t|B|\psi_t\rangle \approx M_{\mu B} = \mathbb{E}_\mu \overline{\langle\psi_t|B|\psi_t\rangle}$ provided that $\mathcal{H}_\mu$ and T are sufficiently large and $D_E, G(\kappa)$ and $\|B\|$ are not too large. Unfortunately, as we have already discussed in Section 1.6.1, the equilibration times, i.e., the time scales on which $\langle\psi_t|B|\psi_t\rangle$ (with ψ_0 such that $\langle\psi_0|B|\psi_0\rangle$ is far away from $M_{\mu B}$) reaches the equilibrium value $M_{\mu B}$, that we obtain from Theorem 3.12 are extremely large. For more results concerning time scales of equlibration and thermalization see Section 1.6.5 and the references therein.

We remark that another way to prove generalized normal typicality is to make use of a result by Short and Farrelly [125] who showed that

$$\left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \leq \frac{G(\kappa)\|B\|^2}{d_{\mathrm{eff}}} \left(1 + \frac{8\log_2 d_E}{\kappa T}\right), \tag{3.157}$$

where d_{eff} is the effective dimension of the initial state ψ_0, see (1.148). One can show that $\mathbb{E}_\mu d_{\mathrm{eff}}^{-1} \leq 2D_E/(d_\mu + 1)$ which then gives the first bound from (3.106) in Proposition 3.28. The second bound in (3.106) is sharper if $\mathrm{tr}(B^*B)/d_\mu < \|B\|^2$, which, e.g., $B = P_\nu$ is the case if $d_\nu < d_\mu$.

In Corollary 3.13 we consider the case that $B = P_\nu$ and and times $t \in [0,\infty)$. We obtained that as soon as d_μ is much larger than D_E and D_G, the superposition weight

$\|P_\nu\psi_t\|^2$ is close to $M_{\mu\nu}$ for most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ and most times $t \geq 0$. Put another way, the superposition weights $\|P_\nu\psi_t\|^2$ are nearly constant in the long run. Note, however, that due to recurrence, $\|P_\nu\psi_t\|^2$ does not converge for $t \to \infty$. We also say that the superposition weights $\|P_\nu\psi_t\|^2$ *equilibrate* in the long run. However, this does not necessarily mean that the system thermalizes as thermal equilibrium at time t would correspond to $\|P_\nu\psi_t\|^2 \approx 1$ if ν is the thermal equilibrium macro state and $\|P_\nu\psi_t\|^2 \approx 0$ otherwise.

Note that our result is a statement about most initial wave functions from any (possibly far from thermal equilibrium) macro space. As we want to understand how non-equilibrium initial states evolve towards thermal equilibrium, it is important that we do not just have a result for most initial states from $\mathbb{S}(\mathcal{H})$ as most elements of $\mathbb{S}(\mathcal{H})$ are in thermal equilibrium anyway.

In comparison, von Neumann's [144] normal typicality result, see also Section 1.2, is a statement which is true for all initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$ but the Hamiltonian is assumed to have a purely random eigenbasis. As we have discussed in Section 1.2.3, this assumption on the Hamiltonian is physically unrealistic as it leads to extremely short thermalization times. Therefore we wanted to generalize normal typicality to more realistic Hamiltonians and Theorem 3.12 and Corollary 3.13 in fact hold for arbitrary Hamiltonians with not too highly degenerate eigenvalues and gaps.

Suppose that there is one very large macro space in the decomposition of the Hilbert space that corresponds to thermal equilibrium. Our expectation was that if the Hamiltonian has a band structure in a basis that diagonalizes the projections P_ν to the macro spaces $\mathcal{H}_\nu$ (see Figure 3.1), then a system that is initially in a non-equilibrium macro state (corresponding to a small macro space) does not immediately evolve into the thermal equilibrium subspace but goes through larger and larger macro spaces until it finally reaches thermal equilibrium. This was confirmed by a numerical simulation with a random band matrix, see Figure 3.2.

As predicted by generalized normal typicality, after a sufficiently long equilibration time, the superposition weights $\|P_\nu\psi_t\|^2$ in Figure 3.3 reach and stay close to certain values. We remark that the values to which the superposition weights equilibrate differ slightly from what normal typicality predicts as, e.g., $d_4/D \approx 0.90$ while $M_{24} \approx 0.82$.

As a second corollary of Theorem 3.12 we showed in Corollary 3.14 bounds for the difference between $\|P_\nu\psi_t\|^2$ and $M_{\mu\nu}$ for "realistic dimensions" of the macro spaces. We have seen that for fixed entropies per particle $s_{\nu_\pm}$ and s_μ, the errors are exponentially small in the number of particles N. The upper bounds show that the errors are controlled by the entropy of the initial macro state if a transition to a larger macro space is considered while they are controlled by $s_\mu - s_{\nu_-}/2$ if a transition into a smaller macro state ν_- is studied. Note that these findings are consistent with our numerical simulation in Figure 3.3.

Our numerical simulation also shows that the average entropy $\langle\psi_t|S_{\mathrm{qB}}|\psi_t\rangle$ increases over time (up to small fluctuations), see Figure 3.4, which reflects the second law of thermodynamics. Here, S_{qB} is the quantum Boltzmann entropy from (1.217) with

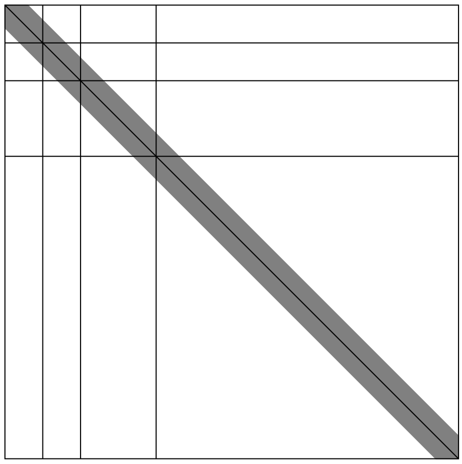

Figure 3.1.: Example of a band matrix: We consider a matrix acting on $\mathbb{C}^D = \bigoplus_{\nu=1}^4 \mathbb{C}^{d_\nu}$, i.e., the Hilbert space $\mathbb{C}^D$ is decomposed into four macro spaces $\mathbb{C}^{d_\nu}$. The first macro space is spanned by the first d_1 basis vectors, the second one by the next d_2 basis vectors, etc.. The matrix has a band structure if only entries close to the diagonal, i.e., in the grey region, differ significantly from zero. (This figure already appeared in [138].)

$k_B = 1$ and therefore

$$\langle\psi_t|S_{\mathrm{qB}}|\psi_t\rangle = \sum_\nu \|P_\nu\psi_t\|^2 S_{\mathrm{qB}}(\nu), \tag{3.158}$$

where $S_{\mathrm{qB}}(\nu) = \log\dim\mathcal{H}_\nu$.

Besides our generalization of normal typicality, we also proved dynamical typicality in [136], see Theorem 3.21 and Corollary 3.22 and see Section 1.3 for related results. Theorem 3.21 shows that for every $t \in \mathbb{R}$, most initial wave functions $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that the expectation $\langle\psi_t|B|\psi_t\rangle$ is close to $w_{\mu B}(t) = \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle$ provided that d_μ is sufficiently large and $\|B\|$ is not too large. Moreover, under the same assumptions we find that for most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ the whole curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ on any finite time interval is close to the curve $t \mapsto w_{\mu B}(t)$ in L^2-norm. Therefore, the curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ is nearly ψ_0-independent.

We remark that our result concerning dynamical typicality in Theorem 3.21 also is contained in a general result by Balz et al. [5], see also Theorem 1.12. However, in our situation the proof is particularly simple and transparent which is why we still gave it in [136]. Moreover, note that while our theorem is formulated for time-independent Hamiltonians, it actually also implies to time-dependent ones. Additionally, we note that by similar arguments as in the proof of Theorem 3.21 one can show that the

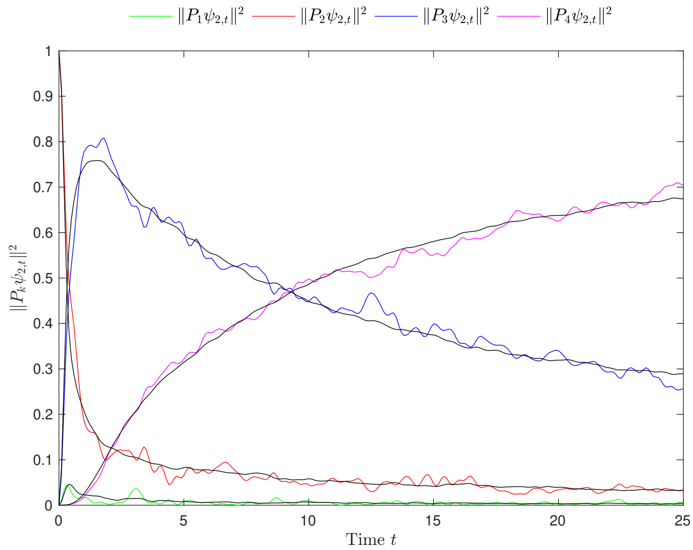

Figure 3.2.: Example of the time evolution of the superposition weights $\|P_\nu\psi_t\|^2$ for a random band matrix as Hamiltonian. Here, $D = 2222$, $d_1 = 2$ (green curve), $d_2 = 20$ (red curve), $d_3 = 200$ (blue curve), and $d_4 = 2000$ (purple curve). The initial state $\psi_0 \in \mathbb{S}(\mathcal{H}_2)$ is chosen purely randomly and the entries of the Hamiltonian $H = (h_{jk})$ satisfy $h_{jj} \sim \mathcal{N}(0, \sigma_{jj}^2)$ and $h_{jk} \sim \mathcal{N}(0, \sigma_{jk}^2/2) + i\mathcal{N}(0, \sigma_{jk}^2/2)$ for $j \neq k$, where $\sigma_{jk}^2 := \exp(-s|j-k|)$ with $s = 0.02$. The initial state first passes through $\mathcal{H}_3$ before reaching the largest macro space $\mathcal{H}_4$. The black curves are the deterministic approximations to the colorful curves according to dynamical typicality, see Theorem 3.21. (This figure already appeared in [136].)

probability current $J_{\nu\nu'}$ between the macro spaces $\mathcal{H}_\nu$ and $\mathcal{H}_{\nu'}$,

$$J_{\nu\nu'} := -i\left(\langle\psi_t|P_\nu H P_{\nu'}|\psi_t\rangle - \langle\psi_t|P_{\nu'} H P_\nu|\psi_t\rangle\right) = 2\,\mathrm{Im}\,\langle\psi_t|P_\nu H P_{\nu'}|\psi_t\rangle, \qquad (3.159)$$

is nearly deterministic. Note that $J_{\nu\nu'}$ can be interpreted as the amount of probability which passes from ν' to ν minus the probability flowing from ν to ν' (per unit time).

In the first part of Theorem 3.21 we obtained three bounds. While the two with $1/\sqrt{\varepsilon}$ are obtained from Chebyshev's inequality, the third bound involving $\sqrt{\log(1/\varepsilon)}$ follows from Lévy's Lemma. Note that in general Lévy's Lemma does not yield better bounds when used instead of Markov's and Chebyshev's inequality in the proof of our generalization of normal typicality. In [143] we discuss two different ways how Lévy's Lemma could be used to prove generalized normal typicality.[26] The first

[26]The work [143] is concerned with GAP measures but the discussion there also applies to the uniform distribution as a special case.

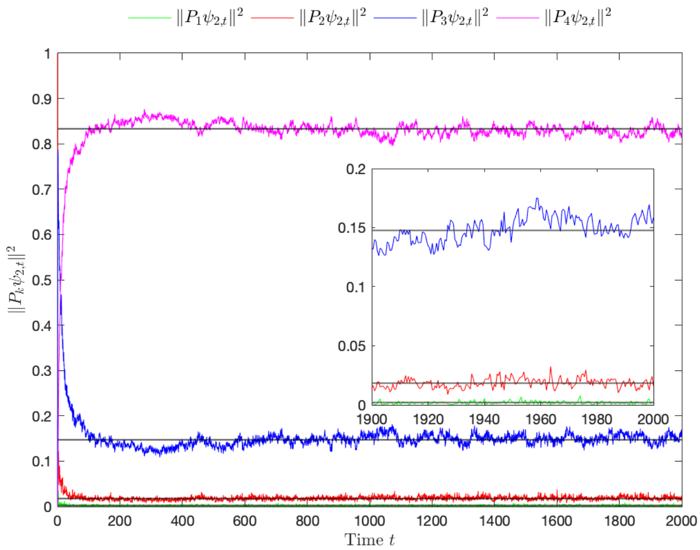

Figure 3.3.: The same numerical simulation as in Figure 3.2 but for longer times. The black horizontal lines are the $M_{2\nu}$ from Corollary 3.13 and we see that after an initial phase of equilibration, the curves $t \mapsto \|P_\nu\psi_t\|^2$ fluctuate closely around these values. (This figure also already appeared in [136].)

one is to consider the function $f(\psi_0) = \langle|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}|^2\rangle_T$ on the sphere $\mathbb{S}(\mathcal{H}_\mu)$ which is Lipschitz continuous with Lipschitz constant η bounded by $8\|B\|^2$. Markov's inequality shows that $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$f(\psi_0) \leq \frac{\mathbb{E}_\mu f}{\varepsilon} \tag{3.160}$$

while it follows from Lévy's Lemma (cf. Theorem 1.14) that

$$f(\psi_0) \leq \mathbb{E}_\mu f + \sqrt{\frac{9\pi^3\eta^2\log(4/\varepsilon)}{2d_\mu}}. \tag{3.161}$$

As $\mathbb{E}_\mu f$ is of order $1/d_\mu$ we find that while the ε-dependence in (3.161) is better than in (3.160), it is the other way around for the dependence on d_μ. Since d_μ is usually very large, ε would have to be extremely small to compensate for the worse d_μ-dependence in (3.161). However, it is of little interest to consider ε as small as 10^{-200} [15] and therefore the bound obtained from Markov's inequality usually is the better one. Note that a similar argumentation can be used in the case of an infinite time average.

Another possible way to make use of Lévy's Lemma is by applying it to the Lipschitz

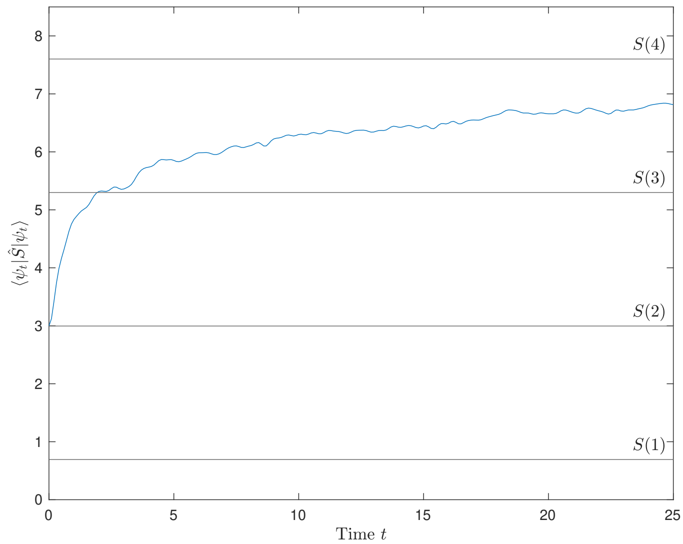

Figure 3.4.: The average entropy curve $t \mapsto \langle\psi_t|S_{\mathrm{qB}}|\psi_t\rangle$ for the same numerical simulation as in Figure 3.2 and Figure 3.3. (This figure also already appeared in [136].)

continuous function $f(\psi_0) = \langle\psi_t|B|\psi_t\rangle$ in order to obtain a bound for $|f(\psi_0) - \mathbb{E}_\mu f|$ and then to use similar estimates as in the proof of Proposition 3.28 in order to bound $|\mathbb{E}_\mu f - M_{\mu B}|$ as well. Altogether we find that $(1-\delta)$-most $t \in [0,T]$ are such that for $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$,

$$\Big|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}\Big| \leq 3\left(\frac{9\pi^3\|B\|^2 D_E G(\kappa)\log(8/\varepsilon)}{d_\mu\delta}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\right)^{1/2}. \tag{3.162}$$

Note that a similar estimate holds true in the case of the infinite time interval $[0,\infty)$ instead of $[0,T]$.

We are interested in a statement concerning the behavior of most initial states at a typical time and therefore have to interchange "most t" and "most ψ_0" which can be done with the help of Footnote 7 from [56] in the case of a finite time interval. We do not directly apply this footnote to the bound for the difference between $\langle\psi_t|B|\psi_t\rangle$ and $M_{\mu B}$ but rather interchange the quantifier in the step where we estimate $|f(\psi_0) - \mathbb{E}_\mu f|$. This yields that $(1-\varepsilon/\delta)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,T]$,

$$\Big|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}\Big| \leq 4\left(\frac{9\pi^3\|B\|^2 D_E G(\kappa)\log(8/\varepsilon)}{d_\mu\delta}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\right)^{1/2} \tag{3.163}$$

We thus see that effectively $1/\varepsilon$ in Theorem 3.12 is replaced by $\log(8/(\varepsilon\delta))$ and if

$\varepsilon \ll \delta$, this bound is slightly better if δ is sufficiently small. Note, however, that in the case of the infinite time interval $[0,\infty)$ we cannot interchange "most t" and "most ψ_0" as the counterexample in Footnote 3 in [143] shows.

Relative Errors

The results regarding generalized normal typicality and dynamical typicality that we discussed so far were all concerned with absolute errors. However, if $\mathcal{H}_\nu$ is small (which should be the case for all non-equilibrium macro spaces) then we expect that $\|P_\nu\psi_t\|^2$, $M_{\mu\nu} \ll 1$ and thus small bounds on the absolute error are not very meaningful in such situations, see also the green line in Figure 3.3 which displays fluctuations around M_{21} that are sometimes larger than M_{21} itself. Note that similar considerations apply to dynamical typicality. Motivated by this, the goal of the second paper of this project was to prove under suitable assumptions on the Hamiltonian that also the *relative* errors are small [138] and to thereby strengthen the previous results from [136].

Following Wigner [146], we model the system's Hamiltonian H as a Hermitian random matrix and we study the superposition weights $\|P_\nu\psi_t\|^2$ for typical H, typical initial state $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ and typical time $t \in [0,\infty)$. As we have already discussed above, we are particularly interested in random matrices with a band structure in a basis that diagonalizes the projections P_ν. Concerning generalized normal typicality we already obtained a small upper bound for $|\|P_\nu\psi_t\|^2 - M_{\mu\nu}|$ in [136] for large d_μ and general Hamiltonians H – we only needed that no eigenvalue and no gap of H is highly degenerate. For a random Hamiltonian whose entries have a joint distribution that is absolutely continuous with respect to the Lebesgue measure, the eigenvalues and eigenvalue gaps are with probability 1 non-degenerate, see Proposition 3.15. Therefore it only remained to prove lower bounds for $M_{\mu\nu}$ and for this we needed lower bounds for $\|P_\nu\phi_n\|^2$, where ϕ_n is an eigenvector of H. To this end we made use of the so-called *no-gaps delocalization* of eigenvectors of certain classes of random matrices which was proved by Rudelson and Vershynin [120] and which we slightly generalized and improved in [138]. Roughly speaking, no-gaps delocalization rules out gaps in the eigenvectors of H, i.e., no significant fraction of the components of an eigenvector can carry only a negligible amount of its mass. We remark that most other results in the literature regarding the delocalization of eigenvectors are concerned with delocalization in the sup norm, i.e., peaks in the components are ruled out, which is another aspect of the delocalization of eigenvectors and neither form implies the other one. Note that from this kind of results we only get upper bounds for the $M_{\mu\nu}$ and only in very special cases lower bounds for certain $M_{\mu\nu}$, see below for an example.

We applied the no-gaps delocalization theorem, see Theorem 3.29, to the situation that the Hamiltonian is of the form $H = H_0 + V$ where H_0 is a deterministic Hermitian matrix (e.g., one with a band structure) and V is a Hermitian random Gaussian perturbation with variances bounded away from 0. Therefore we can think of H as a slightly perturbed band matrix. For such Hamiltonians we proved a lower bound for

$M_{\mu\nu}$ in Theorem 3.16 which led together with Corollary 3.13 to an upper bound for the relative errors in Corollary 3.17. If the variances of the entries of V are bounded by constants the lower bound for $M_{\mu\nu}$ is roughly of the order $(d_\nu/D)^{16}$ and we expect that this is far from optimal. More precisely, we expect that $M_{\mu\nu} \approx d_\nu/D$ (as for normal typicality) as long as the eigenvectors of the random matrix are delocalized. This is also supported by our numerical simulation in Figure 3.3 where the $M_{\mu\nu}$ are not too far away from d_ν/D. Note that in the simulation the bandwidth of the random matrix is roughly given by $2s^{-1} = 400 \approx D^{0.77} \gg \sqrt{D}$ such that we should be in the regime of delocalized eigenvectors. We note further that the simulation is not covered by our theorems as the variances cannot be bounded by a D-independent constant or polynomially in D from below (for the latter see also Remark 2 in [138]); however, the simulation suggests that similar results should be valid in more general situations than the ones covered by our theorems.

Note that besides the results concerning lower bounds for $M_{\mu\nu}$ and upper bounds for the relative errors $|\|P_\nu\psi_t\|^2 - M_{\mu\nu}|/M_{\mu\nu}$ in the special case that the Hamiltonian is of the form $H = H_0 + V$ as above we also consider a more general class of Hamiltonians (which are such that the real and imaginary parts of their entries are independent and continuously distributed random variables and the densities of the real parts are bounded) and arbitrary Hermitian matrices B instead of the projections P_ν in Theorem 3.18, Corollary 3.19 and Corollary 3.20.

Regarding dynamical typicality we proved a simple upper bound on the relative errors in Proposition 3.24 under no assumptions on the Hamiltonian but under the very restrictive assumption that $\max\{b^+_{\min} - b^-_{\max}, b^-_{\min} - b^+_{\max}\} > 0$. A better result was obtained for random Hamiltonians as in Theorem 3.16, see Theorem 3.23 and Theorem 3.25, however, in this situation we were only able to prove bounds for the *comparative error* $|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)|/\overline{\|P_\nu\psi_t\|^2}$ (and similarly for general Hermitian operators B instead of P_ν). If the comparative error is small, then the absolute error at time t is small compared to the long-time average of $\|P_\nu\psi_t\|^2$, however, this does not necessarily mean that it is also small compared to the ensemble average $\mathbb{E}_\mu\|P_\nu\psi_t\|^2$ at time t.

We end the discussion of our results regarding the relative errors with sharper estimates for the quantities $M_{\mu\nu}$ for two ensembles of Hermitian random matrices for which better eigenvector delocalization results are available (however, matrices from these ensembles are not band matrices and they have substantially nonzero entries also far away from the diagonal).

The first one of these results is from Ajanki, Erdős, and Krüger [1] and it allows us to prove a much better lower bound for $M_{\mu\,\mathrm{eq}}, M_{\mathrm{eq}\,\nu}$ and $M_{\mathrm{eq}\,\mathrm{eq}}$ provided that the thermal equilibrium macro space is very dominant. Note that for these results we somewhat shift our perspective and consider sequences $(H^{(D)})$ of random matrices whereas for the previous results $D \in \mathbb{N}$ was fixed. The matrices considered in [1] are Wigner-type matrices satisfying a couple of assumptions, see Section 1.8.1 for the details, and they do not have a band structure. Ajanki, Erdős, and Krüger [1] showed that the eigenvectors of this class of random matrices are (with high probability) delocalized in the sup norm, see Theorem 1.46. Roughly speaking, this theorem shows that no component of an eigenvector ϕ_n of H can be much larger than $\frac{1}{\sqrt{D}}$. While this rules out that a large fraction of the components of ϕ_n is much smaller than $\frac{1}{\sqrt{D}}$ it is still possible that a small fraction of the components of ϕ_n is arbitrarily small. Thus, if d_ν is small compared to D, $\langle\phi_n|P_\nu|\phi_n\rangle$ can be arbitrarily small and thus we expect that also $M_{\mu\nu}$ could be arbitrarily small if both d_μ and d_ν are small compared to D.

In [138, Theorem 13] we showed that for $\tau > 0$, sufficiently large D and any macro states μ and ν we have with high probability the following lower bounds:

$$M_{\mu\nu} \geq \frac{d_\nu}{d_\mu}\left(1 - \frac{D - d_\mu}{D^{1-2\tau}}\right), \qquad M_{\mu\nu} \geq 1 - \frac{D - d_\nu}{D^{1-2\tau}}. \tag{3.164}$$

This shows that if $(D - d_\mu)D^{-1+2\tau} > 1$ or $(D - d_\nu)D^{-1+2\tau} > 1$, the lower bound on $M_{\mu\nu}$ becomes negative and therefore useless as obviously $M_{\mu\nu} \geq 0$. However, if μ or ν corresponds to thermal equilibrium and $\mathcal{H}_{\mathrm{eq}}$ is very dominant in the sense that $d_{\mathrm{eq}} = D - o(D^{1-2\tau})$, then the lower bounds for $M_{\mu\nu}$ are nontrivial and we find, as expected, that $M_{\mathrm{eq}\,\nu} \gtrsim \frac{d_\nu}{d_\mu} \approx \frac{d_\nu}{D}$ and $M_{\mu\,\mathrm{eq}} \gtrsim 1 \approx \frac{d_{\mathrm{eq}}}{D}$.

The second result from random matrix theory that we applied in [138] to obtain better lower bounds for $M_{\mu\nu}$ is due to Cipolloni, Erdős, and Henheik [20]. Their result shows that for Wigner matrices a version of the eigenstate thermalization hypothesis (ETH) is satisfied (with high probability), see Section 1.8.3 and in particular Theorem 1.48 for the details. Note that Wigner matrices do not have a band structure, however, it is expected that their important feature of delocalized eigenvectors is also valid for band matrices whose band width is sufficiently large.

What we showed in [138] was that if the ETH according to Cipolloni, Erdős, and Henheik [20] is satisfied, then for every $\xi > 0$ and any two macro states μ and ν we find for sufficiently large D with high probability that

$$M_{\mu\nu} \geq \frac{d_\nu}{D}\left(1 - \frac{D^\xi}{\sqrt{d_\nu}}\right), \qquad M_{\mu\nu} \geq \frac{d_\nu}{D}\left(1 - \frac{D^\xi}{\sqrt{d_\mu}}\right). \tag{3.165}$$

Thus, if d_ν or d_μ is sufficiently large we find that $M_{\mu\nu} \gtrsim \frac{d_\nu}{D}$, in agreement with our expectations.

Normal Typicality for GAP Measures

The results that we have discussed in this section so far were all concerned with the uniform distribution on the sphere of a finite-dimensional Hilbert space as the measure of typicality. The goal of [143] was to generalize (generalized) normal typicality to GAP measures. Note that these measures can be defined on separable Hilbert spaces, i.e., we need not assume that the system's Hilbert space is finite-dimensional. For any density matrix ρ on $\mathcal{H}$, GAP(ρ) is, roughly speaking, the most spread out distribution over $\mathbb{S}(\mathcal{H})$ with density matrix ρ. If ρ is a canonical density matrix, GAP(ρ) arises as the thermal equilibrium distribution of wave functions and it can be viewed as a quantum analogue of the canonical ensemble [59, 57]. Therefore, generalizing typicality results from the uniform distribution on the sphere to GAP measures can be seen as an expression of equivalence of ensembles.

Our main result, Theorem 3.26, shows that if ρ is a density matrix with $\|\rho\| < 1/4$ and B is an operator with $d_{E,B} < \infty$ on a separable Hilbert space $\mathcal{H}$ of dimension $\dim \mathcal{H} \geq 4$, then GAP(ρ)-most initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for most times $t \in [0, \infty)$, the expectation $\langle \psi_t | B | \psi_t \rangle$ is close to the (ψ_0- and t-independent) quantity $M_{\rho B}$ provided that no eigenvalue and no eigenvalue gap of the Hamiltonian H is hugely degenerate and $\|B\|$ is not too large. Put another way, the curve $t \mapsto \langle \psi_t | B | \psi_t \rangle$ is nearly constant in the long run. We also obtained a finite-time result, however, as already mentioned below Theorem 3.26, the equilibration times we get are usually extremely large. Note that in the case that $\rho = P_\mu / d_\mu$ for some macro state μ, the measure GAP(ρ) is just the uniform distribution over $\mathbb{S}(\mathcal{H}_\mu)$ and then we basically recover the generalized normal typicality result from Theorem 3.12.

As we have already remarked below Theorem 3.26, our result covers the important case that $B = P_\nu$ for some macro state ν. The reason is that usually a coarse-grained version of the energy is among the macro observables which are used to construct the macro spaces $\mathcal{H}_\nu$ in the decomposition (1.12) of $\mathcal{H}$. Therefore, each $\mathcal{H}_\nu$ is a subset of a (finite-dimensional) energy shell corresponding to a small energy interval $[E, E + \Delta E]$ which implies that $|\mathcal{E}_{P_\nu}| < \infty$. Thus Theorem 3.26 shows that for GAP(ρ)-most $\psi_0 \in \mathbb{S}(\mathcal{H})$, the superposition weights $\|P_\nu \psi_t\|^2$ are in the long run $t \to \infty$ close to the fixed value $M_{\rho P_\nu}$.

Note that in [143] we were only concerned with absolute errors, however, if $B = P_\nu$ or $\rho = P_\nu / d_\nu$, then the considerations from [138] apply. We expect that if $D = \dim \mathcal{H} < \infty$, then it should be the case that $M_{\rho P_\nu} \approx d_\nu / D$: For example if H is non-degenerate and satisfies the ETH, i.e., $\langle m | P_\nu | m \rangle \approx \langle P_\nu \rangle_{\text{mic}} = \operatorname{tr}(P_\nu)/D = d_\nu / D$ for any eigenstate $|m\rangle$ of H, where $\langle \cdot \rangle_{\text{mic}}$ is the micro-canonical expectation value, then

$$M_{\rho P_\nu} = \sum_m \langle m | \rho | m \rangle \langle m | P_\nu | m \rangle \approx \frac{d_\nu}{D} \sum_m \langle m | \rho | m \rangle = \frac{d_\nu}{D}. \tag{3.166}$$

We finally remark that alternative strategies of proof of Theorem 3.26 which make use of Lévy's Lemma for GAP measures instead of Markov's and Chebyshev's inequality do in general not give better bounds. Here, the same discussion as above for the uniform distribution applies: If we apply Lévy's Lemma for GAP measures to the function $f(\psi_0) = \langle |\langle \psi_t | B | \psi_t \rangle - M_{\rho B}|^2 \rangle_T$ (and similarly for the infinite time average), then the resulting bound is worse unless ε is so extremely small that it is of little interest. If we apply Lévy's Lemma for GAP measures to the function $f(\psi_0) = \langle\psi_t|B|\psi_t\rangle$ for fixed $t \geq 0$ and then bound $|\operatorname{tr}(e^{iHt}Be^{-iHt}) - M_{\rho B}|$ with the help of similar methods as were used in the proof of Proposition 3.33, then we obtain a result for most t for most ψ_0 and we have to interchange the quantifiers "most t" and "most ψ_0". In the case of a finite time interval $[0, T]$ this leads again to a bound that is, in relevant cases, slightly better than the one obtained in Theorem 3.26 provided that the set of "bad" times is sufficiently small. However, in the case of the infinite time interval $[0, \infty)$ this reasoning is not possible since the quantifiers "most $t \in [0, \infty)$" and "most ψ_0" can in general not be interchanged, see Footnote 3 in [143]. For more details regarding this alternative strategy of proof, see the corresponding discussion for the case of the uniform distribution above and Remark 1 in [143].

3.3. Macroscopic Thermalization for Highly Degenerate Hamiltonians

3.3.1. Results

In this section we present the results obtained in [117]. We first collect some small motivating results, then continue with the thermalization for general (and possibly highly degenerate) Hamiltonians and end with the thermalization of the free, non-relativistic Fermi gas in arbitrary dimensions.

In the following we always assume that the system's Hilbert space $\mathcal{H}$ is finite-dimensional and that there is a subspace $\mathcal{H}_{\mathrm{eq}}$ which has most of the dimensions of $\mathcal{H}$. This is justified by von Neumann's [144] reasoning to think of $\mathcal{H}$ as of an energy shell[27] (which usually is finite-dimensional) and of $\mathcal{H}_{\mathrm{eq}}$ as a subspace corresponding to thermal equilibrium, see Section 1.2 for details.

A Collection of Motivating Results

Recall from Section 1.6 that a system in a pure state $\psi \in \mathbb{S}(\mathcal{H})$ is in macroscopic thermal equilibrium $\mathrm{MATE}_\varepsilon$ with $\varepsilon > 0$ if and only if $\|P_{\mathrm{eq}}\psi\|^2 \geq 1 - \varepsilon$. Moreover, we say that a Hamiltonian H satisfies the eigenstate thermalization hypothesis ETH_ε if

$$\forall \text{ normalized eigenvectors } \phi \text{ of } H : \phi \in \mathrm{MATE}_\varepsilon. \tag{3.167}$$

Roughly speaking, the ETH implies thermalization of every initial state. This is the content of Theorem 1.34 in the case of non-degenerate Hamiltonians. As we could

[27] Note that in some of our results the Hamiltonian varies due to the addition of a small random perturbation. However, for the justification of our assumptions we pretend that the energy shell does not change, which should cause no harm for sufficiently small perturbations as in this case each energy shell stays invariant during any finite time interval to an arbitrary degree of precision.

not find a proof of this well known fact for degenerate Hamiltonians in the literature, we gave its proof in Proposition 1 in [117]; more precisely, we showed the following result:

Proposition 3.34. *Let H_0 be a self-adjoint operator on a finite-dimensional Hilbert space $\mathcal{H}$, let $\mathcal{H}_{\text{eq}}$ be a subspace of $\mathcal{H}$ and P_{eq} the projection to it. Let $\varepsilon, \delta > 0$ and suppose that H_0 satisfies $\text{ETH}_{\varepsilon\delta}$. Then for every $\psi_0 \in \mathbb{S}(\mathcal{H})$ and $(1-\delta)$-most $t \in [0,\infty)$,*

$$\psi_t = e^{-iH_0t}\psi_0 \in \text{MATE}_\varepsilon. \tag{3.168}$$

Therefore, also for degenerate Hamiltonians, it suffices that the ETH is fulfilled with parameter $\varepsilon\delta$ to ensure that every initial wave function is in MATE_ε for $(1-\delta)$-most of the time.

If there is only one eigenbasis of a degenerate Hamiltonian such that all its eigenvectors are in MATE_ε, then we can only guarantee the ETH (for all eigenvectors) with a larger error (Corollary 1 in [117]):

Proposition 3.35. *Let $\mathcal{H}, \mathcal{H}_{\text{eq}}$ and P_{eq} be as in Proposition 3.34. Let H_0 be a Hamiltonian with maximal degeneracy D_E and suppose that it has an eigen-ONB $(\phi_k)_k$ such that $\phi_k \in \text{MATE}_\varepsilon$ for every k. Then for any normalized eigenvector ϕ of H_0,*

$$\phi \in \text{MATE}_{\varepsilon D_E}. \tag{3.169}$$

Moreover, the bound is sharp: For any D_E and $\varepsilon = 1/n$ for some $n \in \mathbb{N}$ there are H_0, P_{eq} and ϕ such that $\|P_{\text{eq}}\phi\|^2 = \max\{1-\varepsilon D_E, 0\}$.

This result is helpful if the eigenvalues of the Hamiltonian H_0 are not too highly degenerate. However, in the case of the free Fermi gas of N particles in $d \geq 1$ the maximal degeneracy is at least 2^{Nd} and the bound from Proposition 3.35 unfortunately becomes useless. A *typical* unit vector, however, is not so bad, even if D_E is large, as the following proposition (Proposition 2 in [117]) shows:

Proposition 3.36. *Let $\varepsilon > 0$, let $\mathcal{H}_e$ be an eigenspace of H_0 of dimension D_e, and suppose that one eigen-ONB of H_0 satisfies the ETH with parameter ε. Then for $\delta = 2\exp(-C\varepsilon^2 D_e)$ with $C = 2/9\pi^3$, $(1-\delta)$-most $\phi \in \mathbb{S}(\mathcal{H}_e)$ lie in $\text{MATE}_{2\varepsilon}$.*

One possibility to overcome the problem of high degeneracy is to add a random perturbation to the Hamiltonian. This lifts, with probability 1, the degeneracy of the eigenvalues (and eigenvalue gaps[28]) as the following lemma (Lemma 2 in [117]) shows:

[28] Note that the statement regarding the eigenvalue gaps is not needed in the following but we included this result anyways as the eigenvalue gaps are connected to the time scales of thermalization and it therefore might be an interesting result for future applications.

Lemma 3.37. *Let H_0 be a (deterministic) and V a random Hermitian $D \times D$ matrix. Suppose that the distribution of V is continuous in the space of Hermitian $D \times D$ matrices. Then, with probability 1, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ the Hamiltonian $H = H_0 + \lambda V$ has non-degenerate eigenvalues and eigenvalue gaps.*

Thermalization for General Hamiltonians

If a Hamiltonian H_0 has an eigenbasis that satisfies the ETH, then Proposition 3.35 and Proposition 3.34 guarantee thermalization of every initial state provided that the maximal degeneracy of the Hamiltonian is not too large. As we are interested in Hamiltonians which are highly degenerate, we proved a theorem which also covers this case (Theorem 1 in [117]):

Theorem 3.38. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, let $\mathcal{H}_{\mathrm{eq}}$ and $\mathcal{H}_\nu$ be two subspaces of $\mathcal{H}$ with associated projections P_{eq} and P_ν and define $P_{\mathrm{neq}} := I - P_{\mathrm{eq}}$, where I is the identity on $\mathcal{H}$. Let H_0 be a self-adjoint operator on $\mathcal{H}$ and assume that H_0 has an orthonormal eigenbasis $(\phi_k)_k$ such that there is an $\varepsilon > 0$ such that $\phi_k \in \mathrm{MATE}_\varepsilon$ for all k. Let V be a self-adjoint operator on $\mathcal{H}$ drawn randomly from a continuous distribution invariant under conjugation with all unitaries commuting with H_0. For $\lambda \in \mathbb{R}$ let $H := H_0 + \lambda V$. Then,*

$$\lim_{\lambda \to 0} \mathbb{E}_V \lim_{T \to \infty} \mathbb{E}_{\psi_0} \frac{1}{T} \int_0^T \|P_{\mathrm{neq}} \psi_t\|^2 \, dt < 2\varepsilon, \tag{3.170}$$

where $\mathbb{E}_{\psi_0}$ denotes the expectation with respect to the uniform distribution on $\mathbb{S}(\mathcal{H}_\nu)$.
As a consequence, for all $\delta, \delta', \delta'' > 0$ there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, for $(1-\delta)-$most V, $(1-\delta')$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ are such that for $(1-\delta'')$-most $t \in [0, \infty)$,

$$\psi_t \in \mathrm{MATE}_{\varepsilon'}, \quad \text{where } \varepsilon' = \frac{3\varepsilon}{\delta\delta'\delta''}. \tag{3.171}$$

In Theorem 3.38 the subspaces $\mathcal{H}_{\mathrm{eq}}$ and $\mathcal{H}_\nu$ are arbitrary, however, from a physical point of view we think of $\mathcal{H}_{\mathrm{eq}}$ as the thermal equilibrium subspace and of $\mathcal{H}_\nu$ as belonging to a (possibly non-equilibrium) macro state ν. If $\mathcal{H}_\nu$ is a non-equilibrium macro space, then Theorem 3.38 shows that for most small perturbations of H_0 most initial states from $\mathcal{H}_\nu$ thermalize. By setting $\mathcal{H}_\nu$ to be the one-dimensional subspace of $\mathcal{H}$ spanned by some fixed initial state ψ_0, we obtain for most perturbations V that this specific ψ_0 thermalizes under a slightly perturbed dynamics (Corollary 2 in [117]):

Corollary 3.39. *Let $\varepsilon, \mathcal{H}, \mathcal{H}_{\mathrm{eq}}, H_0, \phi_k, V$ and H be as in Theorem 3.38. Then for all $\delta, \delta' > 0$ there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and $\psi_0 \in \mathbb{S}(\mathcal{H})$, for $(1-\delta)$-most V, for $(1-\delta')$-most $t \in [0, \infty)$,*

$$\psi_t \in \mathrm{MATE}_{\varepsilon'}, \quad \text{where } \varepsilon' = \frac{3\varepsilon}{\delta\delta'}. \tag{3.172}$$

Thermalization of the Free Fermi Gas

In the following we present the results obtained for the free Fermi gas in arbitrary dimensions by applying the results for general Hamiltonians from above. Moreover, we also discuss a thermalization result for the (unperturbed) free Fermi gas in 1d.

We consider the free, non-relativistic Fermi gas of N particles on a d-dimensional lattice $\Lambda := \{1, \ldots, L\}^d$. Here, $L \in \mathbb{N}$, $d \geq 1$ and we assume periodic boundary conditions. The corresponding Hamiltonian is given by

$$H_0^{\mathrm{fF}} := - \sum_{\substack{x,y \in \Lambda \\ \mathrm{dist}(x,y)=1}} c_x^\dagger c_y, \tag{3.173}$$

where $c_x^\dagger$ and c_x are the creation and annihilation operators of a fermion at site $x \in \Lambda$ which satisfy the canonical anticommutation relations $\{c_x, c_y\} = 0$ and $\{c_x, c_y^\dagger\} = \delta_{x,y}$. Moreover, the corresponding Hilbert space $\mathcal{H}$ is the N-particle sector of fermionic Fock space which has dimension $D = \binom{L^d}{N}$.

For $\Gamma \subset \Lambda$ we define

$$N_\Gamma := \sum_{x \in \Gamma} c_x^\dagger c_x, \tag{3.174}$$

i.e., N_Γ is the number operator of the particles in the sublattice Γ.

Let $\eta > 0$ and $\mu := |\Gamma|/|\Lambda|$. We define the equilibrium subspace $\mathcal{H}_{\mathrm{eq},\eta}$ as being the range of the spectral projection

$$P_{\mathrm{eq},\eta} := \mathbb{1}_{[N(\mu-\eta), N(\mu+\eta)]}(N_\Gamma), \tag{3.175}$$

i.e., $P_{\mathrm{eq},\eta}$ projects to the subspace of $\mathcal{H}$ in which all states have a Born distribution of N_Γ that has support in an $N\eta$-neighborhood of the equilibrium value $N\mu$.

Similarly to Section 1.7.1 for odd L the k-space $\mathcal{K}$ is given by

$$\mathcal{K} := \left\{ \frac{2\pi}{L}\nu \,\middle|\, v \in \left\{0, \pm 1, \ldots, \pm \frac{L-1}{2}\right\}^d \right\} \tag{3.176}$$

and for even L we set

$$\mathcal{K} := \left\{ \frac{2\pi}{L}\nu \,\middle|\, \nu \in \left\{0, \pm 1, \ldots, \pm \left(\frac{L}{2} - 1\right), \frac{L}{2}\right\}^d \right\}. \tag{3.177}$$

For $k \in \mathcal{K}$ we define the creation operator

$$a_k^\dagger := \frac{1}{L^{d/2}} \sum_{x \in \Lambda} e^{ik \cdot x} c_x^\dagger \tag{3.178}$$

and for $k = (k_1, \ldots, k_N) \in \mathcal{K}^N$ the state

$$|\Psi_k\rangle := a^\dagger_{k_1} a^\dagger_{k_2} \ldots a^\dagger_{k_N} |\Phi_{\text{vac}}\rangle, \tag{3.179}$$

where $|\Phi_{\text{vac}}\rangle$ denotes the vaccum state in Fock space. Note that for $k \in \mathcal{K}^N$ the state $|\Psi_k\rangle$ is an eigenstate of the Hamiltonian H_0^{fF} provided that $k_i \neq k_j$ for all $i \neq j$; we denote the set of such $k \in \mathcal{K}^N$ by $\mathcal{K}^N_{\neq}$. Moreover, note that for each $k \in \mathcal{K}^N_{\neq}$ all vectors of the form $k_\pi := (k_{\pi(1)}, \ldots, k_{\pi(N)}) \in \mathcal{K}^N_{\neq}$, where $\pi \in S_N$ with S_N the permutation group on N elements, lead to the same eigenstate. In order to obtain an orthonormal eigenbasis of $\mathcal{H}$ we have to choose D elements from $\mathcal{K}^N_{\neq}$ such that no two elements are permutations of each other. To this end let $\tilde{\mathcal{K}}^N \subset \mathcal{K}^N_{\neq}$ be the set that contains exactly one representative from each permutation class. Then the set

$$B_1 := \left\{ |\Psi_k\rangle : k \in \tilde{\mathcal{K}}^N \right\} \tag{3.180}$$

is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of H_0^{fF}.

Tasaki [134] showed that in one dimension this eigenbasis satisfies an ETH and we proved that his proof generalizes to higher dimensions (Proposition 3 in [117]):

Proposition 3.40 (ETH for one eigenbasis of the free Fermi gas, any dimension d)**.** *Let $d \geq 1$, let $\Gamma \subset \Lambda$ be an arbitrary sublattice, let $0 < \eta < \frac{3}{2}\mu(1-\mu)$, where $\mu = |\Gamma|/|\Lambda|$, and let $\mathcal{H}_{\text{eq},\eta}$ and $P_{\text{eq},\eta}$ be defined as above. Then for every $\Psi_k \in B_1$,*

$$\|P_{\text{neq},\eta}\Psi_k\|^2 < 2e^{-\frac{\eta^2}{3\mu(1-\mu)}N}, \tag{3.181}$$

i.e., $\Psi_k \in \text{MATE}_\varepsilon$ with $\varepsilon = 2e^{-\frac{\eta^2}{3\mu(1-\mu)}N}$. Moreover, if $N < L/(2d)$ then the maximal degeneracy D_E is at least 2^{Nd}.

Because of Proposition 3.35, it follows from Proposition 3.40 that all eigenstates of H_0^{fF} are in MATE with parameter at least $2^{Nd+1}e^{-\frac{\eta^2}{3\mu(1-\mu)}N}$ (provided that $N < L/(2d)$) which is in general larger than 1 and therefore makes the statement useless. But as there is one eigenbasis that satisfies the ETH, we can apply Theorem 3.38 which immediately results in the following corollary (Corollary 4 in [117]):

Corollary 3.41 (Thermalization of the perturbed free Fermi gas, any dimension d)**.** *Let $d \geq 1$, let $\Gamma, \eta, \mu, \mathcal{H}_{\text{eq},\eta}$ and $P_{\text{neq},\eta}$ be as in Proposition 3.40 and let $\mathcal{H}_\nu$ be as in Theorem 3.38. Let V be a self-adjoint $D \times D$ matrix drawn randomly from a continuous distribution that is invariant under conjugation with all unitaries commuting with H_0^{fF}. For $\lambda \in \mathbb{R}$ let $H := H_0^{\text{fF}} + \lambda V$.*

Then for all $\delta, \delta', \delta'' > 0$ there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, for $(1-\delta)$-most perturbations V, $(1-\delta')$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ are such that for $(1-\delta'')$-

most $t \in [0, \infty)$,

$$\|P_{\mathrm{neq},\eta}\psi_t\|^2 < \frac{6}{\delta\delta'\delta''} e^{-\frac{\eta^2}{3\mu(1-\mu)}N}, \tag{3.182}$$

i.e., $\psi_t \in \mathrm{MATE}_\varepsilon$ *with* $\varepsilon = \frac{6}{\delta\delta'\delta''}\exp(-\eta^2 N/(3\mu(1-\mu)))$.

While Corollary 3.41 shows that the free, non-relativistic Fermi gas in any dimension thermalizes under slightly perturbed dynamics, we were able to prove a stronger result in one dimension. More precisely, in this case we were able to prove an ETH (with thermal equilibrium defined by $\mathcal{H}_{\mathrm{eq},\eta}$) for the unperturbed and unmodified Hamiltonian H_0^{fF} (Theorem 2 in [117]):

Theorem 3.42 (ETH for the free Fermi gas in 1d)**.** *Let* $d = 1$*,* L *prime,* $46 \leq N < L/4$ *and* $\eta > \frac{2(\ln N+1)}{N}$*. Moreover, let* $\Gamma \subset \Lambda$ *be an interval, let* $\mathcal{H}_{\mathrm{eq},\eta}$ *and* $P_{\mathrm{neq},\eta}$ *be defined as above and let* $\phi \in \mathbb{S}(\mathcal{H})$ *be an eigenstate of* H_0^{fF}*. Then,*

$$\|P_{\mathrm{neq},\eta}\phi\|^2 \leq \frac{32\ln N}{\eta^2 N}, \tag{3.183}$$

i.e., $\phi \in \mathrm{MATE}_{\varepsilon'}$ *with* $\varepsilon' = \frac{32\ln N}{\eta^2 N}$.

We remark that the technical condition that L is prime ensures that there are only trivial degeneracies, i.e., that all degeneracies are due to changing signs of the components in the k-vector corresponding to an eigenstate $|\Psi_k\rangle$.

With the help of Proposition 3.34 we immediately obtain thermalization of the free, non-relativistic Fermi gas in one dimension for every initial state:

Corollary 3.43 (Thermalization of the free Fermi gas in 1d)**.** *Let* $d = 1$ *and let* $L, N, \eta, \Gamma, \mathcal{H}_{\mathrm{eq},\eta}$ *and* $P_{\mathrm{eq},\eta}$ *be as in Theorem 3.42. Let* $\varepsilon, \delta > 0$ *such that* $\varepsilon\delta \geq \frac{32\ln N}{\eta^2 N}$*. Then for every* $\psi_0 \in \mathbb{S}(\mathcal{H})$ *and* $(1-\delta)$*-most* $t \in [0, \infty)$*,* $\psi_t = e^{-iH_0^{\mathrm{fF}}t}\psi_0$ *is such that*

$$\|P_{\mathrm{neq},\eta}\psi_t\|^2 < \varepsilon, \tag{3.184}$$

i.e., $\psi_t \in \mathrm{MATE}_\varepsilon$.

3.3.2. Strategy of Proof

A Collection of Motivating Results

The proof of Proposition 3.34, which shows that if a Hamiltonian H_0 satisfies $\mathrm{ETH}_{\varepsilon\delta}$, then every initial state spends $(1-\delta)$-most of the time in $\mathrm{MATE}_\varepsilon$, is basically the same as the one of Theorem 1.34 where the additional assumption that H_0 is non-degenerate was made. The only difference lies in the step where $\overline{\langle\psi_t|P_{\mathrm{neq}}|\psi_t\rangle} \leq \varepsilon\delta$ is shown; instead of working with pure states $|n\rangle\langle n|$ where $|n\rangle$ is an eigenstate of H_0, we now have to use spectral projections to the eigenspaces of H_0.

Proposition 3.35 is proved with the help of the simple fact that if the diagonal entries of a positive semi-definite $D \times D$ matrix M are all $\leq \varepsilon$ for some $\varepsilon > 0$, then $\|M\| \leq \varepsilon D$, and the bound is sharp as there exist M for which equality holds, see Lemma 1 in [117]. This result is then applied to the matrix $\Pi_e P_{\text{neq}} \Pi_e$ where Π_e is the projection to the eigenspace of H_0 with eigenvalue e. Note that this matrix has, by assumption, diagonal entries $\leq \varepsilon$ in the basis of the ϕ_k. A possible example which shows that the bound is sharp is given by a $D \times D$ matrix H_0, where $D = \max\{n, D_E\}$, which is diagonal in the basis of the ϕ_k, whose first D_E diagonal entries are equal to zero and all other diagonal entries are non-zero and pairwise distinct. Moreover, P_{neq} is defined as the matrix with all entries equal to $1/D$.

Proposition 3.36 is a simple consequence of the following statement proved by Reimann [107] with the help of Lévy's Lemma: For uniformly distributed $\phi \in \mathbb{S}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$ of dimension D and every self-adjoint operator A on $\mathcal{H}$ (which is not a multiple of the identity) it holds that

$$\mathbb{P}\Big(|\langle\phi|A|\phi\rangle - \operatorname{tr}(A)/D| \geq \varepsilon\Big) \leq 2\exp\left(-\frac{C\varepsilon^2 D}{\Delta_A^2}\right), \tag{3.185}$$

where $C = 2/9\pi^3$ and $\Delta_A > 0$ denotes the difference between the largest and the smallest eigenvalue of A. Applying this inequality to the case that $\mathcal{H} = \mathcal{H}_e$, $D = D_e$ and $A = \Pi_e P_{\text{neq}} \Pi_e$ and making use of the assumption that there is an eigenbasis that satisfies the ETH proves the claim.

The proof of Lemma 3.37 consists of three steps. In the first step we show that the set of Hermitian $n \times n$ matrices with degenerate eigenvalues and the set of Hermitian $n \times n$ matrices with distinct eigenvalues but degenerate gaps can be written as a zero set of a polynomial in the matrix entries. This is the content of Lemma 4 in [117]. The first claim follows immediately from the fact that a matrix has degenerate eigenvalues if and only if its discriminant vanishes and the discriminant can be written as a polynomial in the matrix entries, see, e.g., Lemma 1 in [98]. The second claim is proved by an adaption of the proof of Lemma 1 in [98] to the case of gaps. The idea is to show that a Hermitian matrix A with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ has degenerate gaps if and only if $\det \tilde{V} = 0$ where $\tilde{V}$ is the *Vandermonde matrix*,

$$\tilde{V} = \tilde{V}(x_1, \ldots, x_M) = \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{M-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{M-1} \\ 1 & x_3 & x_3^2 & \ldots & x_3^{M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_M & x_M^2 & \ldots & x_M^{M-1} \end{pmatrix}, \tag{3.186}$$

where $M := n(n-1)$ and $x_1 := \lambda_1 - \lambda_2$, $x_2 := \lambda_1 - \lambda_3, \ldots, x_{n-1} := \lambda_1 - \lambda_n, x_n :=$

$\lambda_2 - \lambda_1, x_{n+1} := \lambda_2 - \lambda_3, \ldots, x_M := \lambda_n - \lambda_{n-1}$. Then one shows that the entries of $B := \tilde{V}^T \tilde{V}$ can be written as a sum over traces of powers of A and as $\det B = (\det \tilde{V})^2$ the claim follows.

The second step is to show that if there is a $\hat{\lambda} > 0$ such that $H = H_0 + \hat{\lambda} V$ has non-degenerate eigenvalues and eigenvalue gaps, then there is a $\lambda_0 > 0$ such that the same is true for *every* $\lambda \in (0, \lambda_0)$. This is the content of Lemma 5 in [117]. For its proof note that by the previous result, there is a polynomial in λ such that its zeros are the matrices with degenerate eigenvalues. By assumption it does not vanish identically and therefore has only finitely many zeros. From this we can conclude the existence of a $\hat{\lambda}_0$ such that $H_0 + \lambda V$ has non-degenerate eigenvalues for all $\lambda \in (0, \hat{\lambda}_0)$. A similar argument shows that the same holds true if "degenerate eigenvalues" is replaced by "degenerate gaps (but non-degenerate eigenvalues)" which proves Lemma 5 in [117]. Note that we also give an alternative proof of the eigenvalue statement in Lemma 5 using arguments from (algebraic) geometry from [41] in Footnote 7 in [117].

In the last step we fix $\hat{\lambda} = 1$, choose a V such that $H = H_0 + V$ has non-degenerate eigenvalues and eigenvalue gaps and apply Lemma 5 from [117]. As the Hamiltonian H has with probability 1 non-degenerate eigenvalues and eigenvalue gaps, see, e.g., Appendix A in [138], the claim of Lemma 3.37 follows.

Thermalization for General Hamiltonians

The first step in the proof of Theorem 3.38 is to show the following identity:

$$\lim_{\lambda \to 0} \mathbb{E}_V \lim_{T \to \infty} \mathbb{E}_{\psi_0} \frac{1}{T} \int_0^T \langle \psi_t | P_{\mathrm{neq}} | \psi_t \rangle \, dt = \frac{1}{d_\nu} \sum_e d_e \, \mathbb{E}_{\psi_e} \left[\langle \psi_e | P_\nu | \psi_e \rangle \langle \psi_e | P_{\mathrm{neq}} | \psi_e \rangle \right]. \tag{3.187}$$

Here, $d_\nu = \dim \mathcal{H}_\nu$, e are the eigenvalues of the Hamiltonian H_0, d_e is the dimension of the eigenspace $\mathcal{H}_e$ of H_0 corresponding to the eigenvalue e, and the ψ_e are uniformly distributed over $\mathbb{S}(\mathcal{H}_e)$.

We first take the limit $T \to \infty$ and then average over $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ (the two operations can be interchanged by dominated convergence); this yields

$$\mathbb{E}_{\psi_0} \lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \psi_t | P_{\mathrm{neq}} | \psi_t \rangle \, dt = \frac{1}{d_\nu} \sum_{j=1}^{D} \langle \psi_j(\lambda) | P_\nu | \psi_j(\lambda) \rangle \langle \psi_j(\lambda) | P_{\mathrm{neq}} | \psi_j(\lambda) \rangle, \tag{3.188}$$

where $D = \dim \mathcal{H}$, $\psi_j(\lambda)$ is an eigenvector of $H = H_0 + \lambda V$ and where V is chosen such that for small λ the Hamiltonian H is non-degenerate. The eigenfunctions $\psi_j(\lambda)$ (with suitable phases) converge for $\lambda \to 0$ to an eigenbasis of H_0, see, e.g., [103, Chapter XII, Problem 17]. To which one they converge depends on the perturbation V. After taking the limit $\lambda \to 0$ in (3.188), we end up with the same expression with the $\psi_j(\lambda)$ replaced by the eigenvectors of some eigenbasis of H_0. Then we take the average over the perturbation V and make use of the fact that its distribution is invariant under conjugation with all unitaries commuting with H_0 to finally obtain (3.187) with the averages over the spheres of the eigenspaces $\mathcal{H}_e$. Note that again the limit $\lambda \to 0$ and the average $\mathbb{E}_V$ can be interchanged using dominated convergence.

The averages in (3.187) can be evaluated with the help of the Lemma 3.27 and together with the fact that one eigenbasis satisfies the ETH we can estimate

$$\lim_{\lambda\to 0} \mathbb{E}_V \lim_{T\to\infty} \mathbb{E}_{\psi_0} \frac{1}{T}\int_0^T \langle\psi_t|P_{\mathrm{neq}}|\psi_t\rangle\, dt < 2\varepsilon. \tag{3.189}$$

Thus for λ small enough, the right-hand side of (3.189) without the limit $\lambda \to 0$ is bounded by 3ε. Then the claim follows by basically applying Markov's inequality three times.

Corollary 3.39 is an almost immediate consequence of Theorem 3.38: by setting $P_\nu = |\psi_0\rangle\langle\psi_0|$ in Theorem 3.38 we find that for every $\psi_0 \in \mathbb{S}(\mathcal{H})$ there is a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and most perturbations, the initial state thermalizes under the slightly perturbed dynamics. In order to get the existence of a $\lambda_0 > 0$ that works for all $\psi_0 \in \mathbb{S}(\mathcal{H})$, we make use of the continuous dependency of $\mathbb{E}_V \lim_{T\to\infty} \frac{1}{T}\int_0^T \langle\psi_t|P_{\mathrm{neq}}|\psi_t\rangle\, dt$ on λ and ψ_0 as well as the compactness of $\mathbb{S}(\mathcal{H})$.

Thermalization of the Free Fermi Gas

The proof of Proposition 3.40 is almost the same as the one given by Tasaki [134] (in an earlier arXiv version) in one dimension, see also Lemma 1.44, where multiple subchains are considered simultaneously. The differences are that here we only consider one sublattice Γ (which is precisely what Tasaki [134] did in his first version of his article and which simplifies the proof), that in the definition of the operators $b_k^\dagger$ from (1.256), $\sqrt{L}$ has to be replaced by $L^{d/2}$ and the scalar multiplication kx becomes the scalar product of two vectors, namely $k \cdot x$. For the second claim in Proposition 3.40 observe that the condition that $N < L/(2d)$ ensures that there is a $k \in \tilde{\mathcal{K}}^N$ such that all $k' \in \tilde{\mathcal{K}}^N$ that differ from k only by a permutation of the N elements and by signs in the Nd components lead to different eigenstates with the same eigenvalue.

For the proof of Theorem 3.42 several propositions are necessary. The basic idea is to compute the expectation and variance of N_Γ in an arbitrary eigenstate of H_0^{fF} and then to use Chebyshev's inequality. As a preparation, we first compute the expectation and variance of N_Γ in eigenstates Ψ_k of H_0^{fF}:

Proposition 3.44 (Expectation and variance of N_Γ in eigenstates Ψ_k)**.** *Let $d \geq 1$, $\Gamma \subset \Lambda$, let $k, k' \in \tilde{\mathcal{K}}^N$ and $x \in \Gamma$. Then*

$$\langle\Psi_k|N_\Gamma|\Psi_k\rangle = N\frac{|\Gamma|}{|\Lambda|}, \tag{3.190}$$

$$\langle\Psi_k|N_\Gamma^2|\Psi_k\rangle - (\langle\Psi_k|N_\Gamma|\Psi_k\rangle)^2 \leq N\frac{|\Gamma|}{|\Lambda|}\left(1-\frac{|\Gamma|}{|\Lambda|}\right). \tag{3.191}$$

Moreover,

$$\langle\Psi_k|c_x^\dagger c_x|\Psi_{k'}\rangle = \frac{1}{|\Lambda|}\mathrm{sgn}(\tilde{\sigma})e^{i(k_{\tilde{\sigma}^{-1}(l)}-k_l)\cdot x} \tag{3.192}$$

if k'_l is the only component of k' which does not appear in k and $\tilde{\sigma} \in \mathcal{S}_N$ is the permutation such that $k'_{\tilde{\sigma}^{-1}(m)} = k_m$ for all $m \neq l$. In particular, if exactly one component of k' does not appear in k, then

$$|\langle\Psi_k|N_\Gamma|\Psi_{k'}\rangle| \leq \frac{|\Gamma|}{|\Lambda|}. \tag{3.193}$$

If more than one component of k' does not appear in k, then

$$\langle\Psi_k|N_\Gamma|\Psi_{k'}\rangle = 0. \tag{3.194}$$

Note that Proposition 3.44 is a combination of Proposition 6 and Proposition 8 from [117]. An important ingredient for the proof of Proposition 3.44 is the following lemma (see Lemma 6 in [117]):

Lemma 3.45. *Let $d \geq 1$, $x_1,\dots,x_N \in \{1,\dots,L\}$, and let $k \in \mathcal{K}^N$ such that there exists a permutation $\tau \in \mathcal{S}_N$ with $\tau(k) := (k_{\tau(1)},\dots,k_{\tau(N)}) \in \tilde{\mathcal{K}}^N$. Then,*

$$\langle\Phi_{\mathrm{vac}}|c_{x_N}\dots c_{x_1}a_{k_1}^\dagger\dots a_{k_N}^\dagger|\Phi_{\mathrm{vac}}\rangle = \frac{1}{L^{Nd/2}}\sum_{\sigma\in\mathcal{S}_N}\mathrm{sgn}(\sigma)\prod_{j=1}^N e^{ik_j\cdot x_{\sigma(j)}}. \tag{3.195}$$

This well-known formula is, e.g., stated in Appendix C in [134] in the case that $d = 1$ and it can be easily proved via induction.

With the help of Lemma 3.45 one finds for $x \in \Gamma$ that

$$\langle\Psi_k|c_x^\dagger c_x|\Psi_{k'}\rangle = \frac{1}{L^{Nd}}\sum_{\sigma\in\mathcal{S}_N}\mathrm{sgn}(\sigma)\sum_{l=1}^N e^{i(k'_{\sigma^{-1}(l)}-k_l)\cdot x}\prod_{\substack{m=1\\m\neq l}}^N\left(\sum_{x_m\in\Lambda}e^{i(k'_{\sigma^{-1}(m)}-k_m)\cdot x_m}\right). \tag{3.196}$$

Because of $\sum_{y\in\Lambda}e^{i(z-z')\cdot y} = L^d\delta_{z,z'}$ for $z,z' \in \mathcal{K}$ we immediately see that (3.196) is equal to zero if k and k' differ in more than one component. If $k = k'$ then only the permutation $\sigma = \mathrm{id}$ yields a non-vanishing contribution and we find that $\langle\Psi_k|c_x^\dagger c_x|\Psi_k\rangle = N/L^d$. If k and k' differ in exactly one component and k'_l is the component of k' which does not appear in k, the only non-vanishing contribution in (3.196) comes from the permutation $\tilde{\sigma} \in \mathcal{S}_N$ such that $k'_{\tilde{\sigma}^{-1}(m)} = k_m$ for all $m \neq l$ and in this case we get $\langle\Psi_k|c_x^\dagger c_x|\Psi_{k'}\rangle = \operatorname{sgn}(\tilde\sigma)e^{i(k_{\tilde\sigma^{-1}(l)}-k_l)\cdot x}/L^d$. From these observations we immediately obtain the claims in Proposition 3.44 concerning $\langle\Psi_k|N_\Gamma|\Psi_{k'}\rangle$.

In order to compute the variance of N_Γ in an eigenstate Ψ_k we have to evaluate

$$\langle\Psi_k|N_\Gamma^2|\Psi_k\rangle = \sum_{x,y\in\Gamma}\langle\Psi_k|c_x^\dagger c_x c_y^\dagger c_y|\Psi_k\rangle. \tag{3.197}$$

We have already seen that $\langle\Psi_k|(c_x^\dagger c_x)^2|\Psi_k\rangle = \langle\Psi_k|c_x^\dagger c_x|\Psi_k\rangle = N/L^d$ and therefore can restrict our considerations to the case that $x \neq y$. Again with the help of Lemma 3.45 we obtain

$$\begin{aligned}&\langle\Psi_k|c_x^\dagger c_x c_y^\dagger c_y|\Psi_k\rangle\\ &\quad= \frac{1}{L^{Nd}}\sum_{\sigma\in\mathcal{S}_N}\operatorname{sgn}(\sigma)\sum_{\substack{l,m=1\\ l\neq m}}^{N} e^{i(k_{\sigma^{-1}(l)}-k_l)\cdot x}e^{i(k_{\sigma^{-1}(m)}-k_m)\cdot y}\prod_{\substack{n=1\\ n\neq l,m}}^{N}\left(\sum_{x_n\in\Lambda}e^{i(k_{\sigma^{-1}(n)}-k_n)\cdot x_n}\right).\end{aligned} \tag{3.198}$$

From (3.198) we see that the only permutations that contribute are the identity and transpositions. After carefully collecting all the contributing terms we arrive at

$$\langle\Psi_k|c_x^\dagger c_x c_y^\dagger c_y|\Psi_k\rangle = \frac{N(N-1)}{L^{2d}} - \frac{2}{L^{2d}}\sum_{\substack{p,q=1\\ p<q}}^{N}\operatorname{Re}\left(e^{i(k_p-k_q)\cdot x}e^{i(k_q-k_p)\cdot y}\right). \tag{3.199}$$

Putting everything together we find that

$$\langle\Psi_k|N_\Gamma^2|\Psi_k\rangle = N\frac{|\Gamma|}{|\Lambda|} + N(N-1)\frac{|\Gamma|^2}{|\Lambda|^2} - \frac{2}{|\Lambda|^2}\sum_{\substack{p,q=1\\ p<q}}^{N}\left|\sum_{x\in\Gamma}e^{i(k_p-k_q)\cdot x}\right|^2 \tag{3.200a}$$

$$\leq N\frac{|\Gamma|}{|\Lambda|} + N(N-1)\frac{|\Gamma|^2}{|\Lambda|^2}. \tag{3.200b}$$

From this and $\langle\Psi_k|N_\Gamma|\Psi_k\rangle = N|\Gamma|/|\Lambda|$ the bound in Proposition 3.44 for the variance follows immediately.

The next step in the proof of Theorem 3.42 is to compute the expectation and variance of N_Γ in an arbitrary eigenstate of H_0^{fF}. Note that while Proposition 3.44 is valid in any dimension, for arbitrary eigenstates we were only able to show bounds in one dimension. Note that if L is prime, then all degeneracies of the eigenvalues of H_0^{fF} in one dimension are trivial, i.e., only due to changing the signs of the components k_i, see the proof of Theorem 3.2. in [134]. We collect the results for arbitrary eigenstates in the following proposition which is a combination of Proposition 5 and Proposition 7 from [117]:

Proposition 3.46 (Expectation and variance of N_Γ in arbitrary eigenstates)**.** *Let $d = 1$ and let $k = (k_1, \ldots, k_N) \in \tilde{\mathcal{K}}^N$. Let $E_k := -2\sum_{i=1}^N \cos k_i$ be the corresponding eigenvalue and $\mathcal{H}_{E_k} = \mathrm{span}\{\Psi_{k'} : k'_j = \pm k_j$ for all $j\}$ the corresponding eigenspace. Let $\Gamma \subset \Lambda$ be an interval and let $\phi \in \mathbb{S}(\mathcal{H}_{E_k})$. Then*

$$\left| \langle\phi|N_\Gamma|\phi\rangle - N\frac{|\Gamma|}{|\Lambda|} \right| \leq \ln N + 1, \tag{3.201}$$

$$\langle\phi|N_\Gamma^2|\phi\rangle - (\langle\phi|N_\Gamma|\phi\rangle)^2 \leq 4N\ln N + 13N + 3(\ln N)^2 + 13\ln N + 10 \tag{3.202}$$

$$\overset{N\geq 46}{\leq} 8N\ln N. \tag{3.203}$$

The assumption that Γ is an interval is just technical and there is no loss of generality in assuming that $\Gamma = \{1, \ldots, |\Gamma|\}$ as the Hamiltonian H_0^{fF} is invariant under cyclic permutations of the chain Λ.

We first discuss the proof of (3.201) concerning the expectation $\langle\phi|N_\Gamma|\phi\rangle$. Without loss of generality we assume that there are no components k_l, k_m such that $k_l = -k_m$ and that $0 < k_j < \pi$ for all j (all other cases lead to less terms and therefore smaller upper bounds). We can express $|\phi\rangle$ in the basis of the eigenfunctions $\Psi_{k'}$ with $k'_j = \pm k_j$ for all j, i.e.,

$$|\phi\rangle = \sum_{k'} \alpha_{k'}|\Psi_{k'}\rangle, \tag{3.204}$$

where $\alpha_{k'} = \langle\Psi_{k'}|\phi\rangle$. Then we find that

$$\langle\phi|N_\Gamma|\phi\rangle = N\frac{|\Gamma|}{|\Lambda|} + \sum_{x=1}^{|\Gamma|}\sum_{k'\neq k''} \alpha_{k'}^*\alpha_{k''}\langle\Psi_{k'}|c_x^\dagger c_x|\Psi_{k''}\rangle \tag{3.205a}$$

$$= N\frac{|\Gamma|}{|\Lambda|} + \frac{2}{L}\mathrm{Re}\left(\sum_{j=1}^N \frac{e^{-2ik_j} - e^{-2ik_j(|\Gamma|+1)}}{1-e^{-2ik_j}} \sum_{k':k'_j>0} \chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha_{k'}^*\alpha_{k''}\right), \tag{3.205b}$$

where the computation for the second line makes use of Proposition 3.44. The sum over k' can be bounded by 1 with the help of the Cauchy-Schwarz inequality. If $-2k_j \in [-\pi, \pi]$, we have that $|1-e^{-2ik_j}| \geq 8\nu_j/L$ and if $-2k_j < -\pi$, then $|1-e^{2ik_j}| \geq 4(1-2\nu_j/L)$, where $\nu_j \in \{1, \ldots, (L-1)/2\}$. With this we obtain

$$\left|\sum_{x=1}^{|\Gamma|}\sum_{k'\neq k''} \alpha_{k'}^*\alpha_{k''}\langle\Psi_{k'}|c_x^\dagger c_x|\Psi_{k''}\rangle\right| \leq \frac{2}{L}\sum_{j=1}^N \frac{2}{|1-e^{-2ik_j}|} \leq \sum_{j=1}^N \frac{1}{j} \leq \ln N + 1, \tag{3.206}$$

which implies (3.201).

For the proof of (3.203) we can again assume that there are no k_l, k_m such that $k_l = -k_m$ and $0 < k_j < \pi$ for all j. As we already have a bound for $\langle\phi|N_\Gamma|\phi\rangle$, it remains to compute $\langle\phi|N_\Gamma^2|\phi\rangle$ which can be written as

$$\langle\phi|N_\Gamma^2|\phi\rangle = \sum_{k'} |\alpha_{k'}|^2 \langle\Psi_{k'}|N_\Gamma^2|\Psi_{k'}\rangle + \sum_{x\in\Gamma}\sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle\Psi_{k'}|c_x^\dagger c_x|\Psi_{k''}\rangle$$
$$+ \sum_{\substack{x,y\in\Gamma \\ x\neq y}} \sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle\Psi_{k'}|c_x^\dagger c_x c_y^\dagger c_y|\Psi_{k''}\rangle. \tag{3.207}$$

For the first sum we can use the bound for $\langle\Psi_{k'}|N_\Gamma^2|\Psi_{k'}\rangle$ from above as well as the fact that the $|\alpha_{k'}|^2$ sum up to 1; this gives $N^2|\Gamma|^2/|\Lambda|^2$ plus terms of order N. The second sum can be bounded as above yielding a term of the order $\ln N$. Concerning the third sum we first note that we have for $x \neq y$ and $k' \neq k''$ that

$$\langle\Psi_{k'}|c_x^\dagger c_x c_y^\dagger c_y|\Psi_{k''}\rangle$$
$$= \frac{1}{L^N} \sum_{\sigma\in\mathcal{S}_N} \operatorname{sgn}(\sigma) \sum_{\substack{l,m=1 \\ l\neq m}}^{N} e^{i(k''_{\sigma^{-1}(l)} - k'_l)x} e^{i(k''_{\sigma^{-1}(m)} - k'_m)y} \prod_{\substack{n=1 \\ n\neq l,m}}^{N} \left(\sum_{x_n\in\Lambda} e^{i(k''_{\sigma^{-1}(n)} - k'_n)x_n} \right). \tag{3.208}$$

It follows that only those k' and k'' which differ in one or two components give a non-vanishing contribution. We first consider the case that k' and k'' differ in exactly one component. Then only the identity and transpositions lead to non-zero terms in (3.208) and after carefully collecting all of them (and considering $k'_j > 0$ and $k'_j < 0$ separately) we obtain

$$\sum_{x,y\in\Gamma} \sum_{\substack{k',k''\text{differ} \\ \text{in 1 component}}} \alpha_{k'}^* \alpha_{k''} \langle\Psi_{k'}|c_x^\dagger c_x c_y^\dagger c_y|\Psi_{k''}\rangle$$
$$= \frac{2}{L^2} \mathrm{Re} \left(\sum_{x,y=1}^{|\Gamma|} \sum_{j=1}^{N} \left[(N-1) \left(e^{-2ik_j x} + e^{-2ik_j y} \right) \sum_{k':k'_j>0} \chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}} \alpha_{k'}^* \alpha_{k''} \right.\right.$$
$$- \sum_{\substack{p=1 \\ p\neq j}}^{N} \left(e^{i(-k_j-k_p)x} e^{i(k_p-k_j)y} + e^{i(k_p-k_j)x} e^{i(-k_j-k_p)y} \right) \sum_{k':k'_j,k'_p>0} \chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}} \alpha_{k'}^* \alpha_{k''}$$
$$- \sum_{\substack{p=1 \\ p\neq j}}^{N} \left(e^{i(-k_j+k_p)x} e^{i(-k_p-k_j)y} + e^{i(-k_p-k_j)x} e^{i(-k_j+k_p)y} \right)$$

$$\times \sum_{k':k'_j>0,k'_p<0} \chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''}\Bigg]\Bigg). \quad (3.209)$$

Note that we sum here over all $x, y \in \Gamma$ and not only over $x \neq y$ to facilitate the computation. By similar calculations as are needed to evaluate the sums in (3.209) one can show that the error we make by summing over all $x, y \in \Gamma$ instead of $x \neq y$ is of order $\ln N$. The first step to evaluate (3.209) is to perform the summation over x and y. Moreover, as before, the absolute value of the sums over k' can be upper bounded by 1. With estimates similar to the ones we used for evaluating $\langle\phi|N_\Gamma|\phi\rangle$ we find that

$$\frac{2}{L^2}\left|\sum_{x,y=1}^{|\Gamma|}\sum_{j=1}^{N}(N-1)\left(e^{-2ik_jx}+e^{-2ik_jy}\right)\right| \leq \frac{2|\Gamma|(N-1)}{L}(\ln N+1). \quad (3.210)$$

The remaining terms in (3.209) can be upper bounded by terms of order N. Roughly speaking, the reason for this is that, due to the products of two exponential functions (one with x and one with y and both with different components of k) in the other terms, after performing the sums over x and y and estimating similarly as in (3.206) we end up with terms of the form $\sum_{j,p=1}^{N} 1/p^2 \leq 2N$ where we used that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6 < 2$.

If k' and k'' differ in two components, all terms appearing in the analogue of (3.209) are products of an exponential function with x and one with y.[29] Some of these exponentials contain sums or differences of components of k as in the third and fourth line in (3.209) and the corresponding sums can again be upper bounded by terms of order N. If the products, however, are of the form $e^{-ik_jx}e^{-ik_py}$ the sums over j and p can be evaluated as in (3.206) independently leading to a bound of order $(\ln N)^2$.

Putting everything together we arrive at

$$\langle\phi|N_\Gamma^2|\phi\rangle \leq N^2\frac{|\Gamma|^2}{|\Lambda|^2} + 2N\ln N + O(N). \quad (3.211)$$

By making use of the identity $a^2 - b^2 = (a-b)(a+b)$ for $a, b \in \mathbb{R}$ we obtain from (3.201) that

$$\langle\phi|N_\Gamma|\phi\rangle^2 \geq N^2\frac{|\Gamma|^2}{|\Lambda|^2} - 2N\ln N + O(N) \quad (3.212)$$

which finally gives that

$$\langle\phi|N_\Gamma^2|\phi\rangle - \langle\phi|N_\Gamma|\phi\rangle^2 \leq 4N\ln N + O(N). \quad (3.213)$$

[29]We remark that the terms with $x = y$ vanish in this case, so we make no error by summing over all x and y instead of only over $x \neq y$.

One can show that if $N \geq 46$ then the terms now hidden in $O(N)$ can be bounded by $4N \ln N$ which then gives (3.203).

After having shown Proposition 3.46, the proof of Theorem 3.42 basically reduces to an application of Chebyshev's inequality.

3.3.3. Discussion

Thermalization for General Hamiltonians

We have seen that in general if a Hamiltonian satisfies the ETH (i.e., all eigenvetors are in MATE), then every initial state thermalizes in the sense that it approaches MATE, see Proposition 3.34. If there is, however, only one eigenbasis of the Hamiltonian which satisfies the ETH and the Hamiltonian is highly degenerate, we cannot conclude that every eigenbasis fulfills the ETH; this is the content of Proposition 3.35. To overcome this difficulty, we propose to add a small random perturbation to the Hamiltonian which lifts the degeneracies with probability 1, see Lemma 3.37. We could then show in Theorem 3.38 that if one eigenbasis of the unperturbed Hamiltonian satisfies the ETH, then most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$, where $\mathcal{H}_\nu$ is an arbitrary macro space, thermalize under a (typical) slightly perturbed dynamics. Moreover, every initial state thermalizes under most small perturbations, see Corollary 3.39.

The assumption on the random perturbation V in Theorem 3.38 is that it follows a continuous distribution that is invariant under conjugation with all unitaries commuting with the unperturbed Hamiltonian H_0. This is the mathematically minimal assumption under which we can prove the thermalization statement in Theorem 3.38. One relevant example that fulfills this assumption is given by the Gaussian unitary ensemble (GUE). Such random perturbations contain super-long-range super-multi-body interactions and therefore they might seem physically unrealistic. We are, however, concerned with very weak perturbations and we argue that in this situation these interactions might not be this physically unrealistic after all: no system is completely isolated and a weak interaction with an environment (e.g., a gas of photons) may be expected to have a similar effect as such a very weak perturbation of H_0.[30]

While adding a small random perturbation lifts the degeneracies of the unperturbed Hamiltonian H_0, we expect that if H_0 possesses one eigenbasis which satisfies the ETH but not every eigen-ONB fulfills it, the small random perturbation will not make the ETH come true (for all eigenbases): For sufficiently small λ the eigen-ONB (with suitable phases) of $H = H_0 + \lambda V$ is arbitrarily close to an eigen-ONB of H_0 that is, in each eigenspace $\mathcal{H}_e$ of H_0, uniformly distributed among the ONBs of $\mathcal{H}_e$ and we conjecture that a random ONB of some $\mathcal{H}_e$ violates the ETH. More precisely, we

[30]Note that in the whole thesis we are concerned with the thermalization of *closed* quantum systems as being open is not necessary for it. However, for the justification of adding a small generic perturbation to the Hamiltonian, it becomes relevant that closed quantum systems are only an idealization that never occur in practice as no system is completely isolated from its environment.

argue that it follows from the fact that typical unit vectors from $\mathbb{S}(\mathcal{H}_e)$ are in MATE if $D_e = \dim \mathcal{H}_e$ is large, see Proposition 3.36, that for large D_e the ETH is satisfied. Moreover, Proposition 3.35 shows that the ETH is satisfied if D_e is sufficiently small. However, there is a critical regime of intermediate D_e in which we do not know whether the ETH is fulfilled and as the proof of Proposition 3.40 suggests, for the free Fermi gas with $d > 1$ there are potentially eigenspaces that fall within this regime, see Section 3.3. in [117] for the details.

We remark that if λV is not small, the situation is different, e.g., if $\lambda = 1$ and V is a GUE matrix, then $H = H_0 + V$ is a *deformed Wigner matrix* and it follows from Theorem 2.7 in [19] that H satisfies a version of ETH.

If not every eigenfunction of H is in MATE, then there are initial states ψ_0 that do not thermalize (take, e.g., ψ_0 to be an eigenfunction of H that is not in MATE). A statement concerning the thermalization of most initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$, however, would not be of much interest as most initial states are in thermal equilibrium anyway, see Theorem 1.31. Therefore we want to consider initial states from "non-equilibrium" subspaces and by taking $\mathcal{H}_\nu$ in Theorem 3.38 to be such a subspace, this theorem tells us that for small typical perturbations most initial states from the non-equilibrium subspace approach MATE and stay there for most of the time. If we consider a fixed initial state $\psi_0 \in \mathbb{S}(\mathcal{H})$, we obtain that most perturbations V are such that ψ_0 thermalizes, see Corollary 3.39. However, if we want to consider the Hamiltonian and therefore also V as fixed (as one usually does), then not every initial state ψ_0 thermalizes. Note further that our results do not give us estimates on *thermalization times*, i.e., on the time scales on which non-equilibrium initial states reach thermal equilibrium.

In general, proving a property for most Hamiltonians with respect to a probability distribution does not show that a particular Hamiltonian that we want to study also has this property. For a random Hamiltonian we can make the model more realistic by choosing the distribution narrower, e.g., we can choose it to be supported closely around some Hamiltonian which we believe is close to the true one or we can condition the distribution on properties that the true Hamiltonian has. Our model (deterministic Hamiltonian plus small random perturbation) can be seen as a combination of a (deterministic) model Hamiltonian H_0 and a random Hamiltonian.

Thermalization of the Free Fermi Gas

As an application of Theorem 3.38 we considered the free, non-relativistic Fermi gas of N particles in arbitrary dimensions. Following Shiraishi and Tasaki [123] (and later Tasaki [134, 133]), we defined the thermal equilibrium subspace $\mathcal{H}_{\mathrm{eq},\eta}$ to consist of the states for which the number of particles in a sublattice $\Gamma \subset \Lambda$ lies within a small tolerance $\eta > 0$ of the equilibrium value $N|\Gamma|/L$. This highly simplified model of thermal equilibrium can be made somewhat more realistic by not only considering one sublattice Γ but multiple sublattices $\Gamma_i \subset \Lambda$ simultaneously and by requiring that the number of particles in each sublattice Γ_i lies within a suitable tolerance of the equilibrium value $N|\Gamma_i|/L$. In fact, Tasaki [134, 133] considered the one-dimensional case, partitioned the lattice into subintervals Γ_i and defined thermal equilibrium in terms of the the number of particles in the Γ_i as just discussed. Note also that as soon as it is shown that for a wave function $\phi \in \mathbb{S}(\mathcal{H})$ the quantity $\|P_{\mathrm{neq}}\phi\|^2$ is small, where thermal equilibrium is only defined in terms of the number of particles in one sublattice Γ, it follows that also $\|\widetilde{P}_{\mathrm{neq}}\phi\|^2$ is small, where $\widetilde{P}_{\mathrm{neq}}$ projects to the subspace of $\mathcal{H}$ in which the numbers of particles in a not too large number of sublattices Γ_i are close to their equilibrium values, see also Remark 1 in [117].

Tasaki [134] showed in one dimension that the eigenfunctions Ψ_k of H_0^{fF} defined in (3.179) satisfy an ETH, more precisely, they are in MATE with parameter exponentially small in N, and we showed that the proof carries over to arbitrary dimensions, see Proposition 3.40. Due to the high degeneracy of the Hamiltonian H_0^{fF} (which is at least 2^{Nd} if $N < L/(2d)$) the ETH for *one* eigenbasis does not imply thermalization; this is a consequence of Proposition 3.35. However, as there is an eigenbasis that satisfies the ETH, we can apply Theorem 3.38 and obtain for typical small perturbations that most initial states from any subspace approach MATE and stay there for most of the time, see Corollary 3.41.

In one dimension we were able to prove a stronger result: We showed in Theorem 3.42 that every eigenstate of the unperturbed Hamiltonian H_0^{fF} is in MATE with parameter of order $\ln N/N$. Then it follows immediately from Proposition 3.34 that every initial state thermalizes (under the unperturbed dynamics), see Corollary 3.43. Note that the proof of Theorem 3.42 makes heavy use of the structure of the degeneracies of H_0^{fF} in one dimension and we were not able to prove an ETH for the unperturbed Hamiltonian in higher dimensions.

We remark that in the classical case, i.e., for the classical free gas of N particles, the situation is different as it obviously does not thermalize for all initial data. However, in the classical setting, thermalization of the free (unperturbed) gas in any dimension can be proved for most initial data, see [10, 11, 24] for the corresponding results, where also estimates on the thermalization time are given.

Our main motivation for proving Theorem 3.38 was to show thermalization of the free Fermi gas in arbitrary dimension. For this model we were able to prove that there is an eigenbasis that satisfies the ETH but we could not show the ETH for all eigenbases. Shortly after our paper appeared as a preprint, Tasaki [135] applied Theorem 3.38 to the Ising model in two dimensions below the critical temperature. He defined thermal equilibrium by requiring that the macroscopic magnetization is close to its microcanonical expectation value. For this system he showed that there is an eigenbasis that satisfies the ETH but not every eigenbasis does so. Tasaki [135] then proved with the help of Theorem 3.38 that every initial state from a given highly degenerate eigenspace of the Hamiltonian thermalizes under typical slightly perturbed dynamics.

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A. Accepted Publications

A.1. Time Evolution of Typical Pure States from a Macroscopic Hilbert Subspace

Time Evolution of Typical Pure States from a Macroscopic Hilbert Subspace

Stefan Teufel[*], Roderich Tumulka[†], Cornelia Vogel[‡]

Abstract

We consider a macroscopic quantum system with unitarily evolving pure state $\psi_t \in \mathcal{H}$ and take it for granted that different macro states correspond to mutually orthogonal, high-dimensional subspaces $\mathcal{H}_\nu$ (macro spaces) of $\mathcal{H}$. Let P_ν denote the projection to $\mathcal{H}_\nu$. We prove two facts about the evolution of the superposition weights $\|P_\nu \psi_t\|^2$: First, given any $T > 0$, for most initial states ψ_0 from any particular macro space $\mathcal{H}_\mu$ (possibly far from thermal equilibrium), the curve $t \mapsto \|P_\nu \psi_t\|^2$ is approximately the same (i.e., nearly independent of ψ_0) on the time interval $[0, T]$. And second, for most ψ_0 from $\mathcal{H}_\mu$ and most $t \in [0, \infty)$, $\|P_\nu \psi_t\|^2$ is close to a value $M_{\mu\nu}$ that is independent of both t and ψ_0. The first is an instance of the phenomenon of dynamical typicality observed by Bartsch, Gemmer, and Reimann, and the second modifies, extends, and in a way simplifies the concept, introduced by von Neumann, now known as normal typicality.

Key words: von Neumann's quantum ergodic theorem; eigenstate thermalization hypothesis; macroscopic quantum system; dynamical typicality; long-time behavior.

1 Introduction

The approach of studying thermalization through the analysis of closed quantum systems with huge numbers of degrees of freedom has led, among other things, to the *eigenstate thermalization hypothesis* (ETH) [4, 27, 6], to the discovery of *canonical typicality* [5, 18, 12], and more recently to the discovery of *dynamical typicality* [1,

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2, 16, 21, 22, 23], which is the fact that most pure states ψ with a given quantum expectation value $\langle\psi|A|\psi\rangle$ of a macroscopic observable A also have nearly the same $\langle\psi|B|\psi\rangle$ for any other observable B (and likewise also nearly the same $\langle\psi_t|B|\psi_t\rangle$). Here, we provide a very simple proof of an important special case of this statement, namely for A a projection and $\langle\psi|A|\psi\rangle = 1$. Put differently, we show that most ψ from a macroscopically large subspace of Hilbert space have almost the same expectation values of bounded observables.

Our second result concerns the long-time behavior of $\langle\psi_t|B|\psi_t\rangle$ under the unitary evolution $\psi_t = \exp(-iHt)\psi_0$ (taking $\hbar = 1$) and extends previous results of Reimann and Gemmer [23] as well as von Neumann's [17] result now known as *normal typicality* [10, 13]. In particular, our result avoids certain unrealistic assumptions of von Neumann's.

As usual for the description of macroscopic closed quantum systems, we restrict our consideration to a micro-canonical energy interval $[E - \Delta E, E]$ that is small in macroscopic units but large enough to contain very many eigenvalues of the Hamiltonian H; for a system of N particles, relevant intervals contain of order $\exp(N)$ eigenvalues. Let $\mathcal{H}$ be the corresponding spectral subspace, i.e., the range of $\mathbb{1}_{[E-\Delta E,E]}(H)$, or *energy shell*, and let $\mathbb{S}(\mathcal{H}) = \{\psi \in \mathcal{H} : \|\psi\| = 1\}$ denote the unit sphere and $D := \dim \mathcal{H} < \infty$. Following von Neumann [17], we assume that different macro states ν of the system correspond to mutually orthogonal subspaces $\mathcal{H}_\nu$ (macro spaces) of $\mathcal{H}$ such that

$$\mathcal{H} = \bigoplus_\nu \mathcal{H}_\nu. \tag{1}$$

Different vectors in the same $\mathcal{H}_\nu$ are regarded as "looking macroscopically equal". For example, the "macroscopic look" could be defined in terms of mutually commuting self-adjoint operators $M_1, \ldots, M_K$ regarded as the "macroscopic observables" [17]; then $\mathcal{H}_\nu$ are the joint eigenspaces and $\nu = (m_1, \ldots, m_K)$ is the corresponding list of eigenvalues. Let P_ν denote the projection onto $\mathcal{H}_\nu$. Although some macro spaces will have much larger dimensions $d_\nu := \dim \mathcal{H}_\nu$ than others, all d_ν will be very large, roughly comparable to $\exp(N)$.

In this setting, it is natural to consider initial states ψ_0 from a certain macro space and ask about the time evolution of the *macroscopic superposition weights* $\|P_\nu\psi_t\|^2$. We present two general, theoretical findings about these weights that mainly arise just from the hugeness of the d_ν's. The first finding (dynamical typicality) is that the curve given by $\|P_\nu\psi_t\|^2$ as a function of t is nearly ψ_0-independent once we fix the macro state of ψ_0. In other words, if ψ_0 is purely random in $\mathcal{H}_\mu$, then the superposition weights are nearly deterministic. The second finding (generalized normal typicality) is that in the long run, as $t \to \infty$, $\|P_\nu\psi_t\|^2$ is nearly constant, meaning it is close for most $t \in [0,\infty)$ to a t-independent and ψ_0-independent value, once we fix the macro state of ψ_0. This does not mean that $\|P_\nu\psi_t\|^2$ converges as $t \to \infty$ (it does not), but that the time periods in which $\|P_\nu\psi_t\|^2$ is far from that value tend to be short compared to the time intervals separating these periods. One can say that the $\|P_\nu\psi_t\|^2$ *equilibrate* in the long run; however, this equilibration does not correspond to thermal equilibrium in the sense of thermodynamics; rather, thermal equilibrium at time t would correspond to $\|P_\nu\psi_t\|^2 \approx 1$ for one particular ν (the macro state of thermal equilibrium, $\mathcal{H}_\nu = \mathcal{H}_{\mathrm{eq}}$) and $\|P_\nu\psi_t\|^2 \approx 0$ for all other ν's. We therefore speak of *normal equilibrium* when $\|P_\nu\psi_t\|^2$ assumes its long-term value for all ν.

Our results are *typicality* statements, i.e., they concern the way *most* ψ_0 behave, notwithstanding the existence of few exceptional ψ_0 that behave differently. However, a statement about most ψ_0 in $\mathbb{S}(\mathcal{H})$ would be of limited interest because it could be violated by every system outside of thermal equilibrium, as usually most ψ_0 in $\mathbb{S}(\mathcal{H})$ are in thermal equilibrium (meaning they are close to $\mathcal{H}_{\mathrm{eq}}$) [11]. Instead, we make more specific statements: we allow an arbitrary initial macro space $\mathcal{H}_\mu$, possibly far from thermal equilibrium, and make statements about most ψ_0 in $\mathbb{S}(\mathcal{H}_\mu)$. Such statements are also naturally of interest when we ask about the increase of the quantum Boltzmann entropy observable [14]

$$\hat{S} = \sum_\nu S(\nu) P_\nu \,, \tag{2}$$

where

$$S(\nu) = k_B \log d_\nu \tag{3}$$

is the quantum Boltzmann entropy of the macro state ν, and k_B is the Boltzmann constant. Note that a quantum system can be in a superposition of different macro states and thus also in a superposition of different entropy values.

In Section 2, we formulate our theorem about dynamical typicality and compare it to related results in the literature. In Section 3, the same for generalized normal typicality. In Section 4, we prove our result on dynamical typicality. In Section 5, we formulate further variants of our results. In Section 6, conclusions for realistic sizes of d_ν are discussed. In Section 7, we outline the proof of generalized normal typicality. In Section 8, we collect the remaining proofs. In Section 9, we conclude.

2 Dynamical Typicality

2.1 Mathematical Description

For formulating theorems, we introduce the following terminology. Suppose that for each $\psi \in \mathbb{S}(\mathcal{H}_\mu)$, the statement $s(\psi)$ is either true or false, and let $\varepsilon > 0$. We say that $s(\psi)$ is true for $(1-\varepsilon)$-*most* $\psi \in \mathbb{S}(\mathcal{H}_\mu)$ if and only if

$$u_\mu\big(\big\{\psi \in \mathbb{S}(\mathcal{H}_\mu) : s(\psi)\big\}\big) \geq 1 - \varepsilon \,, \tag{4}$$

where u_μ is the normalized uniform measure over $\mathbb{S}(\mathcal{H}_\mu)$. Similarly, given $T>0$ and $\delta>0$, we say that a statement $s(t)$ is true for $(1-\delta)$-*most* $t\in[0,T]$ if and only if

$$\tfrac{1}{T}\Big|\big\{t\in[0,T]:s(t)\big\}\Big|\geq 1-\delta\,, \tag{5}$$

where $|S|$ means the length of the set $S\subset\mathbb{R}$; and that $s(t)$ is true for $(1-\delta)$-*most* $t\in[0,\infty)$ if and only if the lim inf of the left-hand side of (5) as $T\to\infty$ is $\geq 1-\delta$.

The first finding we mentioned can be expressed as follows.

Theorem 1 (Dynamical typicality)**.** *Let μ,ν be arbitrary macro states. There is a function $w_{\mu\nu}:\mathbb{R}\to[0,1]$ such that for every $t\in\mathbb{R}$ and every $\varepsilon>0$, for $(1-\varepsilon)$-most $\psi_0\in\mathbb{S}(\mathcal{H}_\mu)$,*

$$\Big|\|P_\nu\psi_t\|^2-w_{\mu\nu}(t)\Big|\leq\frac{1}{\sqrt{\varepsilon d_\mu}}\,. \tag{6}$$

Moreover, for every μ,ν, every $T>0$, and $(1-\varepsilon)$-most $\psi_0\in\mathbb{S}(\mathcal{H}_\mu)$,

$$\frac{1}{T}\int_0^T\big|\|P_\nu\psi_t\|^2-w_{\mu\nu}(t)\big|^2dt\leq\frac{1}{\varepsilon d_\mu}\,. \tag{7}$$

That is, if $d_\mu\gg 1/\varepsilon$, then for any t and purely random ψ_0 from $\mathcal{H}_\mu$, the random value $\|P_\nu\psi_t\|^2$ is very probably close to the non-random value $w_{\mu\nu}(t)$. The latter can in fact be taken to be the average of $\|P_\nu\psi_t\|^2$ over $\psi_0\in\mathbb{S}(\mathcal{H}_\mu)$, which is

$$w_{\mu\nu}(t):=\frac{1}{d_\mu}\operatorname{tr}\Big[P_\mu\exp(iHt)P_\nu\exp(-iHt)\Big]\,. \tag{8}$$

Likewise, the whole curve of $\|P_\nu\psi_t\|^2$ as a function of $t\in[0,T]$ is very probably close, in the L^2 norm, to $w_{\mu\nu}(t)$ as a function of t. (Smallness of the L^2 norm implies further that $\big|\|P_\nu\psi_t\|^2-w_{\mu\nu}(t)\big|$ is small for most t; however, this statement, which is equivalent to saying that the expression is small for most pairs $(t,\psi_0)\in[0,T]\times\mathbb{S}(\mathcal{H}_\mu)$, follows already from (6); note that the quantifiers "most t" and "most ψ_0" commute. Moreover, it also follows from (7) by letting $T\to\infty$ that the long-time average of $\big|\|P_\nu\psi_t\|^2-w_{\mu\nu}(t)\big|^2$ is small, but this statement is actually weaker than for finite T, and it will be superseded below by a more specific statement in our second result, generalized normal typicality.) A more general statement for arbitrary operators B instead of P_ν and a tighter error bound is formulated in Section 5.

As a further remark, we observe that another quantity is also deterministic for purely random ψ_0 from $\mathbb{S}(\mathcal{H}_\mu)$: not only is the probability $\|P_\nu\psi_t\|^2$ associated with $\mathcal{H}_\nu$ at time t nearly deterministic, but also the *probability current* between $\mathcal{H}_\nu$ and $\mathcal{H}_{\nu'}$,

$$J_{\nu\nu'}:=-i\left(\langle\psi_t|P_\nu HP_{\nu'}|\psi_t\rangle-\langle\psi_t|P_{\nu'}HP_\nu|\psi_t\rangle\right)=2\operatorname{Im}\langle\psi_t|P_\nu HP_{\nu'}|\psi_t\rangle\,. \tag{9}$$

This quantity expresses the amount of probability passing, per unit time, from ν' to ν minus that from ν to ν'; it satisfies a discrete version of the continuity equation, viz.,

$$\partial_t \|P_\nu \psi_t\|^2 = \sum_{\nu'} J_{\nu\nu'}. \tag{10}$$

In Section 8.2 we will show that the probability current between two macro spaces is deterministic.

2.2 Previous Results about Dynamical Typicality

Bartsch and Gemmer [2] introduced the name "dynamical typicality" for the following closely related phenomenon: Given an observable A and $a \in \mathbb{R}$, there is a function $a(t)$ such that for every $t \in \mathbb{R}$ and most $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle \approx a$, $\langle\psi_t|A|\psi_t\rangle \approx a(t)$. Müller, Gross, and Eisert [16] proved a rigorous version of this fact that also implies that for every operator B whose operator norm (largest absolute eigenvalue or singular value) is not too large, there is a value b such that for most $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle \approx a$, $\langle\psi_0|B|\psi_0\rangle \approx b$. As Reimann [22] pointed out, this also implies that for every $t \in \mathbb{R}$ and most $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle \approx a$, $\langle\psi_t|B|\psi_t\rangle \approx b(t)$ for suitable $b(t)$. Setting $A = P_\mu$, $a = 1$, and $B = P_\nu$, this yields that for every $t \in \mathbb{R}$ and most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$, $\langle\psi_t|P_\nu|\psi_t\rangle = \|P_\nu\psi_t\|^2$ is nearly deterministic. For technical reasons, the proofs of Müller, Gross, and Eisert [16] and Reimann [22] do not actually cover the case that A is a projection and $a = 1$. As was pointed out to us by one of the referees of our paper, Balz et al. [1] provide a general result that covers Theorem 1 as a special case. Although our proof strategy is similar to the one in [1], we decided to present our proof in this paper, because it is very simple and transparent and could help to make the at first sight striking phenomenon of dynamical typicality a text book result. Theorem 1 can also be obtained through a proof strategy used by Reimann and Gemmer [23].

A further related result is given by Strasberg et al. [28], who consider repeated measurements at $0 < t_1 < t_2 < \ldots < t_r < T$ of all P_ν's and argue that the probability distribution of the outcomes is essentially indistinguishable from the joint distribution of $X_{t_1}, \ldots, X_{t_r}$ for a suitable Markov process X_t on the set of ν's. This includes the claim that omitting one of the measurements does not significantly alter the distribution of the other outcomes, so the distribution of X_t should agree with $\|P_\nu\psi_t\|^2$, which is in line with our result.

3 Generalized Normal Typicality

3.1 Motivation

It is well known that for most $\phi \in \mathbb{S}(\mathcal{H})$,

$$\|P_\nu \phi\|^2 \approx \frac{d_\nu}{D}\,, \tag{11}$$

provided that d_ν and $D := \dim \mathcal{H}$ are large [10]. Under the additional condition that relative to a fixed decomposition (1) into macro spaces the eigenbasis of H is chosen purely randomly among all orthonormal bases (and some further technical conditions that are not very restrictive), (11) holds also for the eigenstates of H, and it can be shown that every $\psi_0 \in \mathbb{S}(\mathcal{H})$ evolves so that for most times t,

$$\|P_\nu \psi_t\|^2 \approx \frac{d_\nu}{D}\,. \tag{12}$$

This fact is known as *normal typicality* [17, 10, 13, 20].

The assumption of a purely random eigenbasis can be regarded as expressing that the energy eigenbasis is unrelated to the orthogonal decomposition (1). In most realistic systems, however, the energy eigenbasis and the macro decomposition (1) are not unrelated. If they were unrelated, then the system would very rapidly go from any macro space $\mathcal{H}_\nu$ directly to the thermal equilibrium macro space $\mathcal{H}_{\mathrm{eq}}$ (a macro space containing most dimensions of $\mathcal{H}$, $d_{\mathrm{eq}}/D \approx 1$) [7, 9, 8]. But that does not happen in most systems because thermal equilibrium requires that energy (and other quantities) is rather evenly distributed over all degrees of freedom, and for getting evenly distributed, it needs to get transported through space, which usually requires time and passage through other macro states, cf. Figure 1.

That is why we are interested in generalizations of normal typicality that apply also to Hamiltonians whose eigenbasis is not unrelated to $\mathcal{H}_\nu$. For such H, eigenvectors ϕ must be expected to have superposition weights $\|P_\nu\phi\|^2$ not always near d_ν/D. Our result actually applies to *all* Hamiltonians, at the expense that it does not apply to *all* initial quantum states ψ_0. As noted already, a statement about *most* $\psi_0 \in \mathbb{S}(\mathcal{H})$ would be limited to systems starting out in thermal equilibrium. Our result states that for any macro state μ, most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ evolve so that for most times t

$$\|P_\nu \psi_t\|^2 \approx M_{\mu\nu}\,, \tag{13}$$

provided that d_μ is large. See Theorem 2 for the precise quantitative statement and the definition of $M_{\mu\nu}$. The proof (see Section 8) builds particularly on techniques developed by Short and Farrelly [24, 25], but is also related to a series of works on quantum equilibration (e.g., [19, 15]) in which the long-time behavior of $\langle\psi_t|B|\psi_t\rangle$ is studied under various assumptions on B and ψ_0.

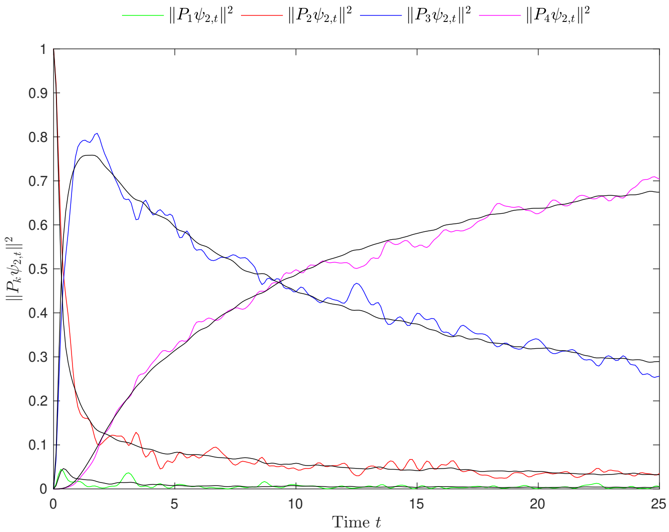

Figure 1: Example of time evolution of superposition weights $\|P_\nu\psi_t\|^2$, here in a Hilbert space of dimension $D = 2222$ decomposed into 4 macro spaces of dimensions $d_1 = 2$ (green curve), $d_2 = 20$ (red curve), $d_3 = 200$ (blue curve), and $d_4 = 2000$ (purple curve). The four curves add up to 1 at each t. At large t, the equilibrium subspace $\mathcal{H}_4$ has the biggest contribution. ψ_0 was chosen purely randomly from $\mathbb{S}(\mathcal{H}_2)$ (i.e., $\mu = 2$, so the red curve starts at 1, all others at 0). The Hamiltonian is a random band matrix (i.e., only entries sufficiently close to the main diagonal are significantly nonzero) in a basis aligned with the macro spaces, but with a wide enough bandwidth to still ensure delocalized eigenfunctions. Thus, parts of ψ_t reach $\mathcal{H}_4$ only after passing through $\mathcal{H}_3$, as mirrored in the fact that the blue curve increases first before it decreases in favor of the purple curve. Along with each of the four curves, also its deterministic approximation $w_{2\nu}(t)$ (in black) is drawn; dynamical typicality asserts that it is a good approximation.

The $M_{\mu\nu}$ are actually the averages of $\|P_\nu\psi_t\|^2$ over $t \in [0,\infty)$ and over $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$. Thus, they depend only on H and the decomposition (1), but not on t or ψ_0.

In this setting, *thermalization* means that $M_{\mu\,\mathrm{eq}} \approx 1$ for every μ, i.e., that for all macro states μ the overwhelming majority of micro states eventually reach thermal equilibrium in the sense that ψ_t lies almost completely in $\mathcal{H}_{\mathrm{eq}}$ and spends most of the time in the long run there. The time scale on which thermalization happens can be read off from the function $w_{\mu\,\mathrm{eq}}(t)$, while the other $w_{\mu\nu}(t)$ provide information about the detailed path to thermal equilibrium passing through intermediate macro states.

3.2 Statement of Result

In the following we consider Hamiltonians with spectral decomposition

$$H = \sum_{e \in \mathcal{E}} e \, \Pi_e, \tag{14}$$

where $\mathcal{E}$ is the set of distinct eigenvalues of H and Π_e the projection onto the eigenspace of H with eigenvalue e. The quantitative bounds in our theorem depend on the Hamiltonian only through the following characteristics of the distribution of its eigenvalues: the maximum degeneracy $D_E := \max_{e \in \mathcal{E}} \operatorname{tr}(\Pi_e)$ of an eigenvalue and the maximal gap degeneracy

$$D_G := \max_{E \in \mathbb{R}} \#\big\{(e, e') \in \mathcal{E} \times \mathcal{E} : e \neq e' \text{ and } e - e' = E\big\}\,. \tag{15}$$

Theorem 2 (Generalized normal typicality)**.** *Let μ, ν be any macro states and define*

$$M_{\mu\nu} := \frac{1}{d_\mu} \sum_{e \in \mathcal{E}} \operatorname{tr}\left(P_\mu \Pi_e P_\nu \Pi_e\right)\,. \tag{16}$$

Then for any $\varepsilon, \delta > 0$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0, \infty)$

$$\Big| \|P_\nu \psi_t\|^2 - M_{\mu\nu} \Big| \leq 4 \sqrt{\frac{D_E D_G}{\delta \varepsilon d_\mu} \min\left\{1, \frac{d_\nu}{d_\mu}\right\}}\,. \tag{17}$$

Thus, as soon as $d_\mu \gg D_E D_G$, i.e., as soon as the dimension of $\mathcal{H}_\mu$ is huge and no eigenvalue and no gap of H is macroscopically degenerate, for most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ the superposition weight $\|P_\nu \psi_t\|^2$ will be close to the fixed value $M_{\mu\nu}$ for most times t.

For comparison, Reimann and Gemmer [23] also concluded that $\langle \psi_t | A | \psi_t \rangle$ is nearly constant, but for a different ensemble based on the condition $\langle \psi_0 | A | \psi_0 \rangle \approx a$. We also provide a statement analogous to Theorem 2 for $\langle \psi_t | B | \psi_t \rangle$ with arbitrary observable B instead of P_ν in Theorem 4 below.

3.3 Example

We illustrate Theorem 2 within a simple random matrix model. We partition the D-dimensional Hilbert space $\mathcal{H} := \mathbb{C}^D = \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \oplus \mathbb{C}^{d_3} \oplus \mathbb{C}^{d_4} =: \bigoplus_{\nu=1}^{4} \mathcal{H}_\nu$ into four macro spaces $\mathcal{H}_\nu$ of dimension d_ν, i.e., $\mathcal{H}_1$ is spanned by the first d_1 canonical basis vectors, $\mathcal{H}_2$ by the next d_2 canonical basis vectors and so on. The Hamiltonian H is a random $D \times D$-matrix H that has a band structure (i.e., mainly near-diagonal entries) and thus couples neighboring macro spaces more strongly than distant ones.

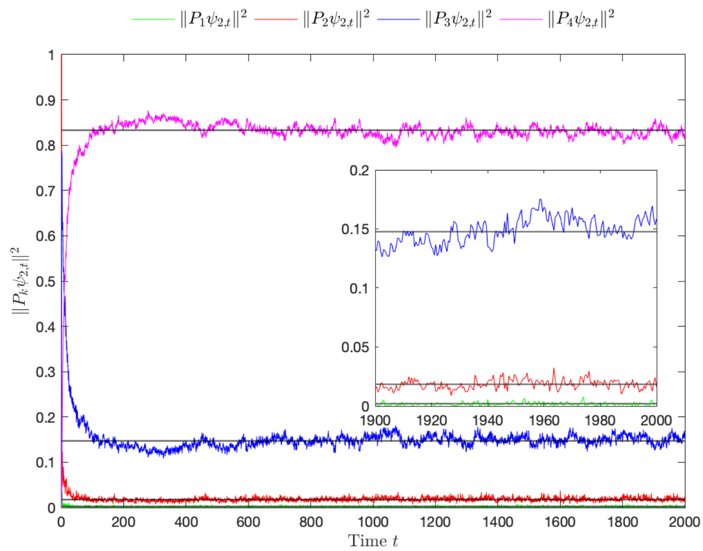

Figure 2: The same simulation as in Figure 1, only for longer times. The horizontal black lines indicate the values of the weights $M_{2\nu}$. The inset shows a part of the figure in magnification. Theorem 2 states that the displayed behavior is typical of initial states in $\mathcal{H}_2$: up to fluctuations that are either small or rare, $\|P_\nu\psi_t\|^2$ is close to $M_{2\nu}$.

More precisely, we choose $H = (h_{ij})_{ij}$ to be a self-adjoint random matrix such that $h_{ii} \sim \mathcal{N}(0, \sigma_{ii}^2)$ and $h_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2/2) + i\mathcal{N}(0, \sigma_{ij}^2/2)$ for $i \neq j$, where

$$\sigma_{ij}^2 := \exp(-s|i-j|) \tag{18}$$

with some $s > 0$ that controls the bandwidth. That is, the variances decrease exponentially in the distance from the diagonal.

In Figures 1 and 2 the weights $\|P_\nu\psi_t\|^2$ are plotted for the values $s = 0.02$, $d_\nu = 2 \times 10^{\nu-1}$, and a random initial vector $\psi_0 \in \mathcal{H}_2$. In Figure 1 the plot shows the initial phase where the system first passes through the 3rd macro state before settling mostly in the "equilibrium space" $\mathcal{H}_4$. Note that the bandwidth is roughly $2s^{-1} = 400 \approx D^{0.77} \gg D^{0.5}$ and we thus expect to be in the regime of delocalized eigenfunctions, which is also confirmed by the numerical results.

Theorem 2 states that the long term behaviour depicted in Figure 2 is typical of initial states $\psi_0 \in \mathcal{H}_2$: after some time the system equilibrates, the superposition weights $\|P_\nu\psi_{2,t}\|^2$ approach values $M_{2\nu}$ independent of the initial state, and stay close to them after the initial phase of equilibration. We also see that these values differ from the ones one would expect if normal typicality would hold: for example while in our simulation $d_4/D \approx 0.90$ one finds that $M_{24} \approx 0.82$.

The average entropy as a function of time is plotted in Figure 3. As expected, it increases up to small fluctuations.

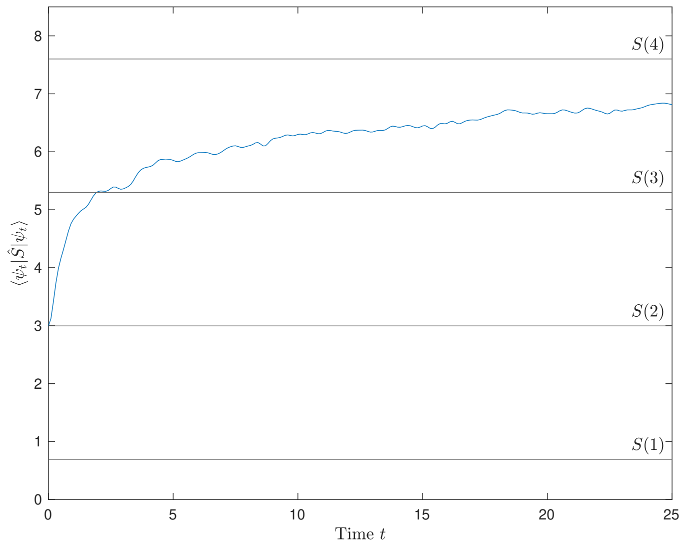

Figure 3: The average entropy $\langle\psi_t|\hat{S}|\psi_t\rangle = \sum_\nu \|P_\nu\psi_t\|^2 S(\nu)$ as a function of time t for $k_{\mathrm{B}} = 1$ and the same simulation as in Figure 1 and Figure 2. The tendency to increase can be regarded as a reflection of the second law of thermodynamics.

4 Proof of Theorem 1

The proof is very simple, based on an application of Chebyshev's, respectively Markov's, inequality to the following formulas for Hilbert space averages and Hilbert space variances [5, App. C]: For any Hilbert space $\mathcal{H}$ of dimension d, uniformly distributed $\psi \in \mathbb{S}(\mathcal{H})$, and any operator B on $\mathcal{H}$,

$$\mathbb{E}\big[\langle\psi|B|\psi\rangle\big] = \frac{1}{d}\,\mathrm{tr}\, B \tag{19}$$

$$\mathrm{Var}\big[\langle\psi|B|\psi\rangle\big] = \frac{1}{d(d+1)}\Big(\mathrm{tr}(B^\dagger B) - \frac{|\,\mathrm{tr}\, B|^2}{d}\Big)\,. \tag{20}$$

(As usual, the variance of a complex random variable Z is defined as $\mathrm{Var}\, Z := \mathbb{E}\big[|Z - \mathbb{E}(Z)|^2\big] = \mathbb{E}\big[|Z|^2\big] - |\mathbb{E}(Z)|^2$.) Dropping the last term and replacing $d+1$ by d, we obtain the trivial upper bound

$$\mathrm{Var}\big[\langle\psi|B|\psi\rangle\big] \leq \frac{\mathrm{tr}(B^\dagger B)}{d^2}\,. \tag{21}$$

Now we insert $\mathcal{H}_\mu$ for $\mathcal{H}$ and $B = P_\mu \exp(iHt) P_\nu \exp(-iHt) P_\mu$; we write $\mathbb{E}_\mu$ and Var_μ for expectation and variance over uniformly distributed $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$. We observe first that

$$\mathbb{E}_\mu\big[\|P_\nu \psi_t\|^2\big] = \frac{1}{d_\mu} \operatorname{tr}\big[P_\mu \exp(iHt) P_\nu \exp(-iHt)\big] = w_{\mu\nu}(t)\,. \tag{22}$$

For the variance, since $|\operatorname{tr}(CD)| \leq \|C\| \operatorname{tr}(|D|)$ for any operators C, D and $\|C\|$ the operator norm of C [26, Thm. 3.7.6], we have that

$$\operatorname{tr}(B^\dagger B) = \operatorname{tr}\Big(P_\mu \exp(-iHt) P_\nu \exp(iHt) P_\mu \exp(iHt) P_\nu \exp(-iHt) P_\mu\Big) \tag{23}$$

$$\leq \|P_\mu\|\, \|\exp(-iHt)\|\, \|P_\nu\|\, \|\exp(iHt)\| \cdots \|\exp(-iHt)\| \, \operatorname{tr} P_\mu \tag{24}$$

$$= d_\mu\,. \tag{25}$$

We thus obtain that

$$\mathrm{Var}_\mu\big[\|P_\nu \psi_t\|^2\big] \leq \frac{1}{d_\mu}\,. \tag{26}$$

The Chebyshev inequality then yields the first claim, (6).

For the second claim, Fubini's theorem allows us to interchange expectation and integral. Thus,

$$\mathbb{E}_\mu\left[\int_0^T \big|\|P_\nu \psi_t\|^2 - w_{\mu\nu}(t)\big|^2 \, dt\right] = \int_0^T \mathbb{E}_\mu\left[\big|\|P_\nu \psi_t\|^2 - w_{\mu\nu}(t)\big|^2\right] dt \tag{27}$$

$$= \int_0^T \mathrm{Var}_\mu\big[\|P_\nu \psi_t\|^2\big]\, dt \tag{28}$$

$$\leq \frac{T}{d_\mu} \tag{29}$$

by (26). Markov's inequality then yields the second claim, (7). □

As a side remark, the arguments of the proof also yield the following upper bound on the Hilbert space variance over subspaces of dimension d_μ for arbitrary B:

$$\mathrm{Var}_\mu\big[\langle\psi|B|\psi\rangle\big] \leq \frac{\operatorname{tr}(P_\mu B^\dagger P_\mu B P_\mu)}{d_\mu^2} \leq \frac{\|B\| \, \operatorname{tr}(|B|)}{d_\mu^2}\,. \tag{30}$$

5 More General Results

5.1 Dynamical Typicality

Here is a variant of Theorem 1 that allows for an arbitrary operator B instead of P_ν and provides a tighter error bound:

Theorem 3. *Let μ, ν be arbitrary macro states and let B be any operator on $\mathcal{H}$. There is a function $w_{\mu B} : \mathbb{R} \to [0,1]$ such that for every $t \in \mathbb{R}$ and every $\varepsilon > 0$, for $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,*

$$\Big|\langle\psi_t|B|\psi_t\rangle - w_{\mu B}(t)\Big| \leq \min\left\{ \frac{\|B\|}{\sqrt{\varepsilon d_\mu}},\ \sqrt{\frac{\|B\|\operatorname{tr}(|B|)}{\varepsilon d_\mu^2}},\ \sqrt{\frac{18\pi^3 \log(4/\varepsilon)}{d_\mu}}\|B\| \right\}. \tag{31}$$

Moreover, for every μ and B, every $T > 0$, and $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,

$$\frac{1}{T}\int_0^T \big|\langle\psi_t|B|\psi_t\rangle - w_{\mu B}(t)\big|^2 dt \leq \frac{\|B\|^2}{\varepsilon d_\mu}\,. \tag{32}$$

In fact, the function $w_{\mu B}(t)$ is the average of $\langle\psi_t|B|\psi_t\rangle$ over $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$, which is

$$w_{\mu B}(t) := \frac{1}{d_\mu}\operatorname{tr}\left[P_\mu \exp(iHt) B \exp(-iHt)\right]. \tag{33}$$

The proof of Theorem 3 (see Section 8.1) is largely analogous to that of Theorem 1. The bound involving $\sqrt{\log(1/\varepsilon)}$ instead of $1/\sqrt{\varepsilon}$ can be obtained by using Lévy's lemma instead of the Chebyshev inequality. However, it turns out that for all other results in this paper, the bounds provided by Markov's and Chebyshev's inequality are better than those provided by Lévy's lemma. That is because in many cases, Lévy's lemma yields a bound that is better in ε but worse in d_μ, which in our situation is worse because d_μ is usually way larger than any relevant $1/\varepsilon$; see Section 8.1 for more detail.

5.2 Generalized Normal Typicality

The next result, Theorem 4, provides a somewhat more general version of Theorem 2 that concerns arbitrary operators B instead of P_ν, as well as finite time intervals instead of $[0,\infty)$. To formulate it, we define the number $d_E := \#\mathcal{E}$ of distinct eigenvalues and the maximal number of gaps in an energy interval of length $\kappa > 0$,

$$G(\kappa) := \max_{E\in\mathbb{R}} \#\big\{(e,e') \in \mathcal{E}\times\mathcal{E} \,:\, e \neq e' \text{ and } e - e' \in [E, E+\kappa)\big\}\,. \tag{34}$$

It follows that $D_G = \lim_{\kappa\to 0^+} G(\kappa)$.

Theorem 4. *Let B be an operator on $\mathcal{H}$, let $\varepsilon, \delta, \kappa, T > 0$, let μ be any macro state, and define*

$$M_{\mu B} := \frac{1}{d_\mu}\sum_{e\in\mathcal{E}} \operatorname{tr}\left(P_\mu \Pi_e B \Pi_e\right)\,. \tag{35}$$

Then $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ *are such that for* $(1-\delta)$*-most* $t \in [0, T]$

$$\Big|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}\Big| \leq \tag{36}$$
$$4\sqrt{\frac{D_E\, G(\kappa)\|B\|}{\delta\varepsilon d_\mu}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}}.$$

Thus, as soon as $d_\mu \gg D_E G(\kappa)\|B\|^2$ and T is large enough, the right-hand side of (36) is small and the expectation $\langle\psi_t|B|\psi_t\rangle$ is close to a fixed value $M_{\mu B}$ for most times $t \in [0, T]$ and most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$. However, the times T required to make the right-hand side of (36) small are usually extremely large. For example, for a system of N particles, $\mathcal{H}$ has dimension of the order $\exp(N)$; provided that no eigenvalue is hugely degenerate, there are of the order $\exp(N)$ energy eigenvalues. In order to obtain a small error, we need to keep $G(\kappa)$ small. For $\kappa \sim \exp(-N)\Delta E$, already the number of nearest-neighbor gaps with $e - e' \in [0, \kappa)$ will be of order $\exp(N)$, and will thus contribute of order $\exp(N)$ to $G(\kappa)$. So, we need $\kappa \ll \exp(-N)$ and therefore $T \gg \exp(N)$ to obtain a small error in (36).

For the proof of Theorems 2 and 4 we need, besides Hilbert space averages and variances, also Hilbert space covariances of two operators. The covariance of two complex random variables X, Y is to be understood as

$$\operatorname{Cov}[X, Y] := \mathbb{E}\big[(X - \mathbb{E}X)^*(Y - \mathbb{E}Y)\big] \tag{37}$$
$$= \mathbb{E}[X^*Y] - (\mathbb{E}X)^*\,\mathbb{E}Y\,. \tag{38}$$

Lemma 1 (Hilbert Space Covariance)**.** *For uniformly distributed* $\psi \in \mathbb{S}(\mathcal{H})$ *with* $\dim\mathcal{H} = d$ *and any two operators* B, C *on* $\mathcal{H}$*,*

$$\operatorname{Cov}\Big[\langle\psi|B|\psi\rangle, \langle\psi|C|\psi\rangle\Big] = \frac{\operatorname{tr}(B^\dagger C)}{d(d+1)} - \frac{\operatorname{tr}(B^\dagger)\operatorname{tr}(C)}{d^2(d+1)}\,. \tag{39}$$

Put differently,

$$\mathbb{E}\big[\langle\psi|B|\psi\rangle^*\langle\psi|C|\psi\rangle\big] = \frac{\operatorname{tr}(B^\dagger)\operatorname{tr}(C) + \operatorname{tr}(B^\dagger C)}{d(d+1)}\,. \tag{40}$$

By inserting $\mathcal{H}_\mu$ for $\mathcal{H}$, it follows that for uniformly distributed $\psi \in \mathbb{S}(\mathcal{H}_\mu)$ and any two operators B, C on $\mathcal{H}$,

$$\mathbb{E}_\mu\big[\langle\psi|B^\dagger|\psi\rangle\langle\psi|C|\psi\rangle\big] = \tag{41}$$
$$\frac{1}{d_\mu(d_\mu+1)}\big(\operatorname{tr}(P_\mu B^\dagger)\operatorname{tr}(P_\mu C) + \operatorname{tr}(P_\mu B^\dagger P_\mu C)\big)\,.$$

6 Realistic Dimensions and Entropy

As indicated before, for a system of N particles or more generally of N degrees of freedom the dimension D is of order $\exp(N)$. We actually expect $D \approx \exp(s_{\mathrm{eq}} N/k_{\mathrm{B}})$, where s_{eq} is the entropy per particle in the thermal equilibrium state, and accordingly for all macro spaces $\mathcal{H}_\mu$,

$$d_\mu = \exp(s_\mu N/k_{\mathrm{B}})\,. \tag{42}$$

The following corollary to Theorem 2 shows that in this situation and assuming that no eigenvalues or gaps are macroscopically degenerate, fluctuations of the time-dependent superposition weights around their expected values are exponentially small in the number of particles with a rate controlled by the entropy per particle in the initial macro state.

Corollary 1. *Assume* (42). *Then, for all macro states* μ, ν_-, ν_+ *with*

$$s_{\nu_-} \leq s_\mu \leq s_{\nu_+} \tag{43}$$

it holds for $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ *for* $(1-\delta)$*-most of the time that*

$$\left| \|P_{\nu_+}\psi_t\|^2 - M_{\mu\nu_+} \right| \leq \frac{4\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{s_\mu N}{2k_{\mathrm{B}}}\right), \tag{44}$$

$$\left| \|P_{\nu_-}\psi_t\|^2 - M_{\mu\nu_-} \right| \leq \frac{4\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{(s_\mu - \frac{s_{\nu_-}}{2})N}{k_{\mathrm{B}}}\right). \tag{45}$$

In particular, if $s_\mu, s_{\nu_\pm}$ are fixed and $N \to \infty$, the error bounds are exponentially small. Note also that the numerical experiment in Figure 2 is consistent with the idea that the fluctuations of the superposition weights in macro spaces ν_+ of larger entropy than the initial state μ are controlled by the entropy s_μ of the initial macro state, while the fluctuations of the superposition weights in macro spaces ν_- of smaller entropy than the initial state μ are controlled by the entropy difference $s_\mu - s_{\nu_-}/2$ and thus even smaller. However, from the green line in Figure 2 (corresponding to $\|P_1\psi_t\|^2$) it is also apparent that the fluctuations of $\|P_\nu\psi_t\|^2$ might exceed the value of $M_{\mu\nu}$. Indeed, if we assume that the weights $M_{\mu\nu}$ scale like in the case of normal typicality, i.e.,

$$M_{\mu\nu} \approx \frac{d_\nu}{D} \approx \exp\left(-\frac{s_{\mathrm{eq}} - s_\nu}{k_{\mathrm{B}}} N\right), \tag{46}$$

then the relative error in (44) is only small if $s_{\nu_+} > s_{\mathrm{eq}} - s_\mu/2$, and the relative error in (45) is only small if $s_{\nu_-} > 2(s_{\mathrm{eq}} - s_\mu)$.

More generally, the question remains under which conditions one can prove that even for $M_{\mu\nu}$ close to 0, the relative error in (17) and thus the relative deviation of $\|P_\nu\psi_t\|^2$ from $M_{\mu\nu}$ will be small. In a separate work [29], we study this question for specific distributions of the random matrix H.

7 Outline of Proof of Theorem 4

Before we provide the technical details of the proof of Theorem 4 in Section 8, we explain now the main strategy and the key ideas. The first step is to control the time variance

$$\left\langle \left|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}\right|^2 \right\rangle_T := \frac{1}{T}\int_0^T \left|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}\right|^2 dt \tag{47}$$

of the quantity $\langle\psi_t|B|\psi_t\rangle$, where

$$M_{\psi_0 B} = \overline{\langle\psi_t|B|\psi_t\rangle} := \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle\psi_t|B|\psi_t\rangle\, dt \tag{48}$$

is just the time-average of $\langle\psi_t|B|\psi_t\rangle$. The time variance (47) was the subject of several earlier investigations concerning thermalization in closed quantum systems. It is usually controlled in terms of the effective dimension [19, 24, 25]

$$d_{\text{eff}} := \Big(\sum_e \langle\psi_0|\Pi_e|\psi_0\rangle^2\Big)^{-1} \tag{49}$$

of the initial state ψ_0, a measure for the number of distinct energies that contribute significantly to ψ_0. In Section 8.7 we slightly improve the bound of [25] (relevant when $d_\nu \ll d_\mu$) so that we can show that, after averaging the initial state over $\mathbb{S}(\mathcal{H}_\mu)$, one obtains that

$$\begin{aligned}\mathbb{E}_\mu\Big[\Big\langle \left|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}\right|^2 \Big\rangle_T\Big] \leq \\ \frac{2D_E G(\kappa)}{d_\mu+1}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|^2, \frac{\operatorname{tr}(B^\dagger B)}{d_\mu}\right\}.\end{aligned} \tag{50}$$

The second step is to show that $M_{\psi_0 B}$ is very close to $M_{\mu B}$ for most states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$. To this end we observe that $\mathbb{E}_\mu(M_{\psi_0 B}) = M_{\mu B}$ and then bound the variance according to

$$\mathbb{E}_\mu\Big[(M_{\psi_0 B} - M_{\mu B})^2\Big] \leq \frac{\|B\|}{d_\mu+1}\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}. \tag{51}$$

A careful application of Markov's inequality then shows that (50) and (51) together imply (36).

8 Remaining Proofs

8.1 Proof of Theorem 3

The phenomenon of concentration of measure, i.e., that on a sphere in high dimension, "nice" functions are nearly constant, is often expressed by means of (e.g., [28, Sec. II.C])

Lemma 2 (Lévy's Lemma)**.** *For any Hilbert space $\mathcal{H}$ with dimension d, any $f : \mathbb{S}(\mathcal{H}) \to \mathbb{R}$ with Lipschitz constant $\eta(f)$, and any $\varepsilon > 0$,*

$$|f(\psi) - \mathbb{E}f| \leq \sqrt{\frac{9\pi^3 \log(4/\varepsilon)}{2d}}\,\eta(f) \tag{52}$$

for $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$.

Alternatively, Chebyshev's inequality yields that

$$|f(\psi) - \mathbb{E}f| \leq \sqrt{\frac{\mathrm{Var}(f)}{\varepsilon}} \tag{53}$$

for $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$. In the important special case $f \geq 0$, Markov's inequality yields that

$$f(\psi) \leq \frac{\mathbb{E}f}{\varepsilon} \tag{54}$$

for $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$, while Lévy's lemma can be used in this situation to obtain that

$$f(\psi) \leq \mathbb{E}f + \sqrt{\frac{9\pi^3 \log(4/\varepsilon)}{2d}}\,\eta(f)\,. \tag{55}$$

Which bound is best depends on $\eta(f)$, $\mathrm{Var}(f)$, and $\mathbb{E}f$. For quadratic functions $f(\psi) = \langle\psi|B|\psi\rangle$, $\eta(f) = 2\|B\|$ on $\mathbb{S}(\mathcal{H})$, while expectation and variance are given by (19) and (20); the first two bounds in (31) arise from the Chebyshev bound (53) with different ways of bounding the variance, and the third from Lévy's lemma (52).

As remarked already, the other results in this paper are not improved by using Lévy's lemma instead of Markov's and Chebyshev's inequality. That is basically because the relevant functions $f \geq 0$ have means that are small like 1/dimension but Lipschitz constants of order 1, so that (55) yields errors of order $1/\sqrt{\text{dimension}}$. Now it is of little interest to make ε smaller than 10^{-200}. (Borel once argued [3, Chap. 6] that events with a probability of 10^{-200} or less can be expected to never occur in the history of the universe.) On the other hand, the dimensions are large like 10^N, so the advantage of (55) over (54) in ε does not compensate for its disadvantage in the dimension.

Proof of Theorem 3. By (19) after inserting $\mathcal{H}_\mu$ for $\mathcal{H}$ and $P_\mu \exp(iHt)B\exp(-iHt)P_\mu$ for B,

$$\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle = \frac{1}{d_\mu}\,\mathrm{tr}\big(P_\mu \exp(iHt)B\exp(-iHt)\big) = w_{\mu B}(t)\,. \tag{56}$$

Lévy's lemma with $\eta = 2\|B\|$ yields the third bound in (31).

By (21) after inserting $\mathcal{H}_\mu$ for $\mathcal{H}$ and $P_\mu \exp(iHt)B\exp(-iHt)P_\mu$ for B,

$$\mathrm{Var}_\mu\langle\psi_t|B|\psi_t\rangle \leq \frac{1}{d_\mu^2}\,\mathrm{tr}\Big(P_\mu \exp(-iHt)B^\dagger \exp(iHt)P_\mu \exp(iHt)B\exp(-iHt)P_\mu\Big)\,. \tag{57}$$

We give two upper bounds for the last expression. First, using $|\operatorname{tr}(CD)| \leq \|C\| \operatorname{tr}(|D|)$ and $\|B^\dagger\| = \|B\|$,

$$(57) \leq \frac{1}{d_\mu^2} \|P_\mu\| \|\exp(-iHt)\| \cdots \|\exp(-iHt)\| \operatorname{tr} P_\mu \tag{58}$$

$$= \frac{1}{d_\mu^2} \|B\|^2 d_\mu = \frac{\|B\|^2}{d_\mu}\,. \tag{59}$$

Second, by leaving B rather than P_μ inside the trace,

$$(57) \leq \frac{1}{d_\mu^2} \|B\| \operatorname{tr}(|B|)\,. \tag{60}$$

From these two bounds on the variance, (53) yields the first two bounds in (31). For the second claim, (32), of Theorem 3, the proof works as for Theorem 1 with the bound (59) for $\operatorname{Var}_\mu \langle \psi_t | B | \psi_t \rangle$. □

8.2 Probability Current

In order to see that also the probability current $J_{\nu\nu'}(t)$ as defined in (9) is deterministic, we verify that $\langle \psi_t | P_\nu H P_{\nu'} | \psi_t \rangle$ is deterministic. This can be obtained in the same way as for Theorem 3 by considering $B = P_\mu \exp(iHt) P_\nu H P_{\nu'} \exp(-iHt) P_\mu$ instead of $B = P_\mu \exp(iHt) P_\nu \exp(-iHt) P_\mu$ and noting that $\|B\| \leq \|H\| = \max\{|E - \Delta E|, |E|\}$. Physically, we expect E to be comparable to the particle number N and thus of order $\log D$, so $|J_{\nu\nu'}(t) - \mathbb{E} J_{\nu\nu'}(t)|$ is bounded by a constant times $\log D / \sqrt{\varepsilon d_\mu}$ (which would be small if we imagine $d_\mu \sim D^\alpha$ with $0 < \alpha < 1$ and fixed ε) for $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$. Likewise, $1/T$ times the L^2 norm over $[0, T]$ is bounded by a constant times $\log^2 D / \varepsilon d_\mu$ (which should be small).

8.3 Hilbert Space Covariance

For the proof of Lemma 1, we need the fourth moments of a random vector that is uniformly distributed over the unit sphere. So consider any Hilbert space $\mathcal{H}$ of dimension d and a uniformly distributed $\psi \in \mathbb{S}(\mathcal{H})$. Let $(\varphi_m)_m$ be an orthonormal basis of $\mathcal{H}$ and $a_m := \langle \varphi_m | \psi \rangle$. Then [17], [5, App. A.2 and C.1]

$$(i)\ \mathbb{E}(a_k^* a_l a_m^* a_n) = 0 \quad \text{if an index occurs only once,} \tag{61a}$$

$$(ii)\ \mathbb{E}\left(a_k^{*2} a_l^2\right) = 0 \quad \text{for } k \neq l, \tag{61b}$$

$$(iii)\ \mathbb{E}\left(|a_k|^4\right) = \frac{2}{d(d+1)}, \tag{61c}$$

$$(iv)\ \mathbb{E}\left(|a_k|^2 |a_l|^2\right) = \frac{1}{d(d+1)} \quad \text{for } k \neq l. \tag{61d}$$

Proof of Lemma 1. Let $(\varphi_m)_m$ be an orthonormal basis of $\mathcal{H}$. Then we can write $\psi \in \mathbb{S}(\mathcal{H})$ as

$$\psi = \sum_m a_m \varphi_m \tag{62}$$

with coefficients $a_m = \langle \varphi_m | \psi \rangle$. By (61), we get that

$$\begin{aligned}
\mathbb{E}\big[\langle\psi|B|\psi\rangle^*\langle\psi|C|\psi\rangle\big] &= \sum_{k,l,k',l'} \langle\varphi_k|B^\dagger|\varphi_l\rangle\langle\varphi_{k'}|C|\varphi_{l'}\rangle\mathbb{E}\left(a_k^* a_l a_{k'}^* a_{l'}\right) && (63)\\
&= \frac{1}{d(d+1)} \sum_{k,l,k',l'} \langle\varphi_k|B^\dagger|\varphi_l\rangle\langle\varphi_{k'}|C|\varphi_{l'}\rangle\left(\delta_{kl}\delta_{k'l'} + \delta_{kl'}\delta_{k'l}\right) && (64)\\
&= \frac{1}{d(d+1)}\left(\sum_{k,l}\langle\varphi_k|B^\dagger|\varphi_k\rangle\langle\varphi_l|C|\varphi_l\rangle + \langle\varphi_k|B^\dagger|\varphi_l\rangle\langle\varphi_l|C|\varphi_k\rangle\right) && (65)\\
&= \frac{1}{d(d+1)}\big(\mathrm{tr}(B^\dagger)\,\mathrm{tr}(C) + \mathrm{tr}(B^\dagger C)\big). && (66)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Cov}\big[\langle\psi|B|\psi\rangle, \langle\psi|C|\psi\rangle\big] &= \mathbb{E}\big[\langle\psi|B|\psi\rangle^*\langle\psi|C|\psi\rangle\big] - \mathbb{E}\big[\langle\psi|B|\psi\rangle^*\big]\,\mathbb{E}\big[\langle\psi|C|\psi\rangle\big] && (67)\\
&= \frac{\mathrm{tr}(B^\dagger)\,\mathrm{tr}(C) + \mathrm{tr}(B^\dagger C)}{d(d+1)} - \frac{\mathrm{tr}(B^\dagger)\,\mathrm{tr}(C)}{d^2} && (68)\\
&= \frac{\mathrm{tr}(B^\dagger C)}{d(d+1)} - \frac{\mathrm{tr}(B^\dagger)\,\mathrm{tr}(C)}{d^2(d+1)}\,. && (69)
\end{aligned}$$

□

8.4 Computing and Estimating some Averages over $\mathbb{S}(\mathcal{H}_\mu)$

As a preparation for the proof of Theorem 4, we derive in this section some upper bounds for relevant time and Hilbert space variances. We first note that it is well known that the limit in

$$M_{\psi_0 B} = \overline{\langle\psi_t|B|\psi_t\rangle} := \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle\psi_t|B|\psi_t\rangle\,dt \tag{70}$$

exists for all B and is given by

$$M_{\psi_0 B} = \langle\psi_0|\sum_{e\in\mathcal{E}}\Pi_e B \Pi_e|\psi_0\rangle\,. \tag{71}$$

From (19), applied to $\mathcal{H}_\mu$, we then obtain that

$$\mathbb{E}_\mu M_{\psi_0 B} = \frac{1}{d_\mu}\sum_{e\in\mathcal{E}}\mathrm{tr}(P_\mu \Pi_e B \Pi_e) = M_{\mu B}\,. \tag{72}$$

Proposition 1. *Let ψ_0 be uniformly distributed in $\mathbb{S}(\mathcal{H}_\mu)$, and let B be any operator on $\mathcal{H}$. Then for every $\kappa, T > 0$,*

$$\mathbb{E}_\mu\left(\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T\right) \leq \frac{2D_E G(\kappa)}{d_\mu + 1}\left(1 + \frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|^2, \frac{\operatorname{tr}(B^\dagger B)}{d_\mu}\right\}, \tag{73}$$

$$\operatorname{Var}_\mu \overline{\langle\psi_t|B|\psi_t\rangle} \leq \frac{\|B\|}{d_\mu + 1}\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}. \tag{74}$$

Proof. We start similarly to the proof of Theorem 1 in [25] and compute

$$\begin{aligned}
&\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T \\
&\quad = \left\langle\left|\sum_{e,e'} e^{i(e-e')t}\langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle - \sum_e \langle\psi_0|\Pi_e B\Pi_e|\psi_0\rangle\right|^2\right\rangle_T && (75)\\
&\quad = \left\langle\left|\sum_{e\neq e'} e^{i(e-e')t}\langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle\right|^2\right\rangle_T && (76)\\
&\quad = \sum_{\substack{e\neq e'\\ e''\neq e'''}} \left\langle e^{i(e-e'-e''+e''')t}\right\rangle_T \langle\psi_0|\Pi_e B\Pi_{e'}|\psi_0\rangle\langle\psi_0|\Pi_{e'''}B^\dagger\Pi_{e''}|\psi_0\rangle. && (77)
\end{aligned}$$

By averaging over $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$, we obtain

$$\begin{aligned}
&\mathbb{E}_\mu\left(\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T\right) \\
&= \sum_{\substack{e\neq e'\\ e''\neq e'''}} \left\langle e^{i(e-e'-e''+e''')t}\right\rangle_T \mathbb{E}_\mu\Big[\langle\psi_0|\Pi_e B\Pi_{e'}|\psi_0\rangle\langle\psi_0|\Pi_{e'''}B^\dagger\Pi_{e''}|\psi_0\rangle\Big] && (78)\\
&= \frac{1}{d_\mu(d_\mu+1)} \sum_{\substack{e\neq e'\\ e''\neq e'''}} \left\langle e^{i(e-e'-e''+e''')t}\right\rangle_T \Big[\operatorname{tr}(P_\mu \Pi_e B \Pi_{e'})\operatorname{tr}(P_\mu\Pi_{e'''}B^\dagger\Pi_{e''}) \\
&\qquad + \operatorname{tr}(P_\mu\Pi_e B\Pi_{e'}P_\mu\Pi_{e'''}B^\dagger\Pi_{e''})\Big], && (79)
\end{aligned}$$

where we applied Lemma 1 in the form (41) in the second equality.

Next we compute the ensemble variance of $\overline{\langle\psi_t|B|\psi_t\rangle}$: By (71) and (20) for $\mathcal{H}_\mu$,

$$\operatorname{Var}_\mu \overline{\langle\psi_t|B|\psi_t\rangle}$$

$$= \frac{1}{d_\mu(d_\mu+1)} \sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_e P_\mu \Pi_{e'} B^\dagger \Pi_{e'}) - \frac{1}{d_\mu^2(d_\mu+1)} \left| \sum_e \operatorname{tr}(P_\mu \Pi_e B \Pi_e) \right|^2 . \quad (80)$$

In the rest of the proof we use the computed expressions to prove the upper bounds for $\mathbb{E}_\mu \left(\langle |\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}|^2 \rangle_T \right)$ and $\operatorname{Var}_\mu \overline{\langle\psi_t|B|\psi_t\rangle}$. To this end, we define for $\alpha = (e,e') \in \mathcal{G} := \{(\bar{e},\bar{e}') \in \mathcal{E}\times\mathcal{E}, \bar{e} \neq \bar{e}'\}$ the vector $v_\alpha := \langle\psi_0|\Pi_{e'} B^\dagger \Pi_e|\psi_0\rangle$. Moreover, we define the Hermitian matrix

$$R_{\alpha\beta} := \left\langle e^{i(G_\alpha - G_\beta)t} \right\rangle_T \quad (81)$$

with $G_\alpha := e - e'$ for $\alpha = (e,e')$. Then we obtain with (77) that

$$\begin{aligned}
\left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T &= \sum_{\alpha,\beta} v_\alpha^* R_{\alpha\beta} v_\beta && (82)\\
&\leq \|R\| \sum_\alpha |v_\alpha|^2 && (83)\\
&= \|R\| \sum_{e\neq e'} \left| \langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle \right|^2 && (84)
\end{aligned}$$

and thus

$$\begin{aligned}
&\mathbb{E}_\mu \left(\left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \right)\\
&\leq \|R\| \sum_{e,e'} \mathbb{E}_\mu \left[\langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle \langle\psi_0|\Pi_{e'} B^\dagger \Pi_e|\psi_0\rangle \right] && (85)\\
&= \frac{\|R\|}{d_\mu(d_\mu+1)} \sum_{e,e'} \left[|\operatorname{tr}(P_\mu \Pi_e B \Pi_{e'})|^2 + \operatorname{tr}(P_\mu \Pi_e B \Pi_{e'} P_\mu \Pi_{e'} B^\dagger \Pi_e) \right] && (86)
\end{aligned}$$

by (41). Short and Farrelly [25] showed for arbitrary $\kappa > 0$ and $T > 0$ that

$$\|R\| \leq G(\kappa) \left(1 + \frac{8 \log_2 d_E}{\kappa T} \right). \quad (87)$$

Moreover, we estimate

$$\begin{aligned}
\sum_{e,e'} |\operatorname{tr}(P_\mu \Pi_e B \Pi_{e'})|^2 &= \sum_{e,e'} |\operatorname{tr}(\Pi_{e'} P_\mu \Pi_e \Pi_e B \Pi_{e'})|^2 && (88)\\
&\leq \sum_{e,e'} \underbrace{\operatorname{tr}(\Pi_{e'} P_\mu \Pi_e P_\mu)}_{\leq \operatorname{tr}(\Pi_{e'}) \leq D_E} \operatorname{tr}(\Pi_{e'} B^\dagger \Pi_e B) && (89)\\
&\leq D_E \operatorname{tr}(B^\dagger B), && (90)
\end{aligned}$$

where we used the Cauchy-Schwarz inequality for operators A, B with scalar product $\operatorname{tr}(A^\dagger B)$ and that $|\operatorname{tr}(CD)| \leq \|C\| \operatorname{tr}(|D|)$. Similarly we find that

$$\sum_{e,e'} |\operatorname{tr}(P_\mu \Pi_e A \Pi_{e'})|^2 \leq \sum_{e,e'} \operatorname{tr}(\Pi_{e'} P_\mu \Pi_e P_\mu) \underbrace{\operatorname{tr}(\Pi_{e'} B^\dagger \Pi_e B)}_{\leq \operatorname{tr}(\Pi_{e'}) \|B\|^2} \tag{91}$$

$$\leq D_E \|B\|^2 d_\mu. \tag{92}$$

This shows that

$$\sum_{e,e'} |\operatorname{tr}(P_\mu \Pi_e B \Pi_{e'})|^2 \leq D_E \min\{\|B\|^2 d_\mu, \operatorname{tr}(B^\dagger B)\}. \tag{93}$$

Next we compute

$$\sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_{e'} P_\mu \Pi_{e'} B^\dagger \Pi_e) = \sum_e \operatorname{tr}\left(\Pi_e P_\mu \Pi_e B \left(\sum_{e'} \Pi_{e'} P_\mu \Pi_{e'}\right) B^\dagger\right) \tag{94}$$

$$\leq \sum_e \operatorname{tr}(\Pi_e P_\mu \Pi_e) \left\| B \left(\sum_{e'} \Pi_{e'} P_\mu \Pi_{e'}\right) B^\dagger \right\| \tag{95}$$

$$\leq \|B\|^2 \sum_e \operatorname{tr}(\Pi_e P_\mu) \tag{96}$$

$$= \|B\|^2 d_\mu, \tag{97}$$

where we used in the third line that $\|\sum_{e'} \Pi_{e'} P_\mu \Pi_{e'}\| \leq 1$, which follows immediately from

$$\left\|\sum_{e'} \Pi_{e'} P_\mu \Pi_{e'} \psi_0\right\|^2 = \sum_{e'} \|\Pi_{e'} P_\mu \Pi_{e'} \psi_0\|^2 \leq \sum_{e'} \|\Pi_{e'} \psi_0\|^2 = \|\psi_0\|^2. \tag{98}$$

Similarly we estimate

$$\sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_{e'} P_\mu \Pi_{e'} B^\dagger \Pi_e) = \sum_{e'} \operatorname{tr}\left(\left(\sum_e \Pi_e P_\mu \Pi_e\right) B \Pi_{e'} P_\mu \Pi_{e'} B^\dagger\right) \tag{99}$$

$$\leq \sum_{e'} \operatorname{tr}(B \Pi_{e'} P_\mu \Pi_{e'} B^\dagger) \tag{100}$$

$$= \sum_{e'} \operatorname{tr}(\Pi_{e'} B^\dagger B \Pi_{e'} \Pi_{e'} P_\mu \Pi_{e'}) \tag{101}$$

$$\leq \sum_{e'} \operatorname{tr}(\Pi_{e'} B^\dagger B) \tag{102}$$

$$= \operatorname{tr}(B^\dagger B). \tag{103}$$

The previous two estimates show that

$$\sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_{e'} P_\mu \Pi_{e'} B^\dagger \Pi_e) \leq \min\{\|B\|^2 d_\mu, \operatorname{tr}(B^\dagger B)\}. \tag{104}$$

Putting everything together, we arrive at the upper bound

$$\begin{aligned}&\mathbb{E}_\mu \left(\left\langle \left| \langle \psi_t | B | \psi_t \rangle - \overline{\langle \psi_t | B | \psi_t \rangle} \right|^2 \right\rangle_T \right) \\ &\qquad \leq \frac{2 D_E G(\kappa)}{d_\mu + 1} \left(1 + \frac{8 \log_2 d_E}{\kappa T} \right) \min \left\{ \|B\|^2, \frac{\operatorname{tr}(B^\dagger B)}{d_\mu} \right\}. \end{aligned} \tag{105}$$

Finally we turn to the upper bound for $\operatorname{Var}_\mu \overline{\langle \psi_t | B | \psi_t \rangle}$. To this end, we estimate

$$\begin{aligned}\sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_e P_\mu \Pi_{e'} B^\dagger \Pi_{e'}) &= \operatorname{tr} \left(P_\mu \left(\sum_e \Pi_e B \Pi_e \right) P_\mu \left(\sum_{e'} \Pi_{e'} B^\dagger \Pi_{e'} \right) \right) && (106)\\ &\leq \operatorname{tr}(P_\mu) \left\| \left(\sum_e \Pi_e B \Pi_e \right) P_\mu \left(\sum_{e'} \Pi_{e'} B^\dagger \Pi_{e'} \right) \right\| && (107)\\ &\leq d_\mu \|B\|^2 && (108)\end{aligned}$$

and

$$\begin{aligned}\sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_e P_\mu \Pi_{e'} B^\dagger \Pi_{e'}) &= \operatorname{tr} \left(B \left(\sum_e \Pi_e P_\mu \left(\sum_{e'} \Pi_{e'} B^\dagger \Pi_{e'} \right) P_\mu \Pi_e \right) \right) && (109)\\ &\leq \operatorname{tr}(|B|) \left\| \sum_e \Pi_e P_\mu \left(\sum_{e'} \Pi_{e'} B^\dagger \Pi_{e'} \right) P_\mu \Pi_e \right\| && (110)\\ &\leq \operatorname{tr}(|B|) \left\| P_\mu \left(\sum_{e'} \Pi_{e'} B^\dagger \Pi_{e'} \right) P_\mu \right\| && (111)\\ &\leq \operatorname{tr}(|B|) \|B\|. && (112)\end{aligned}$$

This shows that

$$\sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_e P_\mu \Pi_{e'} B^\dagger \Pi_{e'}) \leq \|B\| \min\left\{ d_\mu \|B\|, \operatorname{tr}(|B|) \right\} \tag{113}$$

and thus

$$\begin{aligned}\operatorname{Var}_\mu \overline{\langle \psi_t | B | \psi_t \rangle} &\leq \frac{1}{d_\mu (d_\mu + 1)} \sum_{e,e'} \operatorname{tr}(P_\mu \Pi_e B \Pi_e P_\mu \Pi_{e'} B^\dagger \Pi_{e'}) && (114)\\ &\leq \frac{\|B\|}{d_\mu + 1} \min \left\{ \|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu} \right\}. && (115)\end{aligned}$$

□

8.5 Proof of Theorems 2 and 4

Theorem 2 follows immediately from Theorem 4 by setting $B = P_\nu$, choosing κ small enough such that $G(\kappa) = D_G$, and then taking the limit $T \to \infty$.

Proof of Theorem 4. Markov's inequality implies

$$\mathbb{P}_\mu\left(\left\langle|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}|^2\right\rangle_T \geq \frac{4D_E G(\kappa)\|B\|}{\varepsilon d_\mu}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)$$

$$\leq \frac{\mathbb{E}_\mu\left(\left\langle|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}|^2\right\rangle_T\right)}{4D_E G(\kappa)\|B\|\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}}\varepsilon d_\mu \tag{116}$$

$$\leq \frac{\min\{\|B\|^2, \frac{\operatorname{tr}(B^\dagger B)}{d_\mu}\}}{2\|B\|\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}}\varepsilon \tag{117}$$

$$\leq \frac{\varepsilon}{2}, \tag{118}$$

where we used the bounds from Proposition 1 and that $\operatorname{tr}(B^\dagger B) \leq \|B\| \operatorname{tr}(|B|)$. This means that for $(1-\frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,

$$\left\langle|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}|^2\right\rangle_T < \frac{4D_E G(\kappa)\|B\|}{\varepsilon d_\mu}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}. \tag{119}$$

Again with the help of Markov's inequality we obtain that, with λ the Lebesgue measure on $\mathbb{R}$,

$$\frac{\lambda\left\{t \in [0,T] : |\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}|^2 \geq \frac{4D_E G(\kappa)\|B\|}{\delta\varepsilon d_\mu}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right\}}{T} \tag{120}$$

$$\leq \frac{\delta\varepsilon d_\mu\left\langle|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}|^2\right\rangle_T}{4D_E G(\kappa)\|B\|\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}} \tag{121}$$

$$\leq \delta. \tag{122}$$

This shows that for $(1-\frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ we have for $(1-\delta)$-most $t \in [0,T]$ that

$$|\langle\psi_t|B|\psi_t\rangle - M_{\psi_0 B}| \leq 2\left(\frac{D_E G(\kappa)\|B\|}{\delta\varepsilon d_\mu}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2}. \tag{123}$$

Next we prove in a similar way an upper bound for $|M_{\psi_0 B} - M_{\mu B}|$, keeping in mind that $M_{\mu B} = \mathbb{E}_\mu M_{\psi_0 B}$. An application of Chebyshev's inequality and Proposition 1 shows that

$$\mathbb{P}_\mu\left(|M_{\psi_0 B} - M_{\mu B}| \geq \sqrt{\frac{2\|B\| \min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}}{d_\mu \varepsilon}}\right) \leq \frac{\operatorname{Var}_\mu \overline{\langle \psi_t | B | \psi_t \rangle}}{2\|B\| \min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}} d_\mu \varepsilon \tag{124}$$

$$\leq \frac{\varepsilon}{2}. \tag{125}$$

This implies for $(1 - \frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ that

$$|M_{\psi_0 B} - M_{\mu B}| \leq \sqrt{2} \left(\frac{\|B\|}{\varepsilon d_\mu} \min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2}. \tag{126}$$

With the triangle inequality we finally obtain the stated upper bound for $|\langle \psi_t | B | \psi_t \rangle - M_{\mu B}|$. □

8.6 Proof of Corollary 1

From Theorem 4 we obtain immediately that for $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ for $(1-\delta)$-most of the time

$$\left| \|P_{\nu_+} \psi_t\|^2 - M_{\mu\nu_+} \right| \leq 4\sqrt{\frac{D_E D_G}{\delta \varepsilon}} \exp\left(-\frac{s_\mu N}{2k_B}\right) \min\left\{1, \exp\left(\frac{(s_{\nu_+} - s_\mu)N}{2k_B}\right)\right\} \tag{127}$$

$$= 4\frac{\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{s_\mu N}{2k_B}\right). \tag{128}$$

Similarly, we find for $(1 - \varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ for $(1 - \delta)$-most of the time that

$$\left| \|P_{\nu_-} \psi_t\|^2 - M_{\mu\nu_-} \right| \leq 4\sqrt{\frac{D_E D_G}{\varepsilon \delta}} \exp\left(-\frac{s_\mu N}{2k_B}\right) \min\left\{1, \exp\left(\frac{(s_{\nu_-} - s_\mu)N}{2k_B}\right)\right\} \tag{129}$$

$$= 4\frac{\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{(s_\mu - \frac{s_{\nu_-}}{2})N}{k_B}\right). \tag{130}$$

This finishes the proof. □

8.7 Alternative Estimate in Terms of Effective Dimension

In Proposition 1, we have provided two upper bounds (73) for

$$\mathbb{E}_\mu\left(\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T\right).$$

There is an alternative way of obtaining one of the two bounds in (73) using a result of Short and Farrelly [25] based on the concept of effective dimension. We briefly explain this alternative derivation and then comment on why we also need the other bound in (73).

In [25] the authors show that

$$\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T \leq \frac{G(\kappa)\|B\|^2}{d_{\text{eff}}}\left(1+\frac{8\log_2 d_E}{\kappa T}\right), \tag{131}$$

where the effective dimension $d_{\text{eff}} = d_{\text{eff}}(\psi_0)$ of a state ψ_0 is

$$d_{\text{eff}} = \left(\sum_e \langle\psi_0|\Pi_e|\psi_0\rangle^2\right)^{-1}. \tag{132}$$

Taking an average over $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ yields the bound

$$\mathbb{E}_\mu\left(\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T\right) \leq \frac{2D_E G(\kappa)\|B\|^2}{d_\mu+1}\left(1+\frac{8\log_2 d_E}{\kappa T}\right). \tag{133}$$

To see this, note that the only quantity on the right-hand side of (131) that depends on ψ_0 is the effective dimension d_{eff}; therefore, it suffices to estimate $\mathbb{E}_\mu d_{\text{eff}}^{-1}$. With the help of (41) and the usual arguments we find

$$\begin{aligned}
\mathbb{E}_\mu d_{\text{eff}}^{-1} &= \sum_e \mathbb{E}_\mu\left(\langle\psi_0|\Pi_e|\psi_0\rangle\langle\psi_0|\Pi_e|\psi_0\rangle\right) && (134)\\
&= \frac{1}{d_\mu(d_\mu+1)}\left(\operatorname{tr}(P_\mu\Pi_e)^2 + \operatorname{tr}(P_\mu\Pi_e P_\mu \Pi_e)\right) && (135)\\
&\leq \frac{1}{d_\mu(d_\mu+1)}\sum_e\left(\underbrace{\operatorname{tr}(\Pi_e)}_{\leq D_E}\operatorname{tr}(P_\mu\Pi_e) + \operatorname{tr}(P_\mu\Pi_e P_\mu)\right) && (136)\\
&\leq \frac{2D_E}{d_\mu(d_\mu+1)}\sum_e \operatorname{tr}(P_\mu\Pi_e) && (137)\\
&= \frac{2D_E}{d_\mu+1}, && (138)
\end{aligned}$$

and (133) immediately follows.

The second estimate in Proposition 1 is sharper than (133) if and only if $\mathrm{tr}(B^\dagger B)/d_\mu < \|B\|^2$, i.e., roughly speaking, if only few (compared to d_μ) eigenvalues of $B^\dagger B$ are close to the largest eigenvalue and most are much smaller. This becomes relevant, for example, when estimating the transitions from $\mathcal{H}_\mu$ into a lower entropy macro space $\mathcal{H}_\nu$, cf. (45). Then $B = P_\nu$ and

$$\mathrm{tr}(B^\dagger B)/d_\mu = d_\nu/d_\mu \ll 1 = \|P_\nu\|^2 \,.$$

9 Conclusions

Our results concern the behavior of typical pure states ψ_0 from a high-dimensional subspace $\mathcal{H}_\mu$ of Hilbert space under the unitary time evolution. We find that for any operator B, due to the large dimension, the curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ is nearly deterministic (a fact that can also be obtained from [1, 23]), and that in the long run $t \to \infty$ it is nearly constant. In von Neumann's framework of an orthogonal decomposition $\mathcal{H} = \oplus_\nu \mathcal{H}_\nu$ into macro spaces, this means that the time-dependent distribution over the macro states given by the superposition weights $\|P_\nu\psi_t\|^2$ is nearly deterministic and in the long run nearly constant, i.e., it reaches *normal equilibrium*, a situation analogous (but not identical) to thermal equilibrium. Through our theorems, we have provided explicit error bounds.

Von Neumann's [17] prior result in the same direction was based on unrealistic assumptions, saying essentially that H is unrelated to $\mathcal{H}_\nu$. Our result has the advantage of being applicable regardless of relations between H and $\mathcal{H}_\nu$. The question of whether the deviation from the mean is small compared to the mean even when the mean is small itself, will be analyzed further elsewhere [29].

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Data Availability Statement

The Matlab code used to generate the datasets of the provided examples is available from the corresponding author on request.

Conflict of Interest Statement

The authors have no conflicts of interest.

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A.2. Canonical Typicality for Other Ensembles than Micro-canonical

Canonical Typicality for Other Ensembles than Micro-canonical

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Abstract

We generalize Lévy's lemma, a concentration-of-measure result for the uniform probability distribution on high-dimensional spheres, to a much more general class of measures, so-called GAP measures. For any given density matrix ρ on a separable Hilbert space $\mathcal{H}$, GAP(ρ) is the most spread out probability measure on the unit sphere of $\mathcal{H}$ that has density matrix ρ and thus forms the natural generalization of the uniform distribution. We prove concentration-of-measure whenever the largest eigenvalue $\|\rho\|$ of ρ is small. We use this fact to generalize and improve well-known and important typicality results of quantum statistical mechanics to GAP measures, namely canonical typicality and dynamical typicality. Canonical typicality is the statement that for "most" pure states ψ of a given ensemble, the reduced density matrix of a sufficiently small subsystem is very close to a ψ-independent matrix. Dynamical typicality is the statement that for any observable and any unitary time-evolution, for "most" pure states ψ from a given ensemble the (coarse-grained) Born distribution of that observable in the time-evolved state ψ_t is very close to a ψ-independent distribution. So far, canonical typicality and dynamical typicality were known for the uniform distribution on finite-dimensional spheres, corresponding to the micro-canonical ensemble, and for rather special mean-value ensembles. Our result shows that these typicality results hold also for GAP(ρ), provided the density matrix ρ has small eigenvalues. Since certain GAP measures are quantum analogs of the canonical ensemble of classical mechanics, our results can also be regarded as a version of equivalence of ensembles.

Key words: Lévy's lemma; equivalence of ensembles; thermalization; quantum statistical mechanics; concentration of measure; Gaussian adjusted projected (GAP) measure; Scrooge measure; dynamical typicality; random wave function.

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1 Introduction

In the 21st century, a modern perspective on quantum statistical mechanics is to consider an individual closed system in a pure state and investigate its and its subsystems' thermodynamic behavior; see, e.g., [7, 8, 31, 34, 2, 15, 16, 20, 41, 42, 11, 12, 36, 9, 1, 38, 37, 39, 48, 45] after pioneering work in [51, 40, 4, 44, 47].

Roughly speaking, "canonical typicality" is the statement that the reduced density matrix of a subsystem obtained from a pure state of the total system is nearly deterministic if the pure state is randomly drawn from a sufficiently large subspace and the subsystem is not too large. More precisely, the original statement of canonical typicality [26, 7, 31, 18] asserts that for most pure states ψ from a high-dimensional (e.g., micro-canonical) subspace $\mathcal{H}_R$ of the Hilbert space $\mathcal{H}_S$ of a macroscopic quantum system S and for a subsystem a of $S = a \cup b$ so that $\mathcal{H}_S = \mathcal{H}_a \otimes \mathcal{H}_b$, the reduced density matrix

$$\rho_a^\psi := \operatorname{tr}_b |\psi\rangle\langle\psi| \tag{1}$$

is close to the partial trace of $\rho_R := P_R/d_R$ (the normalized projection to $\mathcal{H}_R$) and thus deterministic, provided that $d_R := \dim \mathcal{H}_R$ is sufficiently large:

$$\rho_a^\psi \approx \operatorname{tr}_b \rho_R\,. \tag{2}$$

Here, the words "most ψ" refer to the *uniform distribution* u_R (normalized surface area measure) over the unit sphere

$$\mathbb{S}(\mathcal{H}_R) := \{\psi \in \mathcal{H}_R : \|\psi\| = 1\} \tag{3}$$

in $\mathcal{H}_R$. The name "canonical typicality" comes from the fact that if $\mathcal{H}_R = \mathcal{H}_{\mathrm{mc}}$ is a micro-canonical subspace and thus $\rho_R = \rho_{\mathrm{mc}}$ a micro-canonical density matrix, then $\operatorname{tr}_b \rho_{\mathrm{mc}}$ is close to the *canonical density matrix*

$$\rho_{a,\mathrm{can}} = \frac{1}{Z_a} e^{-\beta H_a} \tag{4}$$

for a with suitable β, provided b is large and the interaction between a and b is weak; see, e.g., [18] for a summary of the standard derivation of this fact.

In this paper, we replace the uniform distribution by other, much more general distributions, so-called GAP measures, and show that for them a *generalized canonical typicality* remains valid. For any density matrix ρ replacing ρ_R in $\mathcal{H}_S$, GAP(ρ) is the most spread-out distribution over $\mathbb{S}(\mathcal{H}_S)$ with density matrix ρ; the acronym stands for *Gaussian adjusted projected* measure [23, 19]. For $\rho = \rho_{\mathrm{can}}$, it arises as the distribution of wave functions in thermal equilibrium [19, 17]. If a system is initially in thermal equilibrium for the Hamiltonian H_0 but then driven out of equilibrium by means of a time-dependent H_t, its wave function will still be GAP(ρ)-distributed for suitable ρ. For general ρ, we think of GAP(ρ) as the natural ensemble of wave functions with density matrix ρ; for a more detailed description, see Section 2.2.

We prove quantitative bounds asserting that for any ρ with small eigenvalues (so ρ is far from pure) and GAP(ρ)-most $\psi \in \mathbb{S}(\mathcal{H}_S)$,

$$\rho_a^\psi \approx \operatorname{tr}_b \rho \,. \tag{5}$$

Some reasons for seeking this generalization are: first, that it is mathematically natural; second, that in situations in which we can ask what the actual distribution of ψ is (more detail later), this distribution might not be uniform; third, that it shows that the sharp cut-off of energies involved in the definition of $\mathcal{H}_{\mathrm{mc}}$ actually plays no role; and finally, that it informs and extends our picture of the equivalence of ensembles. A more detailed discussion of these reasons is given in Section 2.1.

As a direct consequence of generalized canonical typicality let us mention that, just as canonical typicality implies that for most pure states $\psi \in \mathbb{S}(\mathcal{H}_S)$ the entanglement entropy $-\operatorname{tr}(\rho_a^\psi \log \rho_a^\psi)$ has nearly the maximal value $\log d_a$ with $d_a = \dim \mathcal{H}_a$ [22] (because $\rho_a^\psi \approx \operatorname{tr}_b I_S/D = I_a/d_a$ with I the identity operator and $D = d_a d_b = \dim \mathcal{H}_S$), generalized canonical typicality implies that GAP(ρ)-typical ψ have entanglement entropy $-\operatorname{tr}(\rho_a^\psi \log \rho_a^\psi) \approx -\operatorname{tr}(\rho_a \log \rho_a)$ with $\rho_a = \operatorname{tr}_b \rho$.

Since different probability distributions over the unit sphere in a Hilbert space $\mathcal{H}$ can have the same density matrix, and since the outcome statistics of any experiment depend only on the density matrix, it may seem at first irrelevant to even consider distributions over $\mathbb{S}(\mathcal{H})$. However, for example, an ensemble of spins prepared so that (about) half are in state $|\uparrow\rangle$ and the others in $|\downarrow\rangle$ is physically different from a uniform ensemble over $\mathbb{S}(\mathbb{C}^2)$, even though both ensembles have density matrix $\frac{1}{2}I$. Likewise, for an ensemble of particles prepared by taking them from a system in thermal equilibrium, the wave function is GAP-distributed (see Section 2.2). More basically, probability distributions play a key role in any *typicality statement*, i.e., one saying that some condition is satisfied by most wave functions—"most" relative to a certain distribution; such a statement cannot be formulated in terms of density matrices.

We note that the generalization of canonical typicality from uniform measures to GAP measures is not straightforward. First, not *every* measure μ over $\mathbb{S}(\mathcal{H}_S)$ with a given density matrix ρ with small eigenvalues makes it true that for μ-most ψ, $\rho_a^\psi \approx \operatorname{tr}_b \rho$. We give a counter-example in Remark 15 in Section 3. Second, if ρ is not close to a multiple of a projection, then GAP(ρ) is far from uniform; specifically, its density will at some points be larger than at others by a factor like $\exp(D)$ (see Remark 13). And third, even measures close to uniform (for example the von Mises-Fisher distribution, see again Remark 13), can fail to satisfy generalized canonical typicality.

In this paper, we prove generalized canonical typicality in rigorous form by providing error bounds for (5) at any desired confidence level that is implicit in the word "most," see Theorem 1 and Theorem 3. Compared to the known error bounds based on u_R, we can prove more or less the same bounds with d_R replaced by the reciprocal of the largest eigenvalue of ρ,

$$\frac{1}{p_{\max}} := \frac{1}{\|\rho\|} \tag{6}$$

with $\|\cdot\|$ the operator norm. Thus, the approximation is good as soon as no single direction contributes too much to ρ. In particular, for $\rho = \rho_R$, our results essentially reproduce the known error bounds. As one central part of our proof, we also establish a variant of Lévy's lemma [25, 27, 24] (a statement about the concentration of measure on a high-dimensional sphere, see below) for GAP measures instead of the uniform measure (Theorem 2). In particular, our version of Lévy's lemma holds also on infinite dimensional spheres, where the uniform measure does not exist.

Furthermore, we provide several corollaries. The first one shows that for any observable and GAP(ρ)-most ψ, the coarse-grained Born distribution is near a ψ-independent one (see Remark 4 in Section 3.1 for discussion). The second arises from evolving the observable with time and provides a form of *dynamical typicality* [2], which means that for typical initial wave functions, the time evolution "looks" the same; here, "typical" refers to the GAP(ρ) distribution, and "look" (which in [48] meant the macroscopic appearance) refers to the Born distribution for the observable considered. In fact, Corollary 2 even shows that the relevant kind of closeness (to a t-dependent but ψ-independent distribution) holds jointly for most $t \in [0, T]$. As a further variant (Corollary 3), dynamical typicality also holds when "look" refers to ρ_a^ψ. Put differently, the statement here is that for GAP(ρ)-most ψ and most $t \in [0, T]$,

$$\rho_a^{\psi_t} \approx \operatorname{tr}_b \rho_t\,, \tag{7}$$

where $\psi_t = U_t\,\psi$ and $\rho_t = U_t\,\rho\,U_t^*$ for an arbitrary unitary time evolution U_t (allowing for time-dependent H_t). In the original version of canonical typicality, one particularly considers for ρ_R the micro-canonical density matrix $\rho_{\rm mc}$ for a fixed Hamiltonian H, for which the time evolution yields nothing interesting because it is invariant anyway; but if we consider arbitrary ρ, then ρ can evolve in a non-trivial way even for fixed H.

Another corollary (Corollary 4) concerns the conditional wave function ψ_a of a (which is the natural notion of the subsystem wave function for a, see Section 2.2 for the definition): It is known that if d_R is large, then for u_R-most ψ and most bases of $\mathcal{H}_b$, the Born distribution of ψ_a is approximately GAP($\operatorname{tr}_b \rho_R$). We generalize this statement as follows: if d_b is large and ρ has small eigenvalues, then for GAP(ρ)-most ψ and most bases of $\mathcal{H}_b$, the Born distribution of ψ_a is approximately GAP($\operatorname{tr}_b \rho$).

The results of this paper can also be regarded as a variant of *equivalence-of-ensembles* in quantum statistical mechanics, i.e., as a new instance of the well-known phenomenon in statistical mechanics that it does not make a big difference whether we use the micro-canonical ensemble or the canonical one (for suitable β) or another equilibrium ensemble. Indeed, the uniform distribution over the unit sphere in a micro-canonical subspace can be regarded as a quantum analog of the micro-canonical distribution in classical statistical mechanics, and the GAP measure associated with a canonical density matrix as a quantum analog of the canonical distribution; see also Remark 11 in Section 3.2.

Our results on generalized canonical typicality (5) provide two kinds of error bounds based on two strategies of proof. They are roughly analogous to the following two bounds on the probability that a random variable X deviates from its expectation $\mathbb{E}X$ by more than n standard deviations $\sqrt{\mathrm{Var}(X)}$: First, the Chebyshev inequality yields the bound $1/n^2$, which is valid for *any* distribution of X. Second, the *Gaussian* distribution has very light tails, so if X is Gaussian distributed, then the aforementioned probability is actually smaller than e^{-n} (a type of bound known as a Chernoff bound), so the Chebyshev bound would be very coarse. Likewise, the two kinds of bound we provide are based, respectively, on the Chebyshev inequality and the Chernoff bound (in the form of Lévy's lemma). The former is polynomial in $p_{\max}$, the latter exponential as in $e^{-1/p_{\max}}$. For the original statement of canonical typicality (using u_R), the Chebyshev-type bounds were first given by Sugita [46], the Chernoff-type bounds by Popescu et al. [30]. Our proof of the Chebyshev-type bounds makes heavy use of results of Reimann [35].

A version of Lévy's lemma was also established for the mean-value ensemble on a finite-dimensional Hilbert space $\mathcal{H}$ [28]. This is the uniform distribution on $\mathbb{S}(\mathcal{H})$ restricted to the set $\{\psi \in \mathbb{S}(\mathcal{H}) : \langle\psi|A|\psi\rangle = a\}$ for a given observable A and a value a satisfying further conditions. However, as also the authors of [28] point out, the physical relevance of this ensemble remains unclear. Also dynamical typicality has been established for the mean-value ensemble, see [39] for an overview.

The remainder of this paper is organized as follows: In Section 2, we elucidate the motivation and background. In Section 3, we formulate and discuss our results. In Section 4, we provide the proofs. In Section 5, we conclude.

2 Motivation and Background

2.1 Motivation

Canonical typicality is often (rightly) used as a justification and derivation of the canonical density matrix ρ_{can} from something simpler, viz., from the uniform distribution over the unit sphere in an appropriate subspace $\mathcal{H}_{\mathrm{mc}}$. So it may appear surprising that here we consider other distributions instead of the uniform one. That is why we give some elucidation in this section.

The uniform distribution for ψ can appear in either of two roles: as a measure of probability or a measure of typicality. What is the difference? The concept of probability, in the narrower sense used here, refers to a physical situation that occurs many times or can be made to occur many times, so that one can meaningfully speak of the empirical distribution of part of the physical state, such as ψ, over the ensemble of trials. In contrast, the concept of typicality, in the sense used here, refers to a hypothetical ensemble and applies also in situations that do not occur repeatedly, such as the universe as a whole, or occur at most a few times; it defines what a typical solution of an equation or theory looks like, or the meaning of "most." Typicality is used in defining what counts as thermal equilibrium (e.g., [10] and references therein), but also in certain laws of nature such as the past hypothesis (a proposed law about the initial micro-state of the universe serving as the basis of the arrow of time; see [21, Sec. 5.7] for a formulation in terms of typicality). Moreover, it plays a key role for the *explanation* of certain phenomena by showing that they occur in "most" cases.

The mathematical statements apply regardless of whether we think of the measure as probability or typicality. If we use u_{mc} as probability, then the question naturally arises whether the actual distribution of ψ is uniform, and generalizations to other measures are called for. The GAP measures are then particularly relevant, not just as a natural choice of measures, but also because they arise as the thermal equilibrium distribution of wave functions.

But also for u_{mc} as a measure of typicality, which is perhaps the more important or more widely used case, the generalization is relevant. The way we practically think of canonical typicality is that if ψ is just "any old" wave function of S, then ρ_a^ψ will be approximately canonical. But the theorem of original canonical typicality (using u_{mc}) would require that the coefficients of ψ relative to energy levels of S outside of the micro-canonical energy interval $[E-\Delta E, E]$ are exactly zero, which of course goes against the idea of ψ being "any old" ψ. Of course, we would expect that the canonicality of ρ_a^ψ does not depend much on whether other coefficients are exactly zero or not. And the theorems in this paper show that this is correct! They show that if the ρ we start from is not ρ_{mc}, then the crucial part of the reasoning (the typical-ψ part) still goes through, just with corrections reflected in the deviation of $\mathrm{tr}_b\,\rho$ from $\mathrm{tr}_b\,\rho_{\mathrm{mc}}$ (which, by the way, will be minor for $\rho = \rho_{\mathrm{can}}$ with appropriate inverse temperature β). More generally, the theorems in this paper prove the robustness of canonical typicality towards changes in the underlying measure.

The results of this paper also show that when computing the typical reduced state ρ_a^ψ for "any old" ψ, we can start from various choices of ρ of the whole, as long as they yield approximately the same $\mathrm{tr}_b\,\rho$. The results thus provide researchers with a new angle of looking at canonical typicality: it is OK to imagine "any old" ψ, and not crucial to start from u_{mc}.

More generally, our results are a kind of equivalence-of-ensembles statement in the quantum case, and thus add to the picture consisting of various senses in which different thermal equilibrium ensembles are practically equivalent, in this case with "ensemble" meaning ensemble of wave functions (i.e., measures over the unit sphere). Again, it plays a role that the GAP measures arise as the thermal equilibrium distribution of wave functions, and thus as an analog of the canonical ensemble in classical statistical mechanics. This means also that if ψ is itself a conditional wave function, a case in which we know [19, 17] that (for high dimension and most orthonormal bases) ψ is approximately GAP distributed, then canonical typicality applies. A special application concerns the thermodynamic limit, for which it is desirable to think of the conditional wave function ψ_A of a region A in 3-space as obtained from $\psi_{A'}$ for a larger $A' \supset A$, which in turn is obtained from $\psi_{A''}$ for an even larger $A'' \supset A'$, and so on. Then for each step, $\psi_{A'}$ (etc.) is GAP-distributed.

By the way, the results here also have the converse implication of supporting the naturalness of the GAP measures. One might even consider a version of the past hypothesis that uses, as the measure of typicality, a GAP measure instead of the uniform distribution over the unit sphere in some subspace of the Hilbert space of the universe.

2.2 Mathematical Setup and Some Background

One often considers the uniform distribution over the unit sphere in a subspace $\mathcal{H}'$ of a system's Hilbert space $\mathcal{H}$. While this distribution is associated with a density matrix given by the normalized projection to $\mathcal{H}'$, the measure GAP(ρ) forms an analog of it for an *arbitrary* density matrix. We now give its definition and that of some other mathematical concepts we use.

Throughout this paper, all Hilbert spaces $\mathcal{H}$ are assumed to be separable, i.e., to have either a finite or a countably infinite orthonormal basis (ONB). The unit sphere $\mathbb{S}(\mathcal{H})$ is always equipped with the Borel σ-algebra.

Density matrix. To any probability measure μ on $\mathbb{S}(\mathcal{H})$ we can associate a density matrix ρ_μ by

$$\rho_\mu := \int_{\mathbb{S}(\mathcal{H})} \mu(d\psi)|\psi\rangle\langle\psi| \tag{8}$$

(which always exists [49, Lemma 1]). Note that if μ has mean zero then ρ_μ is the covariance matrix of μ. It will turn out for $\mu = \mathrm{GAP}(\rho)$ that $\rho_\mu = \rho$.

GAP measure. The measure GAP(ρ) was first introduced for finite-dimensional $\mathcal{H}$ by Jozsa, Robb, and Wootters [23], who named it *Scrooge measure.*[1] Among several equivalent definitions [17], we use the following one based on Gaussian measures. Let $\mathcal{H}$ be separable and ρ a density matrix on $\mathcal{H}$ with eigenvalues p_n and eigen-ONB

[1]Named after Ebenezer Scrooge, a fictional character in and the protagonist of Charles Dickens' novella *A Christmas Carol* (1843) who is known for being very stingy. As Jozsa et al. argue, the gap measure is in some sense the most spread-out distribution on $\mathbb{S}(\mathcal{H})$ with density matrix ρ and they choose the name “Scrooge measure” because the measure is “particularly stingy with its information.”

$(|n\rangle)_{n=1\dots\dim\mathcal{H}}$, i.e.,

$$\rho = \sum_n p_n |n\rangle\langle n|. \tag{9}$$

A complex-valued random variable Z will be said to be Gaussian with mean $z \in \mathbb{C}$ and variance $\sigma^2 > 0$ if and only if $\operatorname{Re} Z$ and $\operatorname{Im} Z$ are independent real Gaussian random variables with mean $\operatorname{Re} z$ respectively $\operatorname{Im} z$ and each with variance $\sigma^2/2$. Let $(Z_n)_{n=1\dots\dim\mathcal{H}}$ be a sequence of independent $\mathbb{C}$-valued Gaussian random variables with mean 0 and variances

$$\mathbb{E}|Z_n|^2 = p_n. \tag{10}$$

Then we define $\mathrm{G}(\rho)$ to be the distribution of the random vector

$$\Psi^{\mathrm{G}} := \sum_n Z_n |n\rangle, \tag{11}$$

i.e., the Gaussian measure on $\mathcal{H}$ with mean 0 and covariance operator ρ. (It is known [32] in general that for every $\phi \in \mathcal{H}$ and every positive trace-class operator ρ there exists a unique Gaussian measure on $\mathcal{H}$ with mean ϕ and covariance operator ρ.) Note that

$$\mathbb{E}\|\Psi^{\mathrm{G}}\|^2 = \sum_n \mathbb{E}|Z_n|^2 = \sum_n p_n = 1, \tag{12}$$

which also shows that $\|\Psi^{\mathrm{G}}\| < \infty$ almost surely, but in general $\|\Psi^{\mathrm{G}}\| \neq 1$, i.e., $\mathrm{G}(\rho)$ is not a distribution on the sphere $\mathbb{S}(\mathcal{H})$. Projecting the measure $\mathrm{G}(\rho)$ to the sphere $\mathbb{S}(\mathcal{H})$ would not result in a measure with density matrix ρ; therefore we first adjust the density of $\mathrm{G}(\rho)$ and define the *adjusted Gaussian measure* $\mathrm{GA}(\rho)$ on $\mathcal{H}$ as the measure that has density $\|\psi\|^2$ relative to $\mathrm{G}(\rho)$, i.e.,

$$\mathrm{GA}(\rho)(d\psi) := \|\psi\|^2\, \mathrm{G}(\rho)(d\psi), \tag{13}$$

which is a probability measure by virtue of (12). It will turn out below that $\|\psi\|^2$ is the right factor to ensure that $\rho_{\mathrm{GAP}(\rho)} = \rho$.

Let Ψ^{GA} be a $\mathrm{GA}(\rho)$-distributed random vector. We define $\mathrm{GAP}(\rho)$ to be the distribution of

$$\Psi^{\mathrm{GAP}} := \frac{\Psi^{\mathrm{GA}}}{\|\Psi^{\mathrm{GA}}\|}. \tag{14}$$

Note that the denominator is almost surely non-zero (because every 1-element subset of $\mathcal{H}$ has $\mathrm{G}(\rho)$-measure 0 because every Z_n has continuous distribution). With this we find that indeed

$$\rho_{\mathrm{GAP}(\rho)} = \int_{\mathbb{S}(\mathcal{H})} \mathrm{GAP}(\rho)(d\psi)\, |\psi\rangle\langle\psi| \tag{15a}$$

$$= \int_{\mathcal{H}} \mathrm{GA}(\rho)(d\psi) \frac{1}{\|\psi\|^2} |\psi\rangle\langle\psi| \tag{15b}$$

$$= \int_{\mathcal{H}} \mathrm{G}(\rho)(d\psi)\ |\psi\rangle\langle\psi| = \rho. \tag{15c}$$

See [49] for a complete proof of existence and uniqueness of $\mathrm{GAP}(\rho)$ for every density matrix ρ.

$\mathrm{GAP}(\rho)$ can also be characterized as the minimizer of the "accessible information" functional under the constraint that its density matrix is ρ [23]. If all eigenvalues of ρ are positive and $D := \dim \mathcal{H} < \infty$, then $\mathrm{GAP}(\rho)$ possesses a density relative to the uniform distribution u on $\mathbb{S}(\mathcal{H})$ [19, 17],

$$\mathrm{GAP}(\rho)(d\psi) = \frac{D}{\det \rho} \langle\psi|\rho^{-1}|\psi\rangle^{-D-1}\, u(d\psi)\,. \tag{16}$$

It was argued in [19] and mathematically justified in [17] that GAP measures describe the thermal equilibrium distribution of the (conditional) wave function of the system if ρ is a canonical density matrix.

It was also shown in [19] that GAP is equivariant under unitary transformations, i.e., for all density matrices ρ, all unitary operators U on $\mathcal{H}$, and all measurable sets $M \subset \mathbb{S}(\mathcal{H})$ one has

$$\mathrm{GAP}(U\rho U^*)(M) = \mathrm{GAP}(\rho)(UM)\,. \tag{17}$$

In particular, GAP is equivariant under unitary time evolution, and, as a consequence, $\mathrm{GAP}(\rho_t)$ is the relevant distribution on $\mathbb{S}(\mathcal{H})$ whenever the system starts in thermal equilibrium with respect to some Hamiltonian H_0 and evolves according to any Hamiltonian H_t at later times. More generally, the results of [17] (and their extension in Corollary 4) show that if a system has density matrix ρ arising from entanglement, then its (conditional) wave function (relative to a typical basis, see below) is asymptotically GAP-distributed. Thus, GAP is the correct distribution in many practically relevant cases. On top of that, when we have no further restriction than that the density matrix is ρ, then the natural concept of a "typical ψ" should refer to the most spread-out distribution compatible with ρ, which is $\mathrm{GAP}(\rho)$.

Finally, let us remark that $\mathrm{GAP}(\rho)$ is also invariant under global phase changes, i.e., $\mathrm{GAP}(\rho)(M) = \mathrm{GAP}(\rho)(e^{i\varphi}M)$ for all measurable $M \subset \mathbb{S}(\mathcal{H})$ and $\varphi \in \mathbb{R}$. Hence, $\mathrm{GAP}(\rho)$ naturally also defines a probability distribution on the projective space of complex rays in $\mathcal{H}$ and all results presented in the following can be equivalently formulated for rays instead of vectors.

Remark 1. In terms of ρ_μ, we can easily formulate and prove a weaker version of our main result (5); this version is related to (5) in more or less the same way as the statement that in a certain population, the *average* height is 170 cm, is related to the stronger statement that in that population, *most* people are 170 cm tall. The weaker version asserts that the *average* of ρ_a^ψ over ψ using the $\mathrm{GAP}(\rho)$ distribution is equal to $\mathrm{tr}_b\,\rho$, whereas the statement about (5) was that *most* ψ relative to GAP(ρ) have ρ_a^ψ (approximately) equal to $\mathrm{tr}_b\,\rho$. On the other hand, the statement about the average is stronger because it asserts, not *approximate* equality, but *exact* equality. On top of that, the average statement is not limited to the GAP measure but holds for *any* probability measure μ. Here is the full statement: *for separable $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, any probability measure μ on $\mathbb{S}(\mathcal{H})$, and a random vector ψ with distribution μ,*

$$\mathbb{E}_\mu \rho_a^\psi = \mathrm{tr}_b\,\rho_\mu\,. \tag{18}$$

Indeed, tr_b commutes with μ-integration,[2] so

$$\mathbb{E}_\mu \rho_a^\psi = \int_{\mathbb{S}(\mathcal{H})} \mu(d\psi)\ \mathrm{tr}_b\,|\psi\rangle\langle\psi| \tag{19a}$$

$$= \mathrm{tr}_b \int_{\mathbb{S}(\mathcal{H})} \mu(d\psi)\,|\psi\rangle\langle\psi| \tag{19b}$$

$$= \mathrm{tr}_b\,\rho_\mu\,. \tag{19c}$$

◇

Norms. The distance between two density matrices will be measured in the *trace norm*

$$\|M\|_{\mathrm{tr}} := \mathrm{tr}\,|M| = \mathrm{tr}\,\sqrt{M^*M}, \tag{20}$$

where M^* denotes the adjoint operator of M. If M can be diagonalized through an orthonormal basis (ONB), then $\|M\|_{\mathrm{tr}}$ is the sum of the absolute eigenvalues. We will also sometimes use the *operator norm*

$$\|M\| := \sup_{\|\psi\|=1} \|M\psi\|\,, \tag{21}$$

which, if M can be diagonalized through an ONB, is the largest absolute eigenvalue.

Purity. For a density matrix ρ, its *purity* is defined as $\mathrm{tr}\,\rho^2$. In terms of the spectral decomposition $\rho = \sum_n p_n |n\rangle\langle n|$, the purity is $\mathrm{tr}\,\rho^2 = \sum_n p_n^2$, which can be thought of as the average size of p_n. In particular, the purity is positive and ≤ 1; it is $= 1$ if and only if ρ is pure, i.e., a 1d projection; for a normalized projection $\rho_R = P_R/d_R$, the purity is $1/d_R$; conversely, 1/purity can be thought of as the effective number of dimensions over which ρ is spread out. It also easily follows that

$$\mathrm{tr}\,\rho^2 \leq \|\rho\| \leq \sqrt{\mathrm{tr}\,\rho^2} \leq \sqrt{\|\rho\|} \tag{22}$$

because $p_n^2 \leq p_n\|\rho\|$, and if p_{n_0} is the largest eigenvalue, then $p_{n_0}^2 \leq \sum_n p_n^2$ because all other terms are ≥ 0. In words, the average p_n is no greater than the maximal p_n, which is bounded by the square root of the average p_n (and the square root of the maximal p_n).

[2] Since we could not find a good reference for this fact, we have included a proof in Section 4.1.

Conditional wave function. For $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, an ONB $B = (|m\rangle_b)_{m=1\dots\dim\mathcal{H}_b}$ of $\mathcal{H}_b$, and $\psi \in \mathbb{S}(\mathcal{H})$, the conditional wave function ψ_a [5, 6, 19] of system a is a random vector in $\mathcal{H}_a$ that can be constructed by choosing a random one of the basis vectors $|m\rangle_b$, let us call it $|M\rangle_b$, with the Born distribution

$$\mathbb{P}(M = m) = \big\|{}_b\langle m|\psi\rangle\big\|_a^2 \,, \tag{23}$$

taking the partial inner product of $|M\rangle_b$ and ψ, and normalizing:

$$\psi_a := \frac{{}_b\langle M|\psi\rangle}{\|{}_b\langle M|\psi\rangle\|_a} \,. \tag{24}$$

(Note that the event that $\|{}_b\langle M|\psi\rangle\|_a = 0$ has probability 0 by (23). In the context of Bohmian mechanics, the expression "conditional wave function" refers to the position basis and the Bohmian configuration of b [5]; but for our purposes, we can leave it general.)

We can also think of ψ_a as arising from ψ through a quantum measurement with eigenbasis B on system b, which leads to the collapsed quantum state $\psi_a \otimes |M\rangle_b$. Correspondingly, we call the distribution of ψ_a in $\mathbb{S}(\mathcal{H}_a)$ the *Born distribution of* ψ_a and denote it by $\mathrm{Born}_a^{\psi,B}$. However, when considering ψ_a, we will not assume that any observer actually, physically carries out such a quantum measurement; rather, we use ψ_a as a theoretical concept of a wave function associated with the subsystem a. It is related to the reduced density matrix ρ_a^ψ in a way similar to how a conditional probability distribution is to a marginal distribution,

$$\mathbb{E}|\psi_a\rangle\langle\psi_a| = \rho_a^\psi \,. \tag{25}$$

ψ_a is also related to the GAP measure, in fact in two ways. First, when we average $\mathrm{Born}_a^{\psi,B}$ over all ONBs B (using the uniform distribution corresponding to the Haar measure), then we obtain $\mathrm{GAP}(\rho_a^\psi)$ [17, Lemma 1]. Put differently, if we think of both M and B as random and ψ_a thus as doubly random, then its (marginal) distribution is $\mathrm{GAP}(\rho_a^\psi)$; put more briefly, $\mathrm{GAP}(\rho_a^\psi)$ is the distribution of the collapsed pure state in a after a purely random quantum measurement in b on ψ. Second, if d_b is large, then even conditionally on a single given B, the distribution of ψ_a is close to a GAP measure for most B and most ψ according to a GAP measure on $\mathcal{H}_a \otimes \mathcal{H}_b$; this is the content of Corollary 4 below.

3 Main Results

In this section, we present and discuss our main results about generalized canonical typicality. In the following, we use the notation $\mu(f)$ for the average of the function f under the measure μ,

$$\mu(f) := \int \mu(d\psi)\, f(\psi) \,. \tag{26}$$

Note that, by (18),

$$\mathrm{GAP}(\rho)(\rho_a^\psi) = \mathrm{tr}_b\, \rho\,. \tag{27}$$

The statement of our generalized canonical typicality differs in that it concerns *approximate* equality and holds for the *individual* ρ_a^ψ, not only for its average.

3.1 Statements

We first formulate our main theorem on canonical typicality for GAP measures and the underlying variant of Lévy's lemma for GAP measures. We then give a list of further consequences of this generalized version of Lévy's lemma, including results on dynamical typicality and the fact that the typical Born distribution of conditional wave functions is itself a GAP measure. At the end of this section we also state a slightly weaker version of our main theorem that is not based on Lévy's lemma but instead allows for a rather elementary proof based on the Chebyshev inequality. Finally, the known bounds for uniformly distributed ψ will be stated in Remark 12 in Section 3.2 for comparison.

Theorem 1 (Generalized canonical typicality, exponential bounds)**.** *Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be Hilbert spaces with $\mathcal{H}_a$ having finite dimension d_a and $\mathcal{H}_b$ being separable, and let ρ be a density matrix on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. Then for every $\delta > 0$,*

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \left\|\rho_a^\psi - \mathrm{tr}_b\, \rho\right\|_{\mathrm{tr}} \leq c d_a \sqrt{\ln\left(\frac{12 d_a^2}{\delta}\right)\|\rho\|}\right\} \geq 1 - \delta, \tag{28}$$

where $c = 48\pi$.

Remark 2. The relation (28) can equivalently be formulated as a bound on the confidence level, given the allowed deviation: *For every $\varepsilon \geq 0$,*

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \left\|\rho_a^\psi - \mathrm{tr}_b\, \rho\right\|_{\mathrm{tr}} > \varepsilon\right\} \leq 12 d_a^2 \exp\left(-\frac{\tilde{C}\varepsilon^2}{d_a^2\|\rho\|}\right), \tag{29}$$

where $\tilde{C} = \frac{1}{2304\pi^2}$. This form makes it visible why we call Theorem 1 an "exponential bound": because the bound on the probability of too large a deviation is exponentially small in $1/\|\rho\|$. In contrast, the bound (37) is polynomially small in $\mathrm{tr}\,\rho^2$. ◇

A key tool for proving Theorem 1 is Theorem 2 below, a variant of Lévy's lemma for GAP measures. Recall the notation (26).

Theorem 2 (Lévy's lemma for GAP measures)**.** *Let $\mathcal{H}$ be a separable Hilbert space, let $f : \mathbb{S}(\mathcal{H}) \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant*[3] *η, let ρ be a density matrix on $\mathcal{H}$, and let $\varepsilon \geq 0$. Then*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \big|f(\psi) - \mathrm{GAP}(\rho)(f)\big| > \varepsilon\Big\} \leq 6 \exp\left(-\frac{C\varepsilon^2}{\eta^2\|\rho\|}\right), \tag{30}$$

where $C = \frac{1}{288\pi^2}$.

Remark 3. The statement remains true for *complex*-valued f if we replace the constant factor 6 in (30) by 12 and C by $C/2$, as follows from considering the real and imaginary parts of f separately. ◇

As an immediate consequence of Theorem 2 for $f(\psi) = \langle\psi|B|\psi\rangle$, which has Lipschitz constant $\eta \leq 2\|B\|$ [30, Lemma 5], we obtain:

Corollary 1. *Let ρ be a density matrix and B a bounded operator on the separable Hilbert space $\mathcal{H}$. For every $\varepsilon \geq 0$,*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \big|\langle\psi|B|\psi\rangle - \mathrm{tr}(\rho B)\big| > \varepsilon\Big\} \leq 12 \exp\left(-\frac{\tilde{C}\varepsilon^2}{\|B\|^2\|\rho\|}\right) \tag{31}$$

with $\tilde{C} = \frac{1}{2304\pi^2}$.

Remark 4. Corollary 1 provides an extension to GAP measures of the known fact [33] that $\langle\psi|B|\psi\rangle$ has nearly the same value for most ψ relative to the uniform distribution. This kind of near-constancy is different from the near-constancy property of a macroscopic observable, viz., that most of its eigenvalues (counted with multiplicity) in the micro-canonical energy shell are nearly equal. Here, in contrast, nothing (except boundedness) is assumed about the distribution of eigenvalues of B. In particular, if B is a self-adjoint observable, then a typical ψ may well define a non-trivial probability distribution over the spectrum of B, not necessarily a sharply peaked one. The near-constancy property asserted here is that the *average* of this probability distribution is the same for most ψ. In fact, it also follows that the *probability distribution itself* is the same for most ψ ("distribution typicality"), at least on a coarse-grained level (by covering the spectrum of B with not-too-many intervals) and provided that many dimensions participate in ρ. This follows from inserting spectral projections of the observable for B in (31). ◇

[3] A Lipschitz constant refers to a metric on the domain, and two metrics are often considered on the sphere: the spherical metric (distance along the sphere, $d_{\mathrm{sph}}(\psi, \phi) = \arccos \mathrm{Re}\langle\psi|\phi\rangle$) and the Euclidean metric (distance in the ambient space across the interior of the sphere, $d_{\mathrm{Eucl}}(\psi, \phi) = \|\psi - \phi\|$). We use the spherical metric, as did [27, 30, 31], but since $d_{\mathrm{Eucl}}(\psi, \phi) \leq d_{\mathrm{sph}}(\psi, \phi) \leq \frac{\pi}{2}\, d_{\mathrm{Eucl}}(\psi, \phi)$, using the Euclidean metric would at most change the Lipschitz constants by a factor of $\frac{\pi}{2}$.

In contrast to the uniform distribution on the sphere in the micro-canonical subspace, which is invariant under the unitary time evolution, $\mathrm{GAP}(\rho_0)$ will in general evolve, in fact to $\mathrm{GAP}(\rho_t)$ by (17). This leads to questions about what the history $t \mapsto \psi_t$ looks like. Inserting $U_t^* B U_t$ for B in (31) leads us to the first equation in the following variant of "dynamical typicality" for GAP measures.

Corollary 2. *Let $\mathcal{H}$ be a separable Hilbert space, B a bounded operator and ρ a density matrix on $\mathcal{H}$, and $t \mapsto U_t$ a measurable family of unitary operators. Then for every $\varepsilon, t \geq 0$,*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : |\langle\psi_t|B|\psi_t\rangle - \mathrm{tr}(\rho_t B)| > \varepsilon\Big\} \leq 12 \exp\left(-\frac{\tilde{C}\varepsilon^2}{\|B\|^2\|\rho\|}\right), \qquad (32)$$

where $\rho_t = U_t\,\rho\,U_t^*$, $\psi_t = U_t\psi$ and $\tilde{C} = \frac{1}{2304\pi^2}$. Moreover, for every $\varepsilon, T > 0$,

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \frac{1}{T}\int_0^T |\langle\psi_t|B|\psi_t\rangle - \mathrm{tr}(\rho_t B)|\; dt > \varepsilon\Big\} \leq 9 \exp\left(-\frac{\tilde{C}\varepsilon^2}{36\|B\|^2\|\rho\|}\right). \qquad (33)$$

Clearly, for U_t we have in mind either a unitary group $U_t = \exp(-iHt)$ generated by a time-independent Hamiltonian H, or a unitary evolution family U_t satisfying $i\frac{d}{dt}U_t = H_t U_t$ and $U_0 = I$ generated by a time-dependent Hamiltonian H_t. However, the group resp. co-cycle structure play no role in the proof. (In [48], a similar result for the uniform distribution over the sphere in a large subspace was formulated only for time-*independent* Hamiltonians, but the proof given there actually applies equally to time-*dependent* ones.)

The last two corollaries were applications of Lévy's lemma that did not involve reduced density matrices. We now turn to bi-partite systems again and present two further corollaries. We first ask whether, for $\mathrm{GAP}(\rho_0)$-typical ψ_0, the reduced density matrix $\rho_a^{\psi_t}$ remains close to $\mathrm{tr}_b\,\rho_t$ over a whole time interval $[0, T]$. The following corollary answers this question affirmatively for most times in this interval.

Corollary 3. *Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be Hilbert spaces with $\mathcal{H}_a$ having finite dimension d_a and $\mathcal{H}_b$ being separable, ρ a density matrix on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, and $t \mapsto U_t$ a measurable family of unitary operators on $\mathcal{H}$. Then for every $\varepsilon, T > 0$,*

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \frac{1}{T}\int_0^T \left\|\rho_a^{\psi_t} - \mathrm{tr}_b\,\rho_t\right\|_{\mathrm{tr}} dt > \varepsilon\right\} \leq 9 d_a^2 \exp\left(-\frac{\tilde{C}\varepsilon^2}{36 d_a^2\|\rho\|}\right), \qquad (34)$$

where $\rho_t = U_t\,\rho\,U_t^*$, $\psi_t = U_t\psi$ and $\tilde{C} = \frac{1}{2304\pi^2}$.

The next corollary expresses that for $\mathrm{GAP}(\rho)$-typical ψ, large d_b, and small $\mathrm{tr}\,\rho^2$, the conditional wave function ψ_a (relative to a typical basis) has Born distribution close to $\mathrm{GAP}(\mathrm{tr}_b\,\rho)$. (Note that we are considering the distribution of ψ_a *conditionally* on a given ψ, rather than the *marginal* distribution of ψ_a for random ψ, which would be $\int_{\mathbb{S}(\mathcal{H})} \mathrm{GAP}(\rho)(d\psi)\,\mathrm{Born}_a^{\psi,B}(\cdot)$.) Recall the notation (26).

Corollary 4. *Let $\varepsilon,\delta \in (0,1)$, let $\mathcal{H}_a$ be a Hilbert space of dimension $d_a \in \mathbb{N}$, let $f : \mathbb{S}(\mathcal{H}_a) \to \mathbb{R}$ be any continuous (test) function, and let $\mathcal{H}_b$ be a Hilbert space of finite dimension $d_b \geq \max\{4, d_a, 32\|f\|_\infty^2/\varepsilon^2\delta\}$. Then there is $p > 0$ such that for every density matrix ρ on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ with $\|\rho\| \leq p$,*

$$\mathrm{GAP}(\rho) \times u_{\mathrm{ONB}}\Big\{(\psi, B) \in \mathbb{S}(\mathcal{H}) \times \mathrm{ONB}(\mathcal{H}_b) : \\ \big|\mathrm{Born}_a^{\psi,B}(f) - \mathrm{GAP}(\mathrm{tr}_b\,\rho)(f)\big| < \varepsilon\Big\} \geq 1-\delta\,, \quad (35)$$

where $\mathrm{Born}_a^{\psi,B}$ is the distribution of the conditional wave function, $\mathrm{ONB}(\mathcal{H}_b)$ is the set of all orthonormal bases on $\mathcal{H}_b$, and u_{ONB} the uniform distribution over this set.

Remark 5. We conjecture that the closeness between $\mathrm{Born}_a^{\psi,B}$ and $\mathrm{GAP}(\mathrm{tr}_b\,\rho)$ is even better than stated in Corollary 4, at least when 0 is not an eigenvalue of $\mathrm{tr}_b\,\rho$, in the sense that (35) holds not only for continuous f but even for bounded measurable f, and in fact uniformly in f with given $\|f\|_\infty$. This conjecture is suggested by using Lemma 6 of [17] instead of Lemma 5, or rather a variant of it with more explicit bounds. ◇

Whereas Theorem 1 is based on the rather technical concentration of measure result Theorem 2, a slightly weaker statement can be obtained using only the Chebychev inequality and a bound on the variance of random variables of the form $\psi \mapsto \langle\psi|A|\psi\rangle$ with respect to $\mathrm{GAP}(\rho)$ given in Proposition 1 in Section 4.2. The latter bound is also of interest in its own right and has already been established for self-adjoint A by Reimann in [35].

Theorem 3 (Generalized canonical typicality, polynomial bounds)**.** *Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be Hilbert spaces with $\mathcal{H}_a$ having finite dimension d_a and $\mathcal{H}_b$ being separable. Let ρ be a density matrix on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ with $\|\rho\| < 1/4$. Then for every $\delta > 0$,*

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \big\|\rho_a^\psi - \mathrm{tr}_b\,\rho\big\|_{\mathrm{tr}} \leq \sqrt{\frac{28 d_a^5\,\mathrm{tr}\,\rho^2}{\delta}}\right\} \geq 1-\delta. \quad (36)$$

Remark 6. Again, we can equivalently express Theorem 3 as a bound on the confidence level $1-\delta$ for any given allowed deviation of ρ_a^ψ from $\mathrm{tr}_b\,\rho$: *For every ρ with $\|\rho\| < 1/4$ and every $\varepsilon > 0$,*

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \big\|\rho_a^\psi - \mathrm{tr}_b\,\rho\big\|_{\mathrm{tr}} > \varepsilon\Big\} \leq \frac{28 d_a^5\,\mathrm{tr}\,\rho^2}{\varepsilon^2}\,. \quad (37)$$

◇

Remark 7. While our main motivation for developing Theorem 3 is the different strategy of proof, and while the exponential bound of Theorem 1 will usually be tighter than the polynomial bound of Theorem 3, this is not always the case: the bound of Theorem 3 is actually sometimes better, as the following example shows. Suppose that $\|\rho\| = \frac{1}{\sqrt{D}} = p_1$ and that all other p_j are equal, i.e.,

$$p_j = \frac{1 - \frac{1}{\sqrt{D}}}{D - 1} \tag{38}$$

for all $j > 1$. Then,

$$\operatorname{tr} \rho^2 = \frac{1}{D} + \frac{1}{D-1}\left(1 - \frac{1}{\sqrt{D}}\right)^2 \approx \frac{2}{D}, \tag{39}$$

and for, e.g., $d_a = 1000$ and $\varepsilon = 0.01$ we find that

$$\frac{28 d_a^5}{\varepsilon^2}\frac{2}{D} < 12 d_a^2 \exp\left(-\frac{\tilde{C}\varepsilon^2\sqrt{D}}{d_a^2}\right) \tag{40}$$

for $4.67 \cdot 10^{13} < D < 9.17 \cdot 10^{31}$, i.e., in this example there is a regime in which D is already very large but still the polynomial bound is smaller than the exponential one. $\diamond$

3.2 Discussion

Remark 8. *System size.* Theorem 3 shows, roughly speaking, that as soon as

$$\operatorname{tr} \rho^2 \ll d_a^{-5}, \tag{41}$$

GAP(ρ)-most wave functions ψ have ρ_a^ψ close to $\operatorname{tr}_b \rho$. If we think of $1/\operatorname{tr}\rho^2$ as the effective number of dimensions participating in ρ, and if this number of dimensions is comparable to the full number $D = \dim \mathcal{H} = d_a d_b$ of dimensions, then (41) reduces to

$$d_a^5 \ll D. \tag{42}$$

Since the dimension is exponential in the number of degrees of freedom, this condition roughly means that the subsystem a comprises fewer than 20% of the degrees of freedom of the full system. (The same consideration was carried out in [13, 14] for the original statement of canonical typicality.) The stronger exponential bound yields that a can even comprise up to 50% of the degrees of freedom [13, 14]. $\diamond$

Remark 9. *Canonical density matrix.* A ρ of particular interest is the canonical density matrix

$$\rho_{\text{can}} = \frac{1}{Z(\beta)} e^{-\beta H} \,. \tag{43}$$

The relevant condition for generalized canonical typicality to apply to $\rho = \rho_{\text{can}}$ is that it has small purity $\operatorname{tr}\rho^2$ and small largest eigenvalue $\|\rho\|$. We argue that indeed it does.

One heuristic reason is equivalence of ensembles: since ρ_{mc} has purity $1/d_{\text{mc}}$ and largest eigenvalue $1/d_{\text{mc}}$, which is small, the values for ρ_{can} should be similarly small. Another heuristic argument is based on the idealization that the system consists of many non-interacting constituents, so that $\mathcal{H} = \mathcal{H}_1^{\otimes N}$ and $H = \sum_{j=1}^N I^{\otimes(j-1)} \otimes H_1 \otimes I^{\otimes(N-j)}$, so $\rho_{\text{can}} = \rho_{1\text{can}}^{\otimes N}$. It is a general fact that for tensor products $\rho_1 \otimes \rho_2$ of density matrices, the purities multiply, $\operatorname{tr}(\rho_1 \otimes \rho_2)^2 = (\operatorname{tr}\rho_1^2)(\operatorname{tr}\rho_2^2)$, and the largest eigenvalues multiply, $\|\rho_1 \otimes \rho_2\| = \|\rho_1\|\,\|\rho_2\|$. Thus, the purity of ρ_{can} is the N-th power of that of $\rho_{1\text{can}}$, and likewise the largest eigenvalue. Since $N \gg 1$ and the values of $\rho_{1\text{can}}$ are somewhere between 0 and 1, and not particularly close to 1, the values of ρ_{can} are close to 0, as claimed. We expect that mild interaction does not change that picture very much. ◇

Remark 10. *Classical vs. quantum.* While classically, a typical phase point from a canonical ensemble is also a typical phase point from some micro-canonical ensemble, a typical wave function from $\text{GAP}(\rho_\beta)$ does not lie in any micro-canonical subspace $\mathcal{H}_{\text{mc}}$ (if $\mathcal{H} \neq \mathcal{H}_{\text{mc}}$) and even if it does lie in an $\mathcal{H}_{\text{mc}}$, then it is not typical from that subspace; that is because typical wave functions are superpositions of many energy eigenstates, and the weights of these eigenstates in ρ_{mc} and ρ_{can} are reflected in the weights of these eigenstates in the superposition. Therefore, already in the case that ρ is a canonical density matrix, Theorems 3 and 1 are not just simple consequences of canonical typicality but independent results. ◇

Remark 11. *Equivalence of ensembles.* We can now state more precisely the sense in which our results provide a version of equivalence of ensembles. It is well known that if a and b interact weakly and b is large enough, then both ρ_{mc} and ρ_{can} in $\mathcal{H}_S = \mathcal{H}_a \otimes \mathcal{H}_b$ lead to reduced density matrices close to the canonical density matrix (4) for a, $\operatorname{tr}_b \rho_{\text{mc}} \approx \rho_{a,\text{can}} \approx \operatorname{tr}_b \rho_{\text{can}}$, provided the parameter β of ρ_{can} and $\rho_{a,\text{can}}$ is suitable for the energy E of ρ_{mc}. Hence, Theorems 3 and 1 yield that we can start from either u_{mc} or $\text{GAP}(\rho_{\text{can}})$ and obtain for both ensembles of ψ that ρ_a^ψ is nearly constant and nearly canonical. ◇

Remark 12. *Comparison to original theorems.* The original, known theorems about canonical typicality, which refer to the uniform distribution over a suitable sphere instead of a GAP measure, are still contained in our theorems as special cases, except for worse constants and in some places additional factors of d_a (which we usually think of as constant as well). For more detail, let us begin with the known theorem analogous to Theorem 3 (formulated this way in [14, Eq. (32)], based on arguments from [46]):

Theorem 4 (Canonical typicality, polynomial bounds)**.** *Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be Hilbert spaces of respective dimensions $d_a, d_b \in \mathbb{N}$, $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, $\mathcal{H}_R$ be any subspace of $\mathcal{H}$ of dimension d_R, ρ_R be $1/d_R$ times the projection to $\mathcal{H}_R$, and u_R the uniform distribution over $\mathbb{S}(\mathcal{H}_R)$. Then for every $\delta > 0$,*

$$u_R\left\{\psi \in \mathbb{S}(\mathcal{H}_R) : \left\|\rho_a^\psi - \operatorname{tr}_b \rho_R\right\|_{\mathrm{tr}} \leq \frac{d_a^2}{\sqrt{\delta d_R}}\right\} \geq 1 - \delta. \tag{44}$$

When we apply our Theorem 3 to $\rho = \rho_R$ (and assume $d_R \geq 4$), we obtain that $\mathrm{GAP}(\rho) = u_R$, $\operatorname{tr}\rho^2 = 1/d_R$, and almost exactly the bound (44) except for a (rather irrelevant) factor $\sqrt{28}$ and $d_a^{2.5}$ instead of d_a^2. Further explanation of how this different exponent comes about can be found in Section 4.6.

Theorem 5 (Canonical typicality, exponential bounds [30, 31])**.** *With the notation and hypotheses as in Theorem 4, for every $\delta > 0$ such that*

$$\delta < 4 \exp\left(-d_a^2/(18\pi^3)\right), \tag{45}$$

$$u_R\left\{\psi \in \mathbb{S}(\mathcal{H}_R) : \left\|\rho_a^\psi - \operatorname{tr}_b \rho_R\right\|_{\mathrm{tr}} \leq 2\sqrt{\frac{18\pi^3}{d_R}\ln(4/\delta)}\right\} \geq 1 - \delta\,. \tag{46}$$

This theorem was stated slightly differently in [30, 31]; we give the derivation of this form in Section 4.6. Again, the bound agrees with the one (28) provided by Theorem 1 for $\rho = \rho_R$ (so $\|\rho\| = 1/d_R$) up to worse constants and additional factors of d_a.

Next, here is the standard statement of Lévy's lemma:[4]

Theorem 6 (Lévy's Lemma [27])**.** *Let $\mathcal{H}$ be a Hilbert space of finite dimension $D := \dim\mathcal{H} \in \mathbb{N}$, let $f : \mathbb{S}(\mathcal{H}) \to \mathbb{R}$ be a function with Lipschitz constant η, let u be the uniform distribution over $\mathbb{S}(\mathcal{H})$, and let $\varepsilon > 0$. Then*

$$u\left\{\psi \in \mathbb{S}(\mathcal{H}) : \left|f(\psi) - u(f)\right| > \varepsilon\right\} \leq 4 \exp\left(-\frac{\hat{C} D \varepsilon^2}{\eta^2}\right), \tag{47}$$

where $\hat{C} = \frac{2}{9\pi^3}$.

[4] Lévy's original 1922 statement (reprinted as a second edition in [25, Sec. 3.I.9]) was that if a hypersurface $S \subset \mathbb{S}(\mathbb{R}^d)$ divides the sphere in two regions of equal area then its ε-neighborhood has area greater than or equal to that of the ε-neighborhood of an equator, which in turn [25, Sec. 3.I.6] has nearly full area if the dimension d is large enough. As pointed out by, e.g., Milman and Schechtman [27], it follows for a function $f : \mathbb{S}(\mathbb{R}^d) \to \mathbb{R}$ with Lipschitz constant η (by taking $S = f^{-1}(m)$ and m the median of f) that most points ψ have $f(\psi)$ close to m if d is large enough. The variant quoted here referring to the mean instead of the median is due to Maurey and Pisier [29] and also described in [27, App. V].

When we apply our Theorem 2 to $\rho = I/D$, we obtain that $\mathrm{GAP}(\rho) = u$, $\|\rho\| = 1/D$, and exactly the bound (47) except for worse constants. Note that Theorem 2 holds also for infinite dimensional separable $\mathcal{H}$.

We turn to previous results for dynamical typicality. In [48], an inequality analogous to the bound (32) of Corollary 2 was proven for the uniform distribution over the sphere in a subspace. In [28], variants of Lévy's lemma and dynamical typicality were established for the mean-value ensemble of an observable A for a value $a \in \mathbb{R}$, defined by restricting the uniform distribution on $\mathbb{S}(\mathbb{C}^D)$ to the set $\{\psi \in \mathbb{S}(\mathbb{C}^D) : \langle\psi|A|\psi\rangle = a\}$ and normalizing afterwards. However, the physical relevance of this ensemble is unclear, since, in general, the mean value of an observable is itself no observable, and thus it is unclear how this ensemble could be prepared or occur in an experiment. $\diamond$

Remark 13. *Lévy's lemma for other distributions.* Lévy's lemma, although it applies to the uniform and GAP measures, does not apply to *all* rather-spread-out distributions on the sphere; it is thus a non-trivial property of the family of GAP measures.

This can be illustrated by means of the von Mises-Fisher (VMF) distribution, a well known and natural probability distribution on the unit sphere $\mathbb{S}(\mathbb{R}^D)$ in $\mathbb{R}^D$ that is different from the GAP measure. It has parameters $\kappa \in \mathbb{R}_+$ and $\mu \in \mathbb{S}(\mathbb{R}^D)$ and can be obtained from a Gaussian distribution in $\mathbb{R}^D$ with mean μ and covariance $\kappa^{-1}I$ by conditioning on $\mathbb{S}(\mathbb{R}^D)$. The analog of Lévy's lemma for the von Mises-Fisher distribution is false; this can be seen as follows. Its density

$$g(x) = C(D,\kappa)\,\exp\big(\kappa\,\langle\mu, x\rangle_{\mathbb{R}^D}\big) \tag{48}$$

with respect to the uniform distribution u on $\mathbb{S}(\mathbb{R}^D)$ varies at most by a factor of $e^{2\kappa}$ when varying x (while keeping D and κ fixed). For a given Lipschitz function F on the sphere, insertion of $F(x)\,g(x)$ for $f(x)$ in a real variant of Lévy's lemma for the uniform distribution (Theorem 6 above) yields that $F(x)\,g(x)$ for u-most x is close to the u-average of Fg, which equals the VMF-average of F (where the Lipschitz constant of $f = Fg$ could be a bit worse than that of F). The set of exceptional x has small u-measure, and since $C(D,\kappa) \in [e^{-\kappa}, e^{\kappa}]$ and thus $g(x) \in [e^{-2\kappa}, e^{2\kappa}]$, it also has small VMF-measure (larger at most by a factor of $e^{2\kappa}$). Thus, for VMF-most x, $F(x)$ is close to $\mathrm{VMF}(F)/g(x)$, and thus not constant at all. The same argument shows that Lévy's lemma is violated for any sequence of measures $(\mu_D)_{D\in\mathbb{N}}$ on $\mathbb{S}(\mathbb{R}^D)$ whose density g_D relative to u is bounded uniformly in D, has Lipschitz constant bounded uniformly in D, but deviates significantly from 1 on a non-negligible set in $\mathbb{S}(\mathbb{R}^D)$.

For GAP measures the situation is very different. From (16) one can see, for example, that if the eigenvalue p_{n_2} of $\rho = \sum_n p_n|n\rangle\langle n|$ is twice as large as another eigenvalue p_{n_1}, then the density (16) at $\psi = |n_2\rangle$ is 2^{D+1} times as large as that at $\psi = |n_1\rangle$. Thus, the density and its Lipschitz constant are not (for relevant choices of ρ) bounded uniformly in D; rather, non-uniform GAP measures become more and more singular with respect to the uniform distribution for large D. $\diamond$

Remark 14. *Generalized canonical typicality from conditional wave function?* One might imagine a different strategy of deriving generalized canonical typicality, based on regarding ψ itself as a conditional wave function and using the known fact [19, 17] that conditional wave functions are typically GAP distributed. We could introduce a further big system c, choose a high-dimensional subspace $\mathcal{H}_{Rabc}$ in $\mathcal{H}_{abc} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$ so that $\mathrm{tr}_c\, P_{Rabc}/d_{Rabc}$ coincides with the given ρ on $\mathcal{H}_a \otimes \mathcal{H}_b$, and start from a random wave function from $\mathbb{S}(\mathcal{H}_{Rabc})$. However, we do not see how to make such a derivation work. $\diamond$

Remark 15. *Not every measure does what* GAP(ρ) *does.* Generalized canonical typicality as expressed in Theorems 3 and 1 is not true in general if we replace GAP(ρ) by a different measure: if ρ is a density matrix on $\mathcal{H}$ and μ a probability distribution over $\mathbb{S}(\mathcal{H})$ with density matrix $\rho_\mu = \rho$, then it need not be true for μ-most ψ that $\rho_a^\psi \approx \mathrm{tr}_b\, \rho$.

Here is a counter-example. Let $\rho = \sum_{n=1}^{D} p_n |n\rangle\langle n|$ have eigenvalues p_n and eigen-ONB $(|n\rangle)_{n\in\{1,\dots,D\}}$, and let

$$\mu = \sum_{n=1}^{D} p_n\, \delta_{|n\rangle} \tag{49}$$

be the measure that is concentrated on the finite set $\{|n\rangle : 1 \leq n \leq D\}$ and gives weight p_n to each $|n\rangle$. This measure is the narrowest, most concentrated measure with density matrix ρ, and thus a kind of opposite of GAP(ρ), the most spread-out measure with density matrix ρ. A random vector ψ with distribution μ is a random eigenvector $|n\rangle$. What the reduced density matrix $\rho_a^{|n\rangle}$ looks like depends on the vectors $|n\rangle \in \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. Suppose that the eigenbasis of ρ is the product of ONBs of $\mathcal{H}_a$ and $\mathcal{H}_b$, $|n\rangle = |\ell\rangle_a \otimes |m\rangle_b$; then $\rho_a^{|n\rangle} = \mathrm{tr}_b\, |n\rangle\langle n| = |\ell\rangle_a\langle \ell|$ (in an obvious notation), so $\rho_a^{|n\rangle}$ is always a pure state and thus far away from $\mathrm{tr}_b\, \rho = \sum_{\ell,m} p_{\ell m} |\ell\rangle_a \langle \ell|$ if that is highly mixed. Note, however, that if instead of a product basis, we had taken $(|n\rangle)_{n=1\dots D}$ to be a purely random ONB of $\mathcal{H}$, then (with overwhelming probability if $d_b \gg 1$) $\rho_a^{|n\rangle} \approx d_a^{-1} I_a$ and thus also $\mathrm{tr}_b\, \rho$ (which by (18) is the μ-average of ρ_a^ψ) is close to $d_a^{-1} I_a$, so $\rho_a^\psi \approx \mathrm{tr}_b\, \rho$ for μ-most ψ, despite the narrowness of μ. $\diamond$

Remark 16. *Canonical typicality with respect to* GAP(ρ) *does not hold for every* ρ. Let us consider the special case in which ρ has one eigenvalue that is large (e.g., 10^{-1}), while all others are very small (e.g., 10^{-1000}). Such a situation occurs for example for N-body quantum systems with a gapped ground state $|0\rangle$ at very low temperature, T of order $(\log N)^{-1}$. So call the large eigenvalue p and suppose for definiteness that all other eigenvalues are equal,

$$\rho = p|0\rangle\langle 0| + \frac{1-p}{D-1}(I - |0\rangle\langle 0|) = p|0\rangle\langle 0| + (1-p)\frac{I}{D} + O\Big(\frac{1}{D}\Big) \tag{50}$$

with $O(1/D)$ referring to the trace norm and the limit $D \to \infty$. In that case, $\operatorname{tr}\rho^2 \approx p^2$ (e.g., 10^{-2}, while d_a may be 10^{100}), so the smallness condition (41) for generalized canonical typicality is strongly violated. To investigate ρ_a^ψ, note that any vector $\psi \in \mathbb{S}(\mathcal{H})$ can be written as $\psi = \cos\theta e^{i\alpha}|0\rangle + \sin\theta|\phi\rangle$ with $\theta \in [0, \pi/2]$, $\alpha \in [0, 2\pi)$, and $|\phi\rangle \perp |0\rangle$. If ψ has distribution $\mathrm{GAP}(\rho)$, then ϕ has distribution $u_{\mathbb{S}(|0\rangle^\perp)}$ and is independent of θ and α, α is independent of θ and uniformly distributed, and a lengthy computation shows that the distribution of θ has density

$$\frac{2(1-p)^2}{p}\frac{\cos\theta}{\sin^5\theta}\exp\Bigl((1-\tfrac{1}{p})\cot^2\theta\Bigr) \tag{51}$$

as $D \to \infty$. By an error of order $1/\sqrt{D}$, we can replace ϕ by a $u_{\mathbb{S}(\mathcal{H})}$-distributed vector. If $|0\rangle$ factorizes as in $|0\rangle = |0\rangle_a|0\rangle_b$, then $\operatorname{tr}_b\rho = p|0\rangle_a\langle 0| + (1-p)(I_a/d_a) + O(1/d_b)$ and $\rho_a^\psi = \cos^2\theta|0\rangle_a\langle 0| + \sin^2\theta(I_a/d_a) + O(1/\sqrt{d_b})$. Since the latter depends on θ (and thus is not deterministic but has a non-trivial distribution), it follows that $\rho_a^\psi \not\approx \operatorname{tr}_b\rho$ with high probability. ◇

Remark 17. *Comparison to large deviation theory.* In large deviation theory [50], one studies another version of concentration of measures: one considers a sequence of probability distributions $(\mathbb{P}_N)_{N\in\mathbb{N}}$ on (say) the real line and studies whether (and at which rate) $\mathbb{P}_N\bigl([x,\infty)\bigr)$ tends to 0 exponentially fast as $N \to \infty$ for fixed $x \in \mathbb{R}$. Our situation is a bit similar, with the role of x played by ε in (29), and that of $\mathbb{P}_N$ by the distribution of $\|\rho_a^\psi - \operatorname{tr}_b\rho\|_{\mathrm{tr}}$ in $\mathbb{R}$ for $\mathrm{GAP}(\rho)$-distributed ψ. However, our situation does not quite fit the standard framework of large deviations because we do not necessarily consider a sequence ρ_N of density matrices, but rather a fixed ρ with small $\|\rho\|$. That is why we have provided error bounds in terms of the given ρ. ◇

4 Proofs

4.1 Proof of Remark 1

What needs proof here is that also in infinite dimension, the partial trace commutes with the expectation,

$$\mathbb{E}_\mu \operatorname{tr}_b|\psi\rangle\langle\psi| = \operatorname{tr}_b \mathbb{E}_\mu|\psi\rangle\langle\psi|\,. \tag{52}$$

(For $\dim\mathcal{H}_b < \infty$, tr_b is a finite sum and thus trivially commutes with $\mathbb{E}_\mu$.) So suppose that $\mathcal{H}_b$ has a countable ONB $(|l\rangle_b)_{l\in\mathbb{N}}$, and let $|\phi\rangle_a \in \mathcal{H}_a$. Then

$$\begin{aligned} {}_a\langle\phi|\mathbb{E}_\mu \operatorname{tr}_b\bigl(|\psi\rangle\langle\psi|\bigr)|\phi\rangle_a &= \int_{\mathbb{S}(\mathcal{H})} {}_a\langle\phi|\operatorname{tr}_b\bigl(|\psi\rangle\langle\psi|\bigr)|\phi\rangle_a\,\mu(d\psi) &\text{(53a)}\\ &= \int_{\mathbb{S}(\mathcal{H})}\sum_l \bigl|\langle\phi,l|\psi\rangle\bigr|^2\,\mu(d\psi) &\text{(53b)}\end{aligned}$$

$$= \sum_l \int_{\mathbb{S}(\mathcal{H})} \langle \phi, l|\psi\rangle\langle\psi|l,\phi\rangle\, \mu(d\psi) \tag{53c}$$

$$= \sum_l \langle \phi, l|\rho_\mu|l,\phi\rangle \tag{53d}$$

$$= {}_a\langle\phi|\operatorname{tr}_b \rho_\mu|\phi\rangle_a, \tag{53e}$$

where we used Fubini's theorem in the third and the definition of ρ_μ in the fourth line. Since a bounded operator A is uniquely determined by the quadratic form $\phi \mapsto \langle\phi|A|\phi\rangle$, it follows that $\mathbb{E}_\mu(\rho_a^\psi) = \operatorname{tr}_b \rho_\mu$.

4.2 Proof of Theorem 3

We start with the proof of the polynomial version of generalized canonical typicality and thereby introduce approximation techniques for infinite dimensional Hilbert spaces, which will also be used in the proof of the exponential bounds of Theorem 1 later on. For the proof of Theorem 3 we make use of a result from Reimann [35]. Let $(|n\rangle)_{n=1\ldots D}$ be an orthonormal basis of eigenvectors of ρ and $p_1,\ldots,p_D$ the corresponding (positive) eigenvalues. Reimann used the density of the GAP measure GAP(ρ) to compute expressions of the form

$$\mathbb{E}(c_j^* c_k c_m^* c_n), \tag{54}$$

where the expectation is taken with respect to GAP(ρ) and $c_j = \langle j|\psi\rangle$ are the coordinates of $\psi \in \mathbb{S}(\mathcal{H})$ with respect to the orthonormal basis $(|j\rangle)_{j=1\ldots D}$. With the help of these expressions he derived an upper bound for the variance $\operatorname{Var}\langle\psi|A|\psi\rangle$ (also taken with respect to GAP(ρ)) for self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$. We show that Reimann's upper bound for $\operatorname{Var}\langle\psi|A|\psi\rangle$ remains essentially valid also for non-self-adjoint A and this bound will be a main ingredient in our proof of Theorem 3.

We start by computing the expectation $\mathbb{E}\langle\psi|A|\psi\rangle$ and an upper bound for the variance $\operatorname{Var}\langle\psi|A|\psi\rangle$ for an arbitrary operator $A : \mathcal{H} \to \mathcal{H}$, where the expectation and variance are with respect to the measure GAP(ρ). We closely follow Reimann [35] who did these computations in the case that A is self-adjoint. We arrive at the same bound for the variance (with the distance between the largest and smallest eigenvalue of A replaced by its operator norm), however, one step in the proof needs to be modified to account for A not being necessarily self-adjoint. Moreover, we show that the expression for $\mathbb{E}\langle\psi|A|\psi\rangle$ and the upper bound for $\operatorname{Var}\langle\psi|A|\psi\rangle$ remain valid if $\mathcal{H}$ has countably infinite dimension, i.e., if it is separable.

Proposition 1. *Let ρ be a density matrix on a separable Hilbert space $\mathcal{H}$ with positive eigenvalues p_n such that $p_{\max} = \|\rho\| < 1/4$ and let* $\dim\mathcal{H} \geq 4$. *For* GAP(ρ)*-distributed ψ and any bounded operator $A : \mathcal{H} \to \mathcal{H}$,*

$$\mathbb{E}\langle\psi|A|\psi\rangle = \operatorname{tr}(A\rho) \tag{55}$$

and

$$\operatorname{Var}\langle\psi|A|\psi\rangle \leq \frac{\|A\|^2 \operatorname{tr}\rho^2}{1-p_{\max}}\left(1+\frac{4\sqrt{\operatorname{tr}\rho^2}+2\operatorname{tr}\rho^2}{(1-2p_{\max})(1-3p_{\max})}\right). \tag{56}$$

Proof. We first assume that $D := \dim\mathcal{H} < \infty$. The formula for the expectation follows immediately from the fact that the density matrix of $\mathrm{GAP}(\rho)$ is ρ:

$$\mathbb{E}\langle\psi|A|\psi\rangle = \mathbb{E}\operatorname{tr}(|\psi\rangle\langle\psi|A) = \operatorname{tr}(\mathbb{E}|\psi\rangle\langle\psi|A) = \operatorname{tr}(A\rho). \tag{57}$$

For a complex-valued random variable X the variance can be computed by

$$\operatorname{Var}X = \mathbb{E}\left[(X-\mathbb{E}X)^*(X-\mathbb{E}X)\right] = \mathbb{E}(X^*X) - \mathbb{E}(X^*)\mathbb{E}(X). \tag{58}$$

Since the variance of a random variable does not change when a constant is added, we can assume for its computation without loss of generality that $\mathbb{E}\langle\psi|A|\psi\rangle = 0$. Let $(|n\rangle)_{n=1,\ldots,D}$ be an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of ρ. For $\psi\in\mathbb{S}(\mathcal{H})$ we write

$$\langle\psi|A|\psi\rangle = \sum_{l,m}\langle\psi|m\rangle\langle m|A|l\rangle\langle l|\psi\rangle =: \sum_{l,m} c_m^* A_{ml} c_l \tag{59}$$

with $c_l = \langle l|\psi\rangle$ and $A_{ml} = \langle m|A|l\rangle$. Then for $X = \langle\psi|A|\psi\rangle$ we find that

$$\operatorname{Var}X = \sum_{l,m,l',m'} A_{ml}^* A_{m'l'}\mathbb{E}(c_l^* c_m c_{m'}^* c_{l'}). \tag{60}$$

Reimann [35] showed that the fourth moments $\mathbb{E}(c_l^* c_m c_{m'}^* c_{l'})$ all vanish except for the two cases $l=m, m'=l'$ and $l=m', m=l'$ and that

$$\mathbb{E}(|c_m|^2|c_l|^2) = p_m p_l(1+\delta_{ml})K_{ml}, \tag{61}$$

where

$$K_{ml} = \int_0^\infty (1+xp_m)^{-1}(1+xp_l)^{-1}\prod_{n=1}^{D}(1+xp_n)^{-1}\,dx. \tag{62}$$

This implies

$$\begin{aligned}\operatorname{Var}X = &\sum_{m,l}|A_{ml}|^2 p_m p_l(1+\delta_{ml})K_{ml} + \sum_{m,m'} A_{mm}^* A_{m'm'} p_m p_{m'}(1+\delta_{mm'})K_{mm'}\\ &-2\sum_m |A_{mm}|^2 p_m^2 K_{mm} \qquad (63)\\ =&\sum_{m,l}\left[|A_{ml}|^2 + A_{mm}^* A_{ll}\right] p_m p_l K_{ml}. \qquad (64)\end{aligned}$$

Because of $|A_{mm}| \leq \|A\|$ it follows from the computation in [35] that

$$\sum_{m,l} A^*_{mm} A_{ll} p_m p_l K_{ml} \leq \frac{2\|A\|^2 \operatorname{tr}\rho^2}{(1-p_{\max})(1-2p_{\max})(1-3p_{\max})} \left(2(\operatorname{tr}\rho^2)^{1/2} + \operatorname{tr}\rho^2\right) \quad (65)$$

Moreover, as it was shown in [35], $K_{ml} \leq \frac{1}{1-p_{\max}}$ for all l and m and therefore

$$\sum_{m,l} |A_{ml}|^2 p_m p_l K_{ml} \leq \frac{1}{1-p_{\max}} \operatorname{tr}(A^* \rho A \rho). \quad (66)$$

Since A is not necessarily self-adjoint, we have to proceed in a different way than Reimann [35] did to bound this term. To this end we make use of the Cauchy-Schwarz inequality for the trace, i.e. $\operatorname{tr}(B^*C) \leq \sqrt{\operatorname{tr}(B^*B)\operatorname{tr}(C^*C)}$, and the inequality $|\operatorname{tr}(BC)| \leq \|B\| \operatorname{tr}(|C|)$ for any operators B, C [43, Thm. 3.7.6]. With these inequalities we have that

$$\begin{aligned} \operatorname{tr}(A^*\rho A\rho) &\leq \sqrt{\operatorname{tr}(A^*\rho^2 A)\operatorname{tr}(\rho A^* A \rho)} && \text{(67a)} \\ &= \sqrt{\operatorname{tr}(AA^*\rho^2)\operatorname{tr}(A^*A\rho^2)} && \text{(67b)} \\ &\leq \|A\|^2 \operatorname{tr}\rho^2. && \text{(67c)} \end{aligned}$$

Combining (64), (65), (66) and (67c) proves the bound for the variance and thus finishes the proof in the finite-dimensional case.

Now suppose that $\mathcal{H}$ has a countably infinite ONB. The expectation can be computed as before since $\mathrm{GAP}(\rho)(|\psi\rangle\langle\psi|) = \rho$ remains true in the infinite-dimensional setting [49]. For the variance, we approximate ρ by density matrices ρ_n, $n \in \mathbb{N}$, of finite rank defined by

$$\rho_n := \sum_{m=1}^{n-1} p_m |m\rangle\langle m| + \Big(\sum_{m=n}^{\infty} p_m\Big) |n\rangle\langle n|. \quad (68)$$

Then $\|\rho_n - \rho\|_{\mathrm{tr}} \to 0$ as $n \to \infty$, and therefore Theorem 3 in [49] implies that $\mathrm{GAP}(\rho_n) \Rightarrow \mathrm{GAP}(\rho)$ (weak convergence). Note also that from some n_0 onwards, $\sum_{m=n}^{\infty} p_m \leq p_1$ and thus $\|\rho_n\| = p_1 = \|\rho\|$. Let $f(\psi) := |\langle\psi|A|\psi\rangle - \operatorname{tr}(A\rho)|^2$ and $f_n(\psi) := |\langle\psi|A|\psi\rangle - \operatorname{tr}(A\rho_n)|^2$. Because of $\operatorname{tr}(A\rho_n) \to \operatorname{tr}(A\rho)$ and therefore $f_n \to f$ uniformly in ψ it follows that $\mathrm{GAP}(\rho_n)(f_n) - \mathrm{GAP}(\rho_n)(f) \to 0$. Since f is continuous, it follows from the weak convergence of the measures $\mathrm{GAP}(\rho_n)$ that $\mathrm{GAP}(\rho_n)(f) \to \mathrm{GAP}(\rho)(f)$ and therefore altogether that $\mathrm{GAP}(\rho_n)(f_n) \to \mathrm{GAP}(\rho)(f)$. Since, as one easily verifies, $\operatorname{tr}\rho_n^2 \to \operatorname{tr}\rho^2$, the bound for the variance in the finite-dimensional case remains valid in the infinite-dimensional setting.[5] □

[5] A different way to prove that the bound remains valid in the infinite-dimensional setting is

Proof of Theorem 3. Without loss of generality assume that all eigenvalues of ρ are positive. Proposition 1 together with Chebyshev's inequality implies for any operator A and any $\varepsilon > 0$ that

$$\begin{aligned}
&\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \left|\langle\psi|A|\psi\rangle - \mathrm{tr}(A\rho)\right| > \varepsilon\right\} \\
&\quad\leq \frac{\|A\|^2 \,\mathrm{tr}\,\rho^2}{\varepsilon^2(1-p_{\max})}\left(1 + \frac{4\sqrt{\mathrm{tr}\,\rho^2} + 2\,\mathrm{tr}\,\rho^2}{(1-2p_{\max})(1-3p_{\max})}\right) &\text{(69a)}\\
&\quad\leq \frac{4\|A\|^2 \,\mathrm{tr}\,\rho^2}{3\varepsilon^2}\left(1 + 8\left(4\sqrt{p_{\max}} + 2p_{\max}\right)\right) &\text{(69b)}\\
&\quad\leq \frac{28\|A\|^2 \,\mathrm{tr}\,\rho^2}{\varepsilon^2}. &\text{(69c)}
\end{aligned}$$

Let $(|l\rangle_a)_{l=1\ldots d_a}$ and $(|n\rangle_b)_{n=1\ldots d_b}$, where $d_a := \dim\mathcal{H}_a \in \mathbb{N}$ and $d_b := \dim\mathcal{H}_b \in \mathbb{N}\cup\{\infty\}$, be an orthonormal basis of $\mathcal{H}_a$ and $\mathcal{H}_b$ respectively. For

$$A^{lm} = [|l\rangle_a\langle m|] \otimes I_b, \tag{70}$$

where I_b is the identity on $\mathcal{H}_b$, we find $\|A^{lm}\| = 1$,

$$\begin{aligned}
\langle\psi|A^{lm}|\psi\rangle &= \sum_n \langle\psi|\left(|l\rangle_a\langle m| \otimes |n\rangle_b\langle n|\right)|\psi\rangle &\text{(71a)}\\
&= {}_a\langle m|\left(\sum_n {}_b\langle n|\psi\rangle\langle\psi|n\rangle_b\right)|l\rangle_a &\text{(71b)}\\
&= {}_a\langle m|\rho_a^\psi|l\rangle_a &\text{(71c)}
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathrm{tr}(A^{lm}\rho) &= \sum_{k,n} {}_a\langle k|_b\langle n|\left[\left([|l\rangle_a\langle m|] \otimes I_b\right)\rho\right]|k\rangle_a|n\rangle_b &\text{(72a)}\\
&= {}_a\langle m|\left(\sum_n {}_b\langle n|\rho|n\rangle_b\right)|l\rangle_a &\text{(72b)}\\
&= {}_a\langle m|\,\mathrm{tr}_b\,\rho|l\rangle_a. &\text{(72c)}
\end{aligned}$$

the following: Since $\langle\psi|A|\psi\rangle$ is a continuous function of ψ, it follows from the weak convergence of the measures $\mathrm{GAP}(\rho_n)$ that also the distribution of $\langle\psi|A|\psi\rangle$ under $\psi \sim \mathrm{GAP}(\rho_n)$ converges weakly to that under $\psi \sim \mathrm{GAP}(\rho)$ (where the notation $X \sim \mu$ means that the random variable X has distribution μ). Since $\mathrm{tr}(A\rho_n) \to \mathrm{tr}(A\rho)$, this does not change if we subtract $\mathrm{tr}(A\rho_n)$ respectively $\mathrm{tr}(A\rho)$ (because the test functions f can equivalently be assumed to be bounded and Lipschitz [3, Thm. 2.1] and $\langle\psi|A|\psi\rangle$ is Lipschitz), and take the absolute square. Theorem 3.4 of [3] says that if the distribution of the real random variable X_n converges weakly to that of X, then $\mathbb{E}|X| \leq \liminf_n \mathbb{E}|X_n|$. Thus, the variance of $\langle\psi|A|\psi\rangle$ under $\mathrm{GAP}(\rho)$ is bounded by the limit of the bounds for ρ_n. Since $\mathrm{tr}\,\rho_n^2 \to \mathrm{tr}\,\rho^2$, the variance is bounded by the same upper bound as in the finite-dimensional case.

For any $d_a \times d_a$ matrix $M = (M_{ij})$ it holds that $\|M\|_{\mathrm{tr}} \leq \sqrt{d_a}\|M\|_2$, where $\|M\|_2$ denotes the Hilbert-Schmidt norm of M which is defined by

$$\|M\|_2 = \sqrt{\mathrm{tr}(M^*M)} = \sqrt{\sum_{i,j=1}^{d_a} |M_{ij}|^2}, \tag{73}$$

see, e.g., Lemma 6 in [30]. Therefore, we have that

$$\|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}}^2 \leq d_a \sum_{l,m=1}^{d_a} \left|{}_a\langle m|\rho_a^\psi - \mathrm{tr}_b\,\rho|l\rangle_a\right|^2 \tag{74}$$

and thus

$$\begin{aligned}
&\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \left\|\rho_a^\psi - \mathrm{tr}_b\,\rho\right\|_{\mathrm{tr}} > d_a^{3/2}\varepsilon\right\} \\
&\quad\leq \mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \sum_{l,m=1}^{d_a} \left|{}_a\langle m|\rho_a^\psi - \mathrm{tr}_b\,\rho|l\rangle_a\right|^2 \geq d_a^2\varepsilon^2\right\} && \text{(75a)}\\
&\quad\leq \mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \exists\, l,m : \left|{}_a\langle m|\rho_a^\psi - \mathrm{tr}_b\,\rho|l\rangle_a\right| \geq \varepsilon\right\} && \text{(75b)}\\
&\quad\leq \frac{28 d_a^2\,\mathrm{tr}\,\rho^2}{\varepsilon^2}, && \text{(75c)}
\end{aligned}$$

where we used (69c), (71c), (72c) and $\|A^{lm}\| = 1$ in the last step. By replacing $\varepsilon \to d_a^{-3/2}\varepsilon$ we finally obtain

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}} > \varepsilon\right\} \leq \frac{28 d_a^5\,\mathrm{tr}\,\rho^2}{\varepsilon^2}. \tag{76}$$

Setting

$$\delta = \frac{28 d_a^5\,\mathrm{tr}\,\rho^2}{\varepsilon^2} \tag{77}$$

and solving for ε gives (36) and thus finishes the proof. □

4.3 Proof of Theorem 1

The proof of Theorem 1 follows largely the one of canonical typicality given in [30]; some crucial differences concern our generalization of the Lévy lemma and the steps needed for covering infinite dimension.

Let U_a be a unitary operator on $\mathcal{H}_a$. Then the function $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$, $f(\psi) = \mathrm{tr}_a(U_a\rho_a^\psi) = \langle\psi|U_a \otimes I_b|\psi\rangle$ is Lipschitz continuous with Lipschitz constant $\eta \leq 2\|U_a\| = 2$ (see, e.g., Lemma 5 in [30]). By Theorem 2 and Remark 3,

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \left|\mathrm{tr}_a(U_a\rho_a^\psi) - \mathrm{GAP}(\rho)(\mathrm{tr}_a(U_a\rho_a^\psi))\right| > \varepsilon\right\}$$

$$\leq 12 \exp\left(-\frac{C\varepsilon^2}{8\|\rho\|}\right). \tag{78}$$

By (27),

$$\mathrm{GAP}(\rho)(\operatorname{tr}_a(U_a\rho_a^\psi)) = \operatorname{tr}_a\left(U_a \mathrm{GAP}(\rho)(\rho_a^\psi)\right) = \operatorname{tr}_a\left(U_a \operatorname{tr}_b \rho\right). \tag{79}$$

Let $(U_a^j)_{j=0}^{d_a^2-1}$ be unitary operators that form a basis for the space of operators on $\mathcal{H}_a$ such that[6]

$$\operatorname{tr}_a(U_a^{j*}U_a^k) = d_a\delta_{jk}. \tag{80}$$

Then

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \exists j : \left|\operatorname{tr}_a(U_a^j\rho_a^\psi) - \operatorname{tr}_a(U_a^j \operatorname{tr}_b \rho)\right| > \varepsilon\right\} \leq 12 d_a^2 \exp\left(-\frac{C\varepsilon^2}{8\|\rho\|}\right). \tag{81}$$

As in [30], the density matrix ρ_a^ψ can be expanded as

$$\rho_a^\psi = \frac{1}{d_a}\sum_j C_j(\rho_a^\psi)U_a^j, \tag{82}$$

where $C_j(\rho_a^\psi) = \operatorname{tr}_a(U_a^{j*}\rho_a^\psi)$ and (81) becomes

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \exists j : \left|C_j(\rho_a^\psi) - C_j(\operatorname{tr}_b \rho)\right| > \varepsilon\right\} \leq 12 d_a^2 \exp\left(-\frac{C\varepsilon^2}{8\|\rho\|}\right). \tag{83}$$

If $|C_j(\rho_a^\psi) - C_j(\operatorname{tr}_b \rho)| \leq \varepsilon$ for all j, then

$$\begin{aligned}
\|\rho_a^\psi - \operatorname{tr}_b \rho\|_{\mathrm{tr}}^2 &\leq d_a \|\rho_a^\psi - \operatorname{tr}_b \rho\|_2^2 && \text{(84a)}\\
&= d_a \left\|\frac{1}{d_a}\sum_j \left(C_j(\rho_a^\psi) - C_j(\operatorname{tr}_b \rho)\right) U_a^j\right\|_2^2 && \text{(84b)}\\
&= \frac{1}{d_a}\operatorname{tr}_a \left|\sum_j \left(C_j(\rho_a^\psi) - C_j(\operatorname{tr}_b \rho)\right) U_a^j\right|^2 && \text{(84c)}\\
&= \sum_j \left|C_j(\rho_a^\psi) - C_j(\operatorname{tr}_b \rho)\right|^2 && \text{(84d)}
\end{aligned}$$

[6]One possible choice is given by

$$U_a^j = \sum_{k=0}^{d_a-1} e^{2\pi i k(j-(j \bmod d_a))/d_a^2} |(k+j) \bmod d_a\rangle\langle k|,$$

where $(|k\rangle)_{k=0\ldots d_a-1}$ is an orthonormal basis of $\mathcal{H}_a$, see [30].

$$\leq d_a^2 \varepsilon^2. \tag{84e}$$

This implies that

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}} > d_a\varepsilon\right\} \leq 12 d_a^2 \exp\left(-\frac{C\varepsilon^2}{8\|\rho\|}\right) \tag{85}$$

and, after replacing ε by εd_a^{-1},

$$\mathrm{GAP}(\rho)\left\{\psi \in \mathbb{S}(\mathcal{H}) : \|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}} > \varepsilon\right\} \leq 12 d_a^2 \exp\left(-\frac{C\varepsilon^2}{8d_a^2\|\rho\|}\right). \tag{86}$$

Setting

$$\delta = 12 d_a^2 \exp\left(-\frac{C\varepsilon^2}{8d_a^2\|\rho\|}\right) \tag{87}$$

and solving for ε finishes the proof.

4.4 Proof of Theorem 2

The proofs begins with an auxiliary theorem formulated as Theorem 8 below. For better orientation, we also state the analogous fact about Gaussian distributions as Theorem 7 and start with quoting its real version:[7]

Lemma 1 ([27])**.** *Let $F : \mathbb{R}^D \to \mathbb{R}$ be a Lipschitz function with constant η. Let $X = (X_1, \dots, X_D)$ be a vector of independent (real) standard Gaussian random variables. Then for every $\varepsilon > 0$,*

$$\mathbb{P}\big\{|F(X) - \mathbb{E}F(X)| > \varepsilon\big\} \leq 2\exp\left(-\frac{2\varepsilon^2}{\pi^2\eta^2}\right). \tag{88}$$

Now let $\rho = \sum_{n=1}^D p_n |n\rangle\langle n|$ be a density matrix on the D-dimensional Hilbert space $\mathcal{H}$, and let Z be a random vector in $\mathcal{H}$ whose distribution is $\mathrm{G}(\rho)$, the Gaussian measure with mean 0 and covariance ρ as defined in Section 2.2; equivalently, $Z = \sum_{n=1}^D Z_n|n\rangle$, where the Z_n are independent complex mean-zero Gaussian random variables with variances

$$\mathbb{E}|Z_n|^2 = p_n\,. \tag{89}$$

Then we can write $Z = \sqrt{\rho/2}\tilde{Z}$, where the components $\tilde{Z}_n$ of $\tilde{Z} = \sum_{n=1}^D \tilde{Z}_n|n\rangle$ are D independent complex mean-zero Gaussian random variables with variances

[7] The constant in (88) can actually be improved to $1/2$ instead of $2/\pi^2$ [29, p. 180]. But for us it is not important to obtain the optimal constant, and we use a method of proof for Theorem 8 that yields $2/\pi^2$ in Theorem 7.

$\mathbb{E}|\tilde{Z}_n|^2 = 2$, which can be in a natural way identified with a vector of $2D$ independent real standard Gaussian variables.

If $F : \mathcal{H} \to \mathbb{R}$ is Lipschitz with constant η, then $F \circ \sqrt{\rho/2} : \mathcal{H} \to \mathbb{R}$ is also Lipschitz with constant $\eta\sqrt{\|\rho\|/2}$. This function can also naturally be considered as a function on $\mathbb{R}^{2D}$ and then an application of Lemma 1 immediately proves the following theorem:

Theorem 7. *Let* $\dim \mathcal{H} < \infty$*, let* ρ *be a density matrix on* $\mathcal{H}$*, let* Z *be a random vector with distribution* $\mathrm{G}(\rho)$*, and let* $F : \mathcal{H} \to \mathbb{R}$ *be a Lipschitz function with Lipschitz constant* η*. Then for every* $\varepsilon > 0$*,*

$$\mathbb{P}\Big\{|F(Z) - \mathbb{E}F(Z)| > \varepsilon\Big\} \leq 2\exp\left(-\frac{4\varepsilon^2}{\pi^2\eta^2\|\rho\|}\right). \tag{90}$$

However, instead of using Theorem 7, we will use Theorem 8 below, a similar result for the Gaussian adjusted measure $\mathrm{GA}(\rho)$ defined in Section 2.2, which has density $\|\psi\|^2$ relative to $\mathrm{G}(\rho)$. Its proof closely follows the proof of Lévy's Lemma in [27]; for convenience of the reader we provide all the details.

Theorem 8. *Let* $\dim \mathcal{H} < \infty$*, let* ρ *be a density matrix on* $\mathcal{H}$*, let* Z *be a random vector with distribution* $\mathrm{GA}(\rho)$*, and let* $F : \mathcal{H} \to \mathbb{R}$ *be a Lipschitz function with Lipschitz constant* η*. Then for every* $\varepsilon > 0$*,*

$$\mathrm{GA}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \big|F(\psi) - \mathrm{GA}(\rho)(F)\big| > \varepsilon\Big\} \leq 4\exp\left(-\frac{2\varepsilon^2}{\pi^2\eta^2\|\rho\|}\right). \tag{91}$$

Proof. We identify $\mathcal{H}$ with $\mathbb{C}^D$ by means of the ONB $(|n\rangle)_{n=1\ldots D}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function and let $\tilde{Z} = (\tilde{Z}_1, \ldots, \tilde{Z}_D)$ be a vector with the same distribution as Z but independent of it. With the help of Jensen's inequality and Hölder's inequality we find that

$$\mathrm{GA}(\rho)_\psi\left[\varphi(F(\psi) - \mathrm{GA}(\rho)_\phi(F))\right]$$

$$\leq \mathrm{GA}(\rho)_\psi \mathrm{GA}(\rho)_\phi\left[\varphi(F(\psi) - F(\phi))\right] \tag{92a}$$

$$= \int_{\mathcal{H}}\int_{\mathcal{H}} \varphi(F(\psi) - F(\phi))\|\psi\|^2\|\phi\|^2\,\mathbb{P}(d\psi)\mathbb{P}(d\phi) \tag{92b}$$

$$= \sum_{n,m}\int_{\mathbb{C}^D}\int_{\mathbb{C}^D} \varphi(F(Z) - F(\tilde{Z}))|Z_n|^2|\tilde{Z}_m|^2\mathbb{P}(dZ)\mathbb{P}(d\tilde{Z}) \tag{92c}$$

$$\leq \sum_{n,m}\left(\mathbb{E}_{(Z,\tilde{Z})}(|Z_n|^4|\tilde{Z}_m|^4)\mathbb{E}_{(Z,\tilde{Z})}\left(\varphi(F(Z) - F(\tilde{Z}))^2\right)\right)^{1/2}, \tag{92d}$$

where we use the notation $F(Z)$ and $F(\psi)$ interchangeably. We can write $Z_n = \mathrm{Re}\,Z_n + i\mathrm{Im}\,Z_n$ where $\mathrm{Re}\,Z_n$ and $\mathrm{Im}\,Z_n$ are independent real-valued Gaussian random variables with mean 0 and variance $p_n/2$. Since $\mathbb{E}|\mathrm{Re}\, Z_n|^2 = p_n/2$ and $\mathbb{E}|\mathrm{Re}\, Z_n|^4 = 3p_n^2/4$ we obtain

$$\mathbb{E}|Z_n|^4 = \mathbb{E}|\mathrm{Re}\, Z_n|^4 + 2\mathbb{E}|\mathrm{Re}\, Z_n|^2 \mathbb{E}|\mathrm{Im}\, Z_n|^2 + \mathbb{E}|\mathrm{Im}\, Z_n|^4 = 2p_n^2 \tag{93}$$

and therefore

$$\sum_{n,m} \left(\mathbb{E}_{(Z,\tilde{Z})}(|Z_n|^4 |\tilde{Z}_m|^4) \right)^{1/2} = \sum_{n,m} 2p_n p_m = 2. \tag{94}$$

We identify Z with the vector $X := (\mathrm{Re}\, Z_1, \mathrm{Im}\, Z_1, \mathrm{Re}\, Z_2, \ldots, \mathrm{Re}\, Z_D, \mathrm{Im}\, Z_D)$ of real Gaussian random variables and similarly $\tilde{Z}$ with $Y := (\mathrm{Re}\, \tilde{Z}_1, \mathrm{Im}\, \tilde{Z}_1, \mathrm{Re}\, \tilde{Z}_2, \ldots, \mathrm{Re}\, \tilde{Z}_D, \mathrm{Im}\, \tilde{Z}_D)$. For each $0 \le \theta \le \frac{\pi}{2}$ set $X_\theta = X \sin\theta + Y \cos\theta$. One easily sees that the joint distribution of X and Y, which is the multivariate normal distribution with mean vector 0 and covariance matrix $\mathrm{diag}(p_1, p_1, \ldots, p_D, p_D, , p_1, p_1, \ldots, p_D, p_D)/2$, is the same as the joint distribution of X_θ and $\frac{d}{d\theta} X_\theta = X \cos\theta - Y \sin\theta$ since linear combinations of independent Gaussian random variables are again Gaussian and the entries of the expectation vector and covariance matrix can be easily computed.

Since F can be approximated uniformly by continuously differentiable functions, we can without loss of generality assume that F is continuously differentiable.

Let us now assume that φ is non-negative. Then φ^2 is also convex. Then we find with the help of Jensen's inequality that

$$\begin{aligned}
\mathbb{E}\varphi(F(Z) - F(\tilde{Z}))^2 &= \mathbb{E}\varphi(F(X) - F(Y))^2 && \text{(95a)}\\
&= \mathbb{E}\left[\varphi\left(\int_0^{\pi/2} \frac{d}{d\theta} F(X_\theta)\, d\theta \right)^2 \right] && \text{(95b)}\\
&= \mathbb{E}\left[\varphi\left(\int_0^{\pi/2} \left(\nabla F(X_\theta), \frac{d}{d\theta} X_\theta \right)\, d\theta \right)^2 \right] && \text{(95c)}\\
&\le \frac{2}{\pi}\mathbb{E}\left[\int_0^{\pi/2} \varphi\left(\frac{\pi}{2} \left(\nabla F(X_\theta), \frac{d}{d\theta} X_\theta \right) \right)^2 d\theta \right] && \text{(95d)}\\
&= \mathbb{E}\varphi\left(\frac{\pi}{2} \left(\nabla F(X), Y \right) \right)^2, && \text{(95e)}
\end{aligned}$$

where in the last step we used Fubini's theorem and the fact that the joint distribution of X_θ and $\frac{d}{d\theta} X_\theta$ is the same as the joint distribution of X and Y.

Let $\lambda \in \mathbb{R}$ and set $\varphi(x) = \exp(\lambda x)$. Then we get

$$\mathbb{E} \exp\left[2\lambda(F(X) - F(Y)) \right] \le \mathbb{E} \exp\left(\lambda\pi \sum_{i=1}^{2D} \frac{\partial F}{\partial x_i}(X) Y_i \right) \tag{96a}$$

$$
\begin{aligned}
&= \mathbb{E}_X \prod_{i=1}^{2D} \mathbb{E}_Y \exp\left(\lambda\pi \frac{\partial F}{\partial x_i}(X) Y_i\right) && \text{(96b)}\\
&= \mathbb{E}\exp\left(\frac{\lambda^2\pi^2}{4} \sum_{i=1}^{2D} \left(\frac{\partial F}{\partial x_i}(X)\right)^2 p_i\right) && \text{(96c)}\\
&\leq \mathbb{E}\exp\left(\frac{\lambda^2\pi^2\|\rho\|\|\nabla F(X)\|^2}{4}\right) && \text{(96d)}\\
&\leq \exp\left(\frac{\lambda^2\pi^2\|\rho\|\eta^2}{4}\right). && \text{(96e)}
\end{aligned}
$$

Altogether we obtain

$$
\mathrm{GA}(\rho)\left[\exp(\lambda(F(\psi) - \mathrm{GA}(\rho)(F)))\right] \leq 2\exp\left(\frac{\lambda^2\pi^2\|\rho\|\eta^2}{8}\right). \tag{97}
$$

By Markov's inequality we find that

$$
\begin{aligned}
&\mathrm{GA}(\rho)\left\{|F(Z) - \mathrm{GA}(\rho)(F)| > \varepsilon\right\}\\
&\quad = \mathrm{GA}(\rho)\left\{F(Z) - \mathrm{GA}(\rho)(F) > \varepsilon\right\}\\
&\qquad + \mathrm{GA}(\rho)\left\{\mathrm{GA}(\rho)(F) - F(Z) > \varepsilon\right\} && \text{(98a)}\\
&\quad = \mathrm{GA}(\rho)\left\{\exp(\lambda(F(Z) - \mathrm{GA}(\rho)(F))) > e^{\lambda\varepsilon}\right\}\\
&\qquad + \mathrm{GA}(\rho)\left\{\exp(-\lambda(F(Z) - \mathrm{GA}(\rho)(F))) > e^{\lambda\varepsilon}\right\} && \text{(98b)}\\
&\quad \leq 4\exp\left(-\lambda\varepsilon + \frac{\lambda^2\pi^2\|\rho\|\eta^2}{8}\right). && \text{(98c)}
\end{aligned}
$$

Since $\lambda \in \mathbb{R}$ was arbitrary, we can minimize the right-hand side over λ. The minimum is attained at $\lambda_{\min} = 4\varepsilon/(\pi^2\|\rho\|\eta^2)$ and inserting this value in (98c) finally yields (91). □

The last ingredient we need for the proof of Theorem 2 is the following lemma:

Lemma 2. *For all $r > 0$ it holds that*

$$
\mathrm{GA}(\rho)\left\{\|\psi\| < r\right\} \leq \sqrt{2}\exp\left(-\frac{1/2 - r^2}{2\|\rho\|}\right). \tag{99}
$$

Proof. With the help of Hölder's inequality we find that

$$
\begin{aligned}
\mathrm{GA}(\rho)\left\{\|\psi\| < r\right\} &= \sum_n \int_{\mathcal{H}} |Z_n|^2 \mathbb{1}_{\{\|\psi\|<r\}}\,\mathbb{P}(d\psi) && \text{(100a)}\\
&\leq \sum_n \left(\mathbb{E}|Z_n|^4 \mathbb{P}\left(\|\psi\| < r\right)\right)^{1/2} && \text{(100b)}
\end{aligned}
$$

$$= \sqrt{2}\left(\mathbb{P}\left(\|\psi\| < r\right)\right)^{1/2} \tag{100c}$$

Note that in the third line we used (93) and that $\sum_n p_n = 1$. We can write

$$\|\psi\|^2 = \sum_n |Z_n|^2 = \sum_n p_n |\tilde{Z}_n|^2, \tag{101}$$

where the $\tilde{Z}_n$ are independent complex standard Gaussian random variables. For a random variable Y let $M_Y(t) = \mathbb{E}(e^{tY})$ denote its moment generating function. The Chernoff bound states that for any $a \in \mathbb{R}$,

$$\mathbb{P}\{Y \leq a\} \leq \inf_{t<0} M_Y(t) e^{-ta}. \tag{102}$$

Here we thus obtain

$$\mathbb{P}\left\{\|\psi\| < r\right\} = \mathbb{P}\left\{\|\psi\|^2 < r^2\right\} \leq \inf_{t<0} M_{\|\psi\|^2}(t) e^{-tr^2}. \tag{103}$$

We compute

$$M_{\|\psi\|^2}(t) = \prod_n M_{|\tilde{Z}_n|^2}(p_n t) = \prod_n M_{2(\operatorname{Re}\tilde{Z}_n)^2}\left(\frac{p_n t}{2}\right) M_{2(\operatorname{Im}\tilde{Z}_n)^2}\left(\frac{p_n t}{2}\right). \tag{104}$$

Next note that $2(\operatorname{Re}\tilde{Z}_n)^2$ and $2(\operatorname{Im}\tilde{Z}_n)^2$ are chi-squared distributed random variables with one degree of freedom and that the moment generating function of a random variable Y with distribution χ_1^2 is given by

$$M_Y(t) = (1-2t)^{-1/2} \quad \text{for} \quad t < 1/2. \tag{105}$$

Therefore,

$$M_{\|\psi\|^2}(t) = \prod_n (1 - p_n t)^{-1} \tag{106}$$

and this implies

$$\mathbb{P}\left\{\|\psi\| < r\right\} \leq \inf_{t<0} e^{-tr^2} \prod_n (1-p_n t)^{-1} \tag{107a}$$

$$= \inf_{t<0} \exp\left(-tr^2 - \sum_n \ln(1-p_n t)\right) \tag{107b}$$

$$= \inf_{s>0} \exp\left(sr^2 - \sum_n \ln(1+p_n s)\right) \tag{107c}$$

$$\leq \exp\left(\frac{r^2}{\|\rho\|} - \sum_n \ln\left(1 + \frac{p_n}{\|\rho\|}\right)\right), \tag{107d}$$

where we chose $s = \|\rho\|^{-1}$ in the last line. Because of

$$\ln(1+x) \geq \frac{x}{x+1} \geq \frac{x}{2} \quad \text{for} \quad 0 < x \leq 1 \tag{108}$$

we find that

$$\mathbb{P}\left\{\|\psi\| < r\right\} \leq \exp\left(\frac{r^2}{\|\rho\|} - \sum_n \frac{p_n}{2\|\rho\|}\right) = \exp\left(-\frac{1/2 - r^2}{\|\rho\|}\right). \tag{109}$$

Inserting this into (100c) finishes the proof. □

Proof of Theorem 2. We first assume that $D = \dim \mathcal{H} < \infty$. Without loss of generality we can assume that $\mathrm{GAP}(\rho)(f) = 0$. Due to the continuity of f it follows that there exists a $\varphi \in \mathbb{S}(\mathcal{H})$ such that $f(\varphi) = 0$. This implies for all $\tilde{\varphi} \in \mathbb{S}(\mathcal{H})$ that

$$|f(\tilde{\varphi})| = |f(\tilde{\varphi}) - f(\varphi)| \leq \eta\|\tilde{\varphi} - \varphi\| \leq \pi\eta, \tag{110}$$

where we used in the last step that the distance (in the spherical metric) between two points on the unit sphere is bounded by π. Thus f is bounded by $\pi\eta$.

Let $0 < r < 1$ and define $\tilde{f} : \mathcal{H} \to \mathbb{R}$ by

$$\tilde{f}(\psi) = \begin{cases} f\left(\frac{\psi}{\|\psi\|}\right) & \text{if } \|\psi\| \geq r, \\ r^{-1}\|\psi\| f\left(\frac{\psi}{\|\psi\|}\right) & \text{if } \|\psi\| \leq r. \end{cases} \tag{111}$$

For every $\psi, \varphi \in \mathcal{H}$ such that $\|\psi\|, \|\varphi\| \geq r$ we find that

$$\begin{aligned} \left|\tilde{f}(\psi) - \tilde{f}(\varphi)\right| &= \left|f\left(\frac{\psi}{\|\psi\|}\right) - f\left(\frac{\varphi}{\|\varphi\|}\right)\right| && \text{(112a)} \\ &\leq \eta\left\|\frac{\psi}{\|\psi\|} - \frac{\varphi}{\|\varphi\|}\right\| && \text{(112b)} \\ &\leq \frac{\eta}{r}\|\psi - \varphi\|, && \text{(112c)} \end{aligned}$$

where the last inequality follows from

$$\begin{aligned} \left\|\frac{\psi}{\|\psi\|} - \frac{\varphi}{\|\varphi\|}\right\|^2 &= 2 - \frac{2}{\|\psi\|\|\varphi\|}\mathrm{Re}\langle\psi, \varphi\rangle && \text{(113a)} \\ &= 2 + 2\mathrm{Re}\langle\psi, \varphi\rangle\left(r^{-2} - \frac{1}{\|\psi\|\|\varphi\|}\right) - 2r^{-2}\mathrm{Re}\langle\psi, \varphi\rangle && \text{(113b)} \\ &\leq r^{-2}\left(2\|\psi\|\|\varphi\| - 2\mathrm{Re}\langle\psi, \varphi\rangle\right) && \text{(113c)} \\ &\leq r^{-2}\left(\|\psi\|^2 + \|\varphi\|^2 - 2\mathrm{Re}\langle\psi, \varphi\rangle\right) && \text{(113d)} \\ &= r^{-2}\|\psi - \varphi\|^2. && \text{(113e)} \end{aligned}$$

Thus $\tilde{f}$ is Lipschitz continuous with constant η/r on $\{\psi \in \mathcal{H} : \|\psi\| \geq r\}$.

Now let $\psi, \varphi \in \mathcal{H}$ such that $\|\psi\|, \|\varphi\| \leq r$ and $\|\varphi\| \leq \|\psi\|$. Then we obtain

$$\left|\tilde{f}(\psi) - \tilde{f}(\varphi)\right| = r^{-1} \left| \|\psi\| f\left(\frac{\psi}{\|\psi\|}\right) - \|\varphi\| f\left(\frac{\varphi}{\|\varphi\|}\right)\right| \tag{114a}$$

$$\leq r^{-1} \left| \|\psi\| f\left(\frac{\psi}{\|\psi\|}\right) - \|\varphi\| f\left(\frac{\psi}{\|\psi\|}\right)\right| + r^{-1} \left| \|\varphi\| f\left(\frac{\psi}{\|\psi\|}\right) - \|\varphi\| f\left(\frac{\varphi}{\|\varphi\|}\right)\right| \tag{114b}$$

$$\leq \frac{\pi\eta}{r} \big| \|\psi\| - \|\varphi\| \big| + \frac{\eta}{r} \|\varphi\| \left\| \frac{\psi}{\|\psi\|} - \frac{\varphi}{\|\varphi\|} \right\| \tag{114c}$$

$$\leq \frac{5\eta}{r} \|\psi - \varphi\|, \tag{114d}$$

where the last inequality follows from

$$\|\varphi\|^2 \left\| \frac{\psi}{\|\psi\|} - \frac{\varphi}{\|\varphi\|} \right\|^2 = 2\|\varphi\|^2 + 2\mathrm{Re}\,\langle \psi, \varphi \rangle \left(1 - \frac{\|\varphi\|}{\|\psi\|}\right) - 2\mathrm{Re}\,\langle \psi, \varphi \rangle \tag{115a}$$

$$\leq 2\|\psi\|\|\varphi\| - 2\mathrm{Re}\,\langle \psi, \varphi \rangle \tag{115b}$$

$$\leq \|\psi - \varphi\|^2. \tag{115c}$$

Due to the symmetry of the argument in ψ and φ, one finds the same estimate in the case that $\|\psi\| \leq \|\varphi\|$ and we conclude that $\tilde{f}$ is Lipschitz continuous with constant $5\eta/r$ on $\{\psi \in \mathcal{H} : \|\psi\| \leq r\}$.

Finally, let $\psi, \varphi \in \mathcal{H}$ such that $\|\psi\| \leq r$ and $\|\varphi\| \geq r$ and define $\gamma : [0,1] \to \mathcal{H}, \gamma(t) = (1-t)\psi + t\varphi$. Then there exists a $t_0 \in [0,1]$ such that $\|\gamma(t_0)\| = r$ and

$$\|\psi - \gamma(t_0)\| = t_0 \|\psi - \varphi\| \leq \|\psi - \varphi\|, \tag{116}$$

$$\|\gamma(t_0) - \varphi\| = (1 - t_0)\|\psi - \varphi\| \leq \|\psi - \varphi\|. \tag{117}$$

Therefore, we find that

$$\left|\tilde{f}(\psi) - \tilde{f}(\varphi)\right| = \left| r^{-1} \|\psi\| f\left(\frac{\psi}{\|\psi\|}\right) - f\left(\frac{\varphi}{\|\varphi\|}\right)\right| \tag{118a}$$

$$\leq r^{-1} \left| \|\psi\| f\left(\frac{\psi}{\|\psi\|}\right) - \|\gamma(t_0)\| f\left(\frac{\gamma(t_0)}{\|\gamma(t_0)\|}\right)\right| + \left| f\left(\frac{\gamma(t_0)}{\|\gamma(t_0)\|}\right) - f\left(\frac{\varphi}{\|\varphi\|}\right)\right| \tag{118b}$$

$$\leq \frac{5\eta}{r} \|\psi - \gamma(t_0)\| + \frac{\eta}{r} \|\gamma(t_0) - \varphi\| \tag{118c}$$

$$\leq \frac{6\eta}{r} \|\psi - \varphi\|. \tag{118d}$$

We conclude that $\tilde{f}$ is Lipschitz continuous with Lipschitz constant $6\eta/r$.

Using the definition of $\tilde{f}$, we find that

$$\mathrm{GAP}(\rho)\left\{|f(\psi)|>\varepsilon\right\}=\mathrm{GA}(\rho)\left\{\left|f\left(\frac{\psi}{\|\psi\|}\right)\right|>\varepsilon\right\} \tag{119a}$$

$$\leq \mathrm{GA}(\rho)\left\{\left|f\left(\frac{\psi}{\|\psi\|}\right)\right|>\varepsilon \text{ and } \|\psi\|\geq r\right\}+\mathrm{GA}(\rho)\left\{\|\psi\|<r\right\} \tag{119b}$$

$$=\mathrm{GA}(\rho)\left\{\left|\tilde{f}(\psi)\right|>\varepsilon \text{ and } \|\psi\|\geq r\right\}+\mathrm{GA}(\rho)\left\{\|\psi\|<r\right\} \tag{119c}$$

$$\leq \mathrm{GA}(\rho)\left\{\left|\tilde{f}(\psi)\right|>\varepsilon\right\}+\mathrm{GA}(\rho)\left\{\|\psi\|<r\right\} \tag{119d}$$

$$\leq \mathrm{GA}(\rho)\left\{\left|\tilde{f}(\psi)-\mathrm{GA}(\rho)(\tilde{f})\right|>\varepsilon-|\mathrm{GA}(\rho)(\tilde{f})|\right\} + \mathrm{GA}(\rho)\left\{\|\psi\|<r\right\}. \tag{119e}$$

By Lemma 2, the second term can be bounded by $\sqrt{2}\exp(-(1/2-r^2)/2\|\rho\|)$. In order to estimate the first term in (119e), we first derive an upper bound for $|\mathrm{GA}(\rho)(\tilde{f})|$. We compute

$$\mathrm{GA}(\rho)(\tilde{f})=\int_{\{\|\psi\|<r\}} r^{-1}\|\psi\| f\left(\frac{\psi}{\|\psi\|}\right)\mathrm{GA}(\rho)(d\psi)+\int_{\{\|\psi\|\geq r\}} f\left(\frac{\psi}{\|\psi\|}\right)\mathrm{GA}(\rho)(d\psi) \tag{120}$$

$$=\underbrace{\int_{\mathcal{H}} f\left(\frac{\psi}{\|\psi\|}\right)\mathrm{GA}(\rho)(d\psi)}_{=\mathrm{GAP}(\rho)(f)=0}+\int_{\{\|\psi\|<r\}} r^{-1}\|\psi\| f\left(\frac{\psi}{\|\psi\|}\right) - f\left(\frac{\psi}{\|\psi\|}\right)\mathrm{GA}(\rho)(d\psi) \tag{121}$$

and so we obtain, again by Lemma 2,

$$|\mathrm{GA}(\rho)(\tilde{f})|\leq \pi\eta\,\mathrm{GA}(\rho)\left\{\|\psi\|<r\right\}\leq 5\eta\exp\left(-\frac{1/2-r^2}{2\|\rho\|}\right). \tag{122}$$

This implies with the help of Theorem 8 that

$$\mathrm{GA}(\rho)\left\{\left|\tilde{f}(\psi)-\mathrm{GA}(\rho)(\tilde{f})\right|>\varepsilon-|\mathrm{GA}(\rho)(\tilde{f})|\right\} \tag{123a}$$

$$\leq \mathrm{GA}(\rho)\left\{\left|\tilde{f}(\psi)-\mathrm{GA}(\rho)(\tilde{f})\right|>\varepsilon-5\eta\exp\left(-\frac{1/2-r^2}{2\|\rho\|}\right)\right\} \tag{123b}$$

$$\leq 4\exp\left(-\frac{r^2(\varepsilon - 5\eta\exp(-(1/2-r^2)/2\|\rho\|))^2}{18\pi^2\eta^2\|\rho\|}\right), \tag{123c}$$

provided that $\varepsilon > 5\eta\exp(-(1/2-r^2)/2\|\rho\|)$. Altogether we arrive at

$$\begin{aligned}\mathrm{GAP}(\rho)\left\{|f(\psi)| > \varepsilon\right\} \leq\ & 4\exp\left(-\frac{r^2(\varepsilon - 5\eta\exp(-(1/2-r^2)/2\|\rho\|))^2}{18\pi^2\eta^2\|\rho\|}\right)\\ &+ \sqrt{2}\exp\left(-\frac{1/2-r^2}{2\|\rho\|}\right).\end{aligned} \tag{124}$$

Choosing $r = 1/2$ we obtain

$$\mathrm{GAP}(\rho)\left\{|f(\psi)| > \varepsilon\right\} \leq 4\exp\left(-\frac{(\varepsilon - 5\eta\exp(-1/8\|\rho\|))^2}{72\pi^2\eta^2\|\rho\|}\right) + \sqrt{2}\exp\left(-\frac{1}{8\|\rho\|}\right). \tag{125}$$

We can assume without loss of generality that

$$\varepsilon < \pi\eta \tag{126}$$

because otherwise the left-hand side of (30) vanishes: indeed, the distance between any two points on the sphere is at most π, so their f values can differ at most by $\pi\eta$, and for the same reason $f(\psi)$ can differ from its average relative to any measure by at most $\pi\eta$.

Likewise, we can assume without loss of generality that

$$\varepsilon \geq 10\eta\exp(-1/8\|\rho\|) \tag{127}$$

because otherwise the right-hand side of (30) is greater than 1: indeed, for $\varepsilon < 10\eta\exp(-1/8\|\rho\|)$,

$$6\exp\left(-\frac{\varepsilon^2}{288\pi^2\eta^2\|\rho\|}\right) \geq 6\exp\left(-\frac{25\exp(-1/4\|\rho\|)}{72\pi^2\|\rho\|}\right) > 1. \tag{128}$$

As a consequence of (126) and (127), the first exponent in (125) is greater than the second, so

$$\mathrm{GAP}(\rho)\left\{|f(\psi)| > \varepsilon\right\} \leq 6\exp\left(-\frac{(\varepsilon - 5\eta\exp(-1/8\|\rho\|))^2}{72\pi^2\eta^2\|\rho\|}\right) \tag{129}$$

$$\leq 6\exp\left(-\frac{\varepsilon^2}{288\pi^2\eta^2\|\rho\|}\right) \tag{130}$$

by (127). This finishes the proof in the finite-dimensional case.

Now suppose that $\mathcal{H}$ has a countably infinite ONB. Consider the density matrices ρ_n defined as in (68). Let $\varepsilon' > 0$. Because the set

$$A_\varepsilon := \{\psi \in \mathbb{S}(\mathcal{H}) : |f(\psi)| > \varepsilon\} \tag{131}$$

is open in $\mathbb{S}(\mathcal{H})$, it follows from the weak convergence of the measures $\mathrm{GAP}(\rho_n)$ to $\mathrm{GAP}(\rho)$ by the "portmanteau theorem" [3, Thm. 2.1] that

$$\mathrm{GAP}(\rho)(A_\varepsilon) \leq \liminf_{n\to\infty} \mathrm{GAP}(\rho_n)(A_\varepsilon) \leq \mathrm{GAP}(\rho_N)(A_\varepsilon) + \varepsilon' \tag{132}$$

for some large enough $N \in \mathbb{N}$ with $N \geq n_0$. Recall that $n_0 \in \mathbb{N}$ is chosen such that $\|\rho_n\| = \|\rho\|$ for all $n \geq n_0$. Let $\mathcal{H}_N := \mathrm{span}\{|n\rangle : n = 1, \ldots, N\}$. Then, since ρ_N is a density matrix on $\mathcal{H}_N$ and $\mathrm{GAP}(\rho_N)$ is concentrated on $\mathcal{H}_N$, it follows with what we have already proven in the finite-dimensional case that

$$\mathrm{GAP}(\rho_N)\big\{\psi \in \mathbb{S}(\mathcal{H}) : |f(\psi)| > \varepsilon\big\} = \mathrm{GAP}(\rho_N)\big\{\psi \in \mathbb{S}(\mathcal{H}_N) : |f(\psi)| > \varepsilon\big\} \tag{133a}$$

$$\leq 6\exp\left(-\frac{C\varepsilon^2}{\eta^2\|\rho_N\|}\right), \tag{133b}$$

where $C = \frac{1}{288\pi^2}$. Noting that $\|\rho_N\| = \|\rho\|$ and that $\varepsilon' > 0$ was arbitrary, we can altogether conclude that

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \big|f(\psi) - \mathrm{GAP}(\rho)(f)\big| > \varepsilon\Big\} \leq 6\exp\left(-\frac{C\varepsilon^2}{\eta^2\|\rho\|}\right), \tag{134}$$

i.e., the bound (130) remains true in the infinite-dimensional setting. □

4.5 Proofs of Corollaries 2, 3, 4

Proof of Corollary 2. As already noted before Corollary 2, the first inequality follows immediately from Corollary 1 by inserting $U_t^* B U_t$ for B.

For the proof of the second inequality we define

$$Y_t := |\langle\psi_t|B|\psi_t\rangle - \mathrm{tr}(\rho_t B)|\,. \tag{135}$$

Then, for every $s > 0$ we find that

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : e^{sY_t} > e^{s\varepsilon}\Big\} \leq 12\exp\left(-\frac{\tilde{C}\varepsilon^2}{\|B\|^2\|\rho\|}\right), \tag{136}$$

i.e., with $\delta := e^{s\varepsilon}$,

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : e^{sY_t} > \delta\Big\} \leq 12\exp\left(-\frac{\tilde{C}}{\|B\|^2\|\rho\|}\frac{\ln(\delta)^2}{s^2}\right). \tag{137}$$

This implies

$$\mathrm{GAP}(\rho)\left(e^{sY_t}\right) \leq \sum_{n=0}^{\infty}(n+1)\ \mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : e^{sY_t} \in (n, n+1]\Big\} \tag{138a}$$

$$\leq 1 + 12\sum_{n=1}^{\infty}(n+1)\exp\left(-\frac{\tilde{C}\ln(n)^2}{\|B\|^2\|\rho\|s^2}\right) \tag{138b}$$

$$= 1 + 12\sum_{n=1}^{\infty}(n+1)n^{-\frac{\tilde{C}\ln(n)}{\|B\|^2\|\rho\|s^2}}. \tag{138c}$$

With $a := \frac{\tilde{C}}{\|B\|^2\|\rho\|s^2}$ and assuming that $a \leq 1$ we obtain

$$\mathrm{GAP}(\rho)\left(e^{sY_t}\right) \leq 1 + 12\sum_{n=1}^{\lfloor e^{5/2a}\rfloor}(n+1) + 12\sum_{n=\lceil e^{5/2a}\rceil}^{\infty}(n+1)\frac{1}{n^{5/2}} \tag{139a}$$

$$\leq 1 + 6e^{\frac{5}{2a}}\left(e^{\frac{5}{2a}} + 3\right) + 12 \tag{139b}$$

$$= 13 + 18e^{\frac{5}{2a}} + 6e^{\frac{5}{a}} \tag{139c}$$

$$\leq 9e^{\frac{5}{a}}. \tag{139d}$$

An application of Jensen's inequality and Fubini's theorem shows that

$$\mathrm{GAP}(\rho)\left(\exp\left(\frac{1}{T}\int_0^T Y_t\,dt\ s\right)\right) \leq \mathrm{GAP}(\rho)\left(\frac{1}{T}\int_0^T e^{Y_t s}\,dt\right) \tag{140a}$$

$$= \frac{1}{T}\int_0^T \mathrm{GAP}(\rho)\left(e^{Y_t s}\right)\,dt \tag{140b}$$

$$\leq 9e^{5/a}. \tag{140c}$$

With the help of Markov's inequality we find that

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : \frac{1}{T}\int_0^T Y_t\,dt > \varepsilon\Big\} \leq 9e^{5/a}e^{-\varepsilon s}. \tag{141}$$

and choosing $s := \frac{\varepsilon\tilde{C}}{6\|B\|^2\|\rho\|}$ yields the desired bound provided that $\varepsilon > 0$ and $a \leq 1$, i.e., $\|\rho\| \leq \frac{\tilde{C}\varepsilon^2}{36\|B\|^2}$. However, since the bound becomes trivial for $\|\rho\| > \frac{\tilde{C}\varepsilon^2}{36\|B\|^2}$, this assumption on $\|\rho\|$ can be dropped. Moreover, note that the bound is also trivial if $\varepsilon = 0$. □

Proof of Corollary 3. Let

$$Z_t := \|\rho_a^{\psi_t} - \mathrm{tr}_b\,\rho_t\|_{\mathrm{tr}}. \tag{142}$$

It follows from the equivariance of $\rho \mapsto \mathrm{GAP}(\rho)$ and Remark 2 that

$$\mathrm{GAP}(\rho)\Big\{\psi \in \mathbb{S}(\mathcal{H}) : Z_t > \varepsilon\Big\} = \mathrm{GAP}(\rho_t)\Big\{\psi_t \in \mathbb{S}(\mathcal{H}) : Z_t > \varepsilon\Big\} \tag{143a}$$

$$\leq 12 d_a^2 \exp\left(-\frac{\tilde{C}\varepsilon^2}{d_a^2\|\rho\|}\right). \tag{143b}$$

The rest of the proof now follows along the same lines as the proof of Corollary 2. □

Proof of Corollary 4. Choose ψ and B independently with the distributions mentioned. By Theorem 2 of [17] (which requires that $d_b \geq d_a$ and $d_b \geq 4$), we have that

$$\left|\mathrm{Born}_a^{\psi,B}(f) - \mathrm{GAP}(\rho_a^\psi)(f)\right| < \varepsilon/2 \tag{144}$$

with probability $\geq 1 - 16\|f\|_\infty^2/\varepsilon^2 d_b \geq 1 - \delta/2$ for $d_b \geq 32\|f\|_\infty^2/\varepsilon^2\delta$. By Lemma 5 of [17], there is $r = r(\varepsilon, d_a, f) > 0$ such that

$$\left|\mathrm{GAP}(\rho_a^\psi)(f) - \mathrm{GAP}(\mathrm{tr}_b\,\rho)(f)\right| < \varepsilon/2 \tag{145}$$

whenever $\|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}} < r$. By Theorem 1 in the form (29), the latter condition is fulfilled with probability $\geq 1 - 12d_a^2\exp(-\tilde{C}r^2/d_a^2\|\rho\|) \geq 1 - \delta/2$ for $\|\rho\| \leq p := \tilde{C}r^2/d_a^2\ln(24d_a^2/\delta)$. Now (35) follows. □

4.6 Further Explanations to Remark 12

As discussed after Theorem 4, applying Theorem 3 to $\rho = \rho_R$ yields the worse factor $d_a^{2.5}$ instead of d_a^2. Here we want to give some details why in this special case of Theorem 3, slightly better bounds can be obtained.

First suppose that $\mathcal{H}_R = \mathcal{H}$. Similarly to the proof of Theorem 3 one finds that

$$u\big\{\psi \in \mathbb{S}(\mathcal{H}) : \|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}} > \varepsilon\big\} \leq \frac{d_a^3}{\varepsilon^2}\sum_{l,m}\mathrm{Var}\langle\psi|A^{lm}|\psi\rangle, \tag{146}$$

where $A^{lm} = |l\rangle_a\langle m| \otimes I_b$ and $(|l\rangle_a)_{l=1\ldots d_a}$ is an orthonormal basis of $\mathcal{H}_a$. Instead of bounding the sum by d_a^2 times a uniform bound on the variances $\mathrm{Var}\langle\psi|A^{lm}|\psi\rangle$, one can now make use of the fact that for uniformly distributed $\psi \in \mathbb{S}(\mathcal{H})$, the second and fourth moments of the coefficients c_l of ψ in an orthonormal basis $(|n\rangle)_{n=1\ldots D}$ of eigenvectors of ρ can be computed explicitly. More precisely, they satisfy

$$\mathbb{E}(|c_n|^2) = \frac{1}{D}, \quad \mathbb{E}(|c_n|^2|c_k|^2) = \frac{1+\delta_{nk}}{D(D+1)}, \tag{147}$$

and all other second and fourth moments vanish, see e.g. [8, App. A.2 and C.1]. With this we find that

$$\mathrm{Var}\langle\psi|A^{lm}|\psi\rangle = \sum_{k,n}|A_{kn}^{lm}|^2\frac{1+\delta_{kn}}{D(D+1)} + \sum_{k,n}A_{kk}^{lm*}A_{nn}^{lm}\frac{1+\delta_{kn}}{D(D+1)}$$

$$- \sum_n |A^{lm}_{nn}|^2 \frac{2}{D(D+1)} - \mathrm{tr}(A^{lm}\rho)^2 \tag{148a}$$

$$= \frac{\mathrm{tr}(A^{lm*}A^{lm})}{D(D+1)} - \frac{\left|\mathrm{tr}(A^{lm}\rho)\right|^2}{D+1} \tag{148b}$$

$$\le \frac{\mathrm{tr}(A^{lm*}A^{lm})}{D(D+1)}. \tag{148c}$$

Next note that

$$\sum_{l,m} \mathrm{tr}(A^{(lm)*}A^{(lm)}) = d_a \sum_l \mathrm{tr}(|l\rangle_a\langle l| \otimes I_b) = d_a D \tag{149}$$

and therefore

$$u\left\{\psi \in \mathbb{S}(\mathcal{H}) : \|\rho_a^\psi - \mathrm{tr}_b\,\rho\|_{\mathrm{tr}} > \varepsilon\right\} \le \frac{d_a^4}{\varepsilon^2 D}. \tag{150}$$

If $\mathcal{H}_R \neq \mathcal{H}$ is a subspace of $\mathcal{H}$, then this bound remains valid after replacing ρ by P_R/d_R, u by u_R, $\mathcal{H}$ by $\mathcal{H}_R$ and D by d_R. This follows immediately from the previous computations after noting that

$$\sum_{l,m} \mathrm{tr}(A^{(lm)*}P_R A^{(lm)} P_R) \le \sum_{l,m} \mathrm{tr}(A^{(lm)*}P_R A^{(lm)}) = d_a \sum_l \mathrm{tr}((|l\rangle_a\langle l| \otimes I_b P_R) = d_a d_R. \tag{151}$$

Setting $\delta := d_a^4/(\varepsilon^2 d_R)$ and solving for ε finally gives Theorem 4.

In [30, 31], Theorem 5 was stated in a slightly different form; more precisely, there it was shown that for every $\varepsilon > 0$,

$$u_R\left\{\psi \in \mathbb{S}(\mathcal{H}_R) : \left\|\rho_a^\psi - \mathrm{tr}_b\,\rho_R\right\|_{\mathrm{tr}} > \varepsilon + \sqrt{d_a\,\mathrm{tr}(\mathrm{tr}_a\,\rho_R)^2}\right\} \le 4\exp\Big(-\frac{d_R\varepsilon^2}{18\pi^3}\Big). \tag{152}$$

We now show how this implies the bound in Theorem 5. By setting $\delta := 4\exp(-d_R\varepsilon^2/(18\pi^3))$ and solving for ε, we obtain

$$\varepsilon = \sqrt{\frac{18\pi^3}{d_R}\ln(4/\delta)}. \tag{153}$$

With this and $\mathrm{tr}(\mathrm{tr}_a\,\rho_R)^2 \le d_a/d_R$ we obtain

$$u_R\left\{\psi \in \mathbb{S}(\mathcal{H}_R) : \left\|\rho_a^\psi - \mathrm{tr}_b\,\rho_R\right\|_{\mathrm{tr}} \le \sqrt{\frac{18\pi^3}{d_R}\ln(4/\delta)} + \sqrt{d_a^2/d_R}\right\} \ge 1 - \delta\,. \tag{154}$$

The first square root dominates as soon as

$$\delta < 4\exp\left(-d_a^2/(18\pi^3)\right), \tag{155}$$

which we can, of course, assume without loss of generality since otherwise we would have $\delta > 1$ and then the lower bound on the probability would be trivial. This immediately implies (46).

5 Summary and Conclusions

Typicality theorems assert that, for big systems, some condition is true of *most* points, or here, most wave functions. The word "most" usually refers to a uniform distribution u (say, over the unit sphere $\mathbb{S}(\mathcal{H}_R)$ in some Hilbert subspace $\mathcal{H}_R$), but here we use the GAP measure as the natural analog of the uniform distribution in cases with given density matrix ρ. Since the GAP measure for $\rho = \rho_{\text{can}}$ is the thermal equilibrium distribution of wave functions, our typicality theorems can be understood as expressing a kind of equivalence of ensembles between a micro-canonical ensemble of wave functions ($u_{\mathbb{S}(\mathcal{H}_{\text{mc}})}$) and a canonical ensemble of wave functions ($\text{GAP}(\rho_{\text{can}})$). Yet, our results apply to arbitrary ρ.

The key mathematical step is the generalization of Lévy's lemma to GAP measures, that is, of the concentration of measure on high-dimensional spheres. The fact that the pure states of a quantum system are always the points on a sphere then allows us to deduce very general typicality theorems from this kind of concentration of measure. In particular, these typicality statements are largely independent of the properties of the Hamiltonian and require only that many dimensions participate in ρ.

Specifically, some of these statements concern a bi-partite quantum system $a \cup b$, where b is macroscopically large. We have shown that for most ψ from the $\text{GAP}(\rho)$ ensemble, the reduced density matrix ρ_a^ψ is close to its average $\operatorname{tr}_b \rho$ assuming that the largest eigenvalue (Theorem 1) or at least the average eigenvalue (Theorem 3) of ρ is small. That is, we have established an extension of canonical typicality to GAP measures. This family of measures is particularly natural in this context because it arises anyway in the context of bi-partite systems as the typical asymptotic distribution of the conditional wave function [19, 17], a fact extended further in Corollary 4.

Another important application of concentration-of-measure of GAP yields (Corollary 1) that for any observable B, most ψ from the $\text{GAP}(\rho)$ ensemble have nearly the same Born distribution (when suitably coarse grained). Moreover (Corollaries 2 and 3), if the initial wave function ψ_0 is $\text{GAP}(\rho)$-distributed, then for any unitary time evolution the whole curves $t \mapsto \langle\psi_t|B|\psi_t\rangle$ and $t \mapsto \rho_a^{\psi_t}$ are nearly deterministic (and given by $\operatorname{tr}(B\rho_t)$ and $\operatorname{tr}_b \rho_t$).

All these results contribute different aspects to the picture of how an individual, closed quantum system in a pure state can display thermodynamic behavior [51, 40, 4, 44, 47, 7, 8, 31, 34, 2, 15, 16, 20, 41, 42, 11, 12, 36, 9, 1, 38, 37, 39, 48, 45], and thus help clarify the role of ensembles as defining a concept of typicality, while thermal density matrices arise from partial traces.

In sum, our results describe simple relations between the following concepts: reduced density matrix, many participating dimensions, and GAP measures. That is, if many dimensions participate in ρ, then for $\text{GAP}(\rho)$-most ψ, the reduced density matrix ρ_a^ψ is nearly independent of ψ.

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A.3. Typical Macroscopic Long-Time Behavior for Random Hamiltonians

Typical Macroscopic Long-Time Behavior for Random Hamiltonians

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Abstract

We consider a closed macroscopic quantum system in a pure state ψ_t evolving unitarily and take for granted that different macro states correspond to mutually orthogonal subspaces $\mathcal{H}_\nu$ (macro spaces) of Hilbert space, each of which has large dimension. We extend previous work on the question what the evolution of ψ_t looks like macroscopically, specifically on how much of ψ_t lies in each $\mathcal{H}_\nu$. Previous bounds concerned the *absolute* error for typical ψ_0 and/or t and are valid for arbitrary Hamiltonians H; now, we provide bounds on the *relative* error, which means much tighter bounds, with probability close to 1 by modeling H as a random matrix, more precisely as a random band matrix (i.e., where only entries near the main diagonal are significantly nonzero) in a basis aligned with the macro spaces. We exploit particularly that the eigenvectors of H are delocalized in this basis. Our main mathematical results confirm the two phenomena of generalized normal typicality (a type of long-time behavior) and dynamical typicality (a type of similarity within the ensemble of ψ_0 from an initial macro space). They are based on an extension we prove of a no-gaps delocalization result for random matrices by Rudelson and Vershynin [37].

Key words: normal typicality; dynamical typicality; eigenstate thermalization hypothesis; delocalized eigenvector; macroscopic quantum system.

1 Introduction

We are concerned with an isolated quantum system in a pure state ψ comprising a macroscopically large number of particles. Studying such systems in order to study

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macroscopic behavior is an approach that is particularly used in connection with the *eigenstate thermalization hypothesis* (ETH) which has been investigated by physicists as well as mathematicians [9, 40, 8, 6]. In particular, the phenomena of equilibration and thermalization in closed quantum systems have attracted increasing attention in recent years [3, 4, 16, 17, 18, 19, 21, 30, 31, 32, 33, 34, 38, 39, 42, 43].

We assume that the Hilbert space $\mathcal{H}$ of the system has very large but finite dimension.[1] Following Wigner [46], we model the Hamiltonian as a Hermitian random matrix H. Following von Neumann [45], we regard as given an orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\nu} \mathcal{H}_\nu \tag{1}$$

of the system's Hilbert space $\mathcal{H}$ into subspaces $\mathcal{H}_\nu$ ("macro spaces") associated with different macro states ν [23, 21, 42, 43]. The macroscopic behavior or macroscopic appearance of a vector $\psi \in \mathcal{H}$ is then represented by the weights given by ψ to different macro states ν, i.e., by the sizes of the contributions to ψ from the various $\mathcal{H}_\nu$, $\|P_\nu\psi\|^2$ with P_ν the projection to $\mathcal{H}_\nu$. Here we ask how, under the unitary time evolution $\psi_t = \exp(-iHt)\psi_0$, the macroscopic appearance of ψ_t typically evolves and equilibrates in the long run, more precisely, what $\|P_\nu\psi_t\|^2$ are for typical ψ_0 from a macro space and/or typical large t (where "typical ψ_0" refers to the uniform distribution over the unit sphere; more precise formulations later). Thereby, we extend and refine previous work [43] on the same physical question by means of a deeper mathematical analysis of the behavior of $\|P_\nu\psi_t\|^2$ for typical H (on top of typical ψ_0 and typical t) from suitable random matrix ensembles.

1.1 Normal Typicality

The macroscopic appearance of a system with $\psi \in \mathcal{H}_\nu$ is summarized by the macro state ν; of course, a general state is a superposition of contributions from different macro spaces, which makes the superposition weights $\|P_\nu\psi\|^2$ relevant. It can be useful to compare a given state ψ to a *purely random* vector ϕ, i.e., one that is uniformly distributed over the unit sphere $\mathbb{S}(\mathcal{H}) = \{\psi \in \mathcal{H} : \|\psi\| = 1\}$; ϕ has, with probability near 1, superposition weights approximately proportional to the dimension of $\mathcal{H}_\nu$,

$$\|P_\nu\phi\|^2 \approx \frac{d_\nu}{D}, \tag{2}$$

[1] The physical background is that $\mathcal{H}$ is really the subspace of the full Hilbert space corresponding to a micro-canonical energy interval $[E - \Delta E, E]$ with ΔE the resolution of macroscopic energy measurements, and this interval contains a very large but finite number of eigenvalues of the Hamiltonian, realistically of the order $10^{10^{10}}$. However, in this paper we will not assume that all eigenvalues of H on $\mathcal{H}$ lie between $E - \Delta E$ and E.

provided that $d_\nu := \dim \mathcal{H}_\nu$ and $D := \dim \mathcal{H}$ are large; see, e.g., Lemma 1 in [19] and the references therein. Such a behavior is sometimes called "normal," in analogy to the concept of a normal number [28].

In the cases that von Neumann considered in [45], it turned out that every $\psi_0 \in \mathbb{S}(\mathcal{H})$ evolves so that for most $t \in [0, \infty)$,

$$\|P_\nu \psi_t\|^2 \approx \frac{d_\nu}{D} \quad \forall \nu \tag{3}$$

if d_ν and D are sufficiently large. This behavior is called "normal typicality" and more elaborated in [19, 21].

However, the cases von Neumann considered are not very realistic: His conditions on H are true with probability near 1 if the eigenbasis of H is chosen purely randomly (i.e., uniformly) among all orthonormal bases (and some further technical conditions that are not very restrictive). This can be regarded as expressing that the energy eigenbasis is *unrelated* to the orthogonal decomposition (1). In this case, the system would very rapidly go from any initial macro space $\mathcal{H}_\mu$ to the thermal equilibrium macro space $\mathcal{H}_{\mathrm{eq}}$ [16, 17, 18], which is unphysical.[2] That is why we are interested in generalizations of normal typicality that apply also to Hamiltonians whose eigenbasis is not unrelated to the decomposition (1).

In such more general scenarios, we showed in [43] that the following *generalized normal typicality* still holds: for most ψ_0 from the unit sphere $\mathbb{S}(\mathcal{H}_\mu)$ in the macro space associated with a (possibly non-equilibrium) macro state μ and most $t \in [0, \infty)$,

$$\|P_\nu \psi_t\|^2 \approx M_{\mu\nu} \quad \forall \nu \tag{4}$$

for suitable values $M_{\mu\nu}$, in fact (for non-degenerate H)

$$M_{\mu\nu} = \frac{1}{d_\mu} \sum_n \langle \phi_n | P_\mu | \phi_n \rangle \langle \phi_n | P_\nu | \phi_n \rangle \tag{5}$$

with $\phi_1, \ldots, \phi_D$ an orthonormal eigenbasis of H. Put another way, generalized normal typicality means that the superposition weights $\|P_\nu \psi_t\|^2$ approach certain stable values and stay close to them for most times; we also say that the system *equilibrates.* See Figure 1 and Figure 2 for a numerical example.

One of our goals in this paper is to strengthen the results for suitable types of random Hamiltonians. We are particularly interested in random matrices with band structure (see Figure 3) in a basis that diagonalizes the projections onto the macro spaces. This band structure makes it less likely that a state in a small macro space (far away from equilibrium) goes directly into the thermal equilibrium macro space without passing through some other macro spaces in between.

[2]Thermal equilibrium requires that energy (and other quantities) is rather evenly distributed over all degrees of freedom, and for getting an even distribution, it needs to get transported through space, which requires time and passage through other macro states.

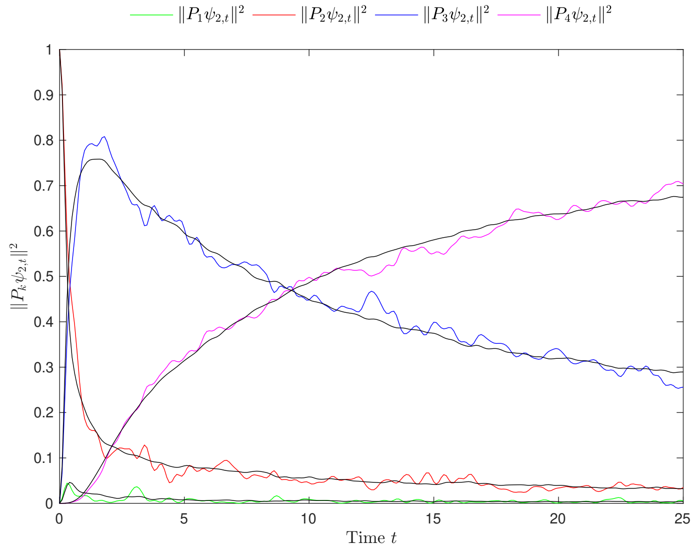

Figure 1: Numerical simulation of the functions $t \mapsto \|P_\nu \psi_t\|^2$ as detailed in Appendix B. A Hilbert space of dimension $D = 2222$ is decomposed into 4 macro spaces of dimensions $d_1 = 2$ (green curve), $d_2 = 20$ (red curve), $d_3 = 200$ (blue curve), and $d_4 = 2000$ (purple curve). At large times, see also Figure 2, the equilibrium subspace $\mathcal{H}_4$ has the biggest contribution to ψ_t. The initial wave function ψ_0 was chosen purely randomly from the unit sphere $\mathbb{S}(\mathcal{H}_2)$, and the Hamiltonian is a random band matrix in a basis aligned with the macro spaces with a band that is wide enough such that the eigenfunctions are still delocalized. Then parts of ψ_t reach $\mathcal{H}_4$ only after passing through the macro space $\mathcal{H}_3$; in this example the blue curve increases first before it decreases and the purple curve increases. The black curves are the deterministic approximations $w_{2\nu}(t)$, see (34) according to dynamical typicality. (This figure already appeared in [43].)

We showed in [43] that for general Hamiltonians under suitable assumptions the *absolute* errors

$$\left|\|P_\nu \psi_t\|^2 - M_{\mu\nu}\right| \tag{6}$$

are small (see Theorems 1, 2 in Section 2). However, if $d_\nu \ll D$ then we expect that $\|P_\nu \psi_t\|^2 \ll 1$ and $M_{\mu\nu} \ll 1$. Then the absolute error would be small indeed, but this would not imply that $\|P_\nu \psi_t\|^2 / M_{\mu\nu} \approx 1$. Therefore we want to show in the following that for suitable random Hamiltonians the *relative* errors

$$\frac{\left|\|P_\nu \psi_t\|^2 - M_{\mu\nu}\right|}{M_{\mu\nu}} \tag{7}$$

are small as well. This is the first goal of this paper: to show for certain tractable models and under suitable conditions on the d_ν that (4) holds in the sense that even

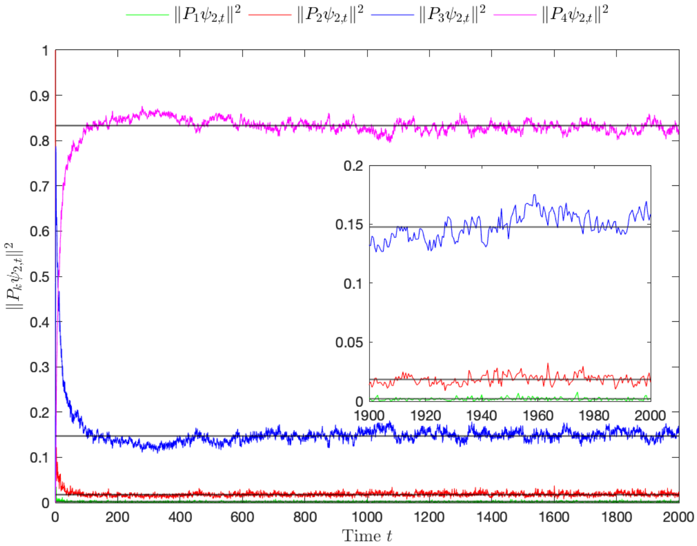

Figure 2: The same simulation as in Figure 1, only for larger times. The inset shows a part of the figure in magnification. In the long run, the curves $t \mapsto \|P_\nu\psi_t\|^2$ reach the values $M_{\mu\nu}$ (indicated by the black horizontal lines) and stay close to them, up to either small or rare fluctuations. (This figure also already appeared in [43].)

the *relative* error is small. And this goal is indeed highly relevant, since we expect that $d_\nu \ll D$ for *all* non-equilibrium macrospaces $\mathcal{H}_\nu$.

Rigorous statements are formulated in Section 3: In Theorem 4 we provide a lower bound on $M_{\mu\nu}$ under the assumption that $H = H_0 + V$ where H_0 is any deterministic Hermitian matrix and V is a Gaussian random matrix with entries that are independent up to Hermitian symmetry and have variances bounded away from 0 (so also far from the main diagonal, entries of H cannot be *too* small). From this lower bound and the upper bound on the absolute error provided in [43] (and recapitulated in Section 2), we obtain an upper bound on the relative error in Corollary 2, valid with high probability (i.e., for most H) for most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ and most $t \in [0,\infty)$. That is, the upshot is that for typical H from the ensemble considered, $\|P_\nu\psi_t\|^2$ is nearly independent of ψ_0 and in the long run nearly independent of t—it equilibrates.

This equilibration is related to the question of the increase of the quantum Boltzmann entropy observable

$$S_B := \sum_\nu S(\nu)P_\nu \tag{8}$$

with entropy values

$$S(\nu) := k_B \log(d_\nu)\,, \tag{9}$$

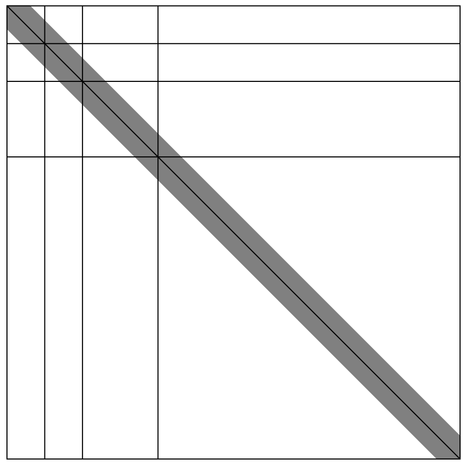

Figure 3: A band matrix has significantly nonzero entries only in the grey region near the main diagonal. The first macro space is spanned by the first d_1 basis vectors, the second macro space by the next d_2 basis vectors, etc.; in the figure, a line is drawn after the first d_1 columns and the first d_1 rows, etc.

where k_B is the Boltzmann constant. Here is how: Physically, in every energy shell corresponding to a small energy interval $[E - \Delta E, E]$, there usually is one macro space $\mathcal{H}_\nu$, the one corresponding to thermal equilibrium, that has the overwhelming majority of dimensions, $\frac{d_\nu}{D} \approx 1$ [20]; we will denote it by $\mathcal{H}_{\mathrm{eq}}$. Now if normal typicality holds as in (3), or already if the $M_{\mu\nu}$ in (4) tend to be bigger for ν with bigger dimensions d_ν, then an initial state $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ from a non-equilibrium (i.e., comparatively small) macro space $\mathcal{H}_\mu$ will evolve so that at most times t, the biggest contribution to ψ_t lies in $\mathcal{H}_{\mathrm{eq}}$, and if $\|P_{\mathrm{eq}}\psi_t\|^2 \approx 1$, this implies further that $\|S_B\psi_t\| \approx S(\mathrm{eq}) \gg S(\mu) = \|S_B\psi_0\|$, that is, the state has evolved to one with higher quantum Boltzmann entropy.

In order to obtain lower bounds for the $M_{\mu\nu}$ (and therefore upper bounds for the relative errors), we need lower bounds on $\|P_\nu\phi_n\|$, where (ϕ_n) is again an orthonormal eigenbasis of H.

We take as fixed an orthonormal basis $(|j\rangle)_{j=1}^D$ of $\mathcal{H}$ that diagonalizes the P_ν (i.e., is such that each $|j\rangle$ lies in some $\mathcal{H}_\nu$). When we talk of H as a matrix, we refer to this basis. In terms of this basis,

$$\|P_\nu\phi_n\|^2 = \sum_{j\in I_\nu} |\langle\phi_n|j\rangle|^2 \tag{10}$$

with I_ν the appropriate subset of $[D] := \{1, \ldots, D\}$. In order to obtain a lower bound on this quantity we ask whether ϕ_n (expressed as an element of $\mathbb{C}^D$ relative to the basis $(|j\rangle)_j$) is localized or delocalized.

Our first main result, Theorem 4, is based on results by Rudelson and Vershynin [37] about the *delocalization of eigenvectors* of a random matrix. While the delocalization of eigenvectors of different types of random matrices has been studied intensively in the literature in the recent years, e.g., [13, 14, 11, 10, 15, 35, 47, 1, 12, 5, 7], most results were concerned with delocalization in the sup-norm, meaning that

$$\|\phi_n\|_\infty = \sup_{j=1}^{D} |\langle \phi_n | j \rangle| \tag{11}$$

cannot be too large, so many entries of ϕ_n must be non-negligible. However, that would still allow a negligible $\|P_\nu \phi_n\|$ for some ν. Thus, for a lower bound on (10), we need that only few entries are negligible. For this kind of reasons, results about the sup-norm only yield upper bounds for the $M_{\mu\nu}$ and, in very special cases, lower bounds for some of the $M_{\mu\nu}$; for example in Section 6 we show that with a sup-norm delocalization result from Ajanki, Erdős, and Krüger (2017) [1] we obtain useful lower bounds (Theorem 13) only if μ or ν is the thermal equilibrium macro state and d_{eq} is extremely large. The result from Rudelson and Vershynin (2016) [37], however, concerns another aspect of the delocalization of eigenvectors: it rules out gaps, i.e., no significant fraction of the coordinates of an eigenvector can carry only a negligible fraction of its mass. Their result gives lower bounds on (10) and enables us to prove nontrivial lower bounds for all $M_{\mu\nu}$. Unfortunately, the lower bounds for $M_{\mu\nu}$ obtained in this way seem to be far from optimal. More precisely, as long as one is in the delocalized regime, one expects that actually $M_{\mu\nu} \approx \frac{d_\nu}{D}$, as for normal typicality, even though the eigenvectors are not uniformly distributed over the sphere. (A matrix with uniformly chosen eigenvectors will be unlikely to have band structure, but a band matrix can very well have $M_{\mu\nu} \approx \frac{d_\nu}{D}$, as the simple example of the discrete Laplacian in 1d shows.) The bounds obtained by Rudelson and Vershynin were recently improved for matrices with independent entries [25, 26] but since we are interested in Hermitian random matrices (since they can serve as Hamiltonians), these improved results are not applicable in our situation.

Our theorems in this paper are proved for a general deterministic Hermitian matrix B instead of a projection P_ν, in parallel to previous works such as [38, 39, 34]; in combination with the results from [43] we can also obtain a finite-time result (that concerns most $t \in [0, T]$ instead of most $t \in [0, \infty)$).

Sharper estimates for all $M_{\mu\nu}$ can be obtained for certain ensembles of Hermitian random matrices for which stronger statements about the delocalization of eigenvectors are available. However, such statements are apparently not currently available for random band matrices, but only for matrices that have very substantially nonzero entries also far from the diagonal. Nevertheless, we report here also what would follow about $M_{\mu\nu}$ for those matrices, based on two results in the literature. First, in Section 6, we use a delocalization result of Ajanki, Erdős, and Krüger [1] to obtain that $M_{\mu\nu} \approx d_\nu/D$ in case $\nu = \mathrm{eq}$ or $\mu = \mathrm{eq}$. Second, in Section 7, we consider a version of the ETH formulated by Cipolloni, Erdős, and Henheik (2023) [6] and show that it implies that $M_{\mu\nu} \approx d_\nu/D$ for *all* μ and ν.

1.2 Dynamical Typicality

In [43] we were not only concerned with generalized normal typicality but also with *dynamical typicality*, which means that for any given t, $\|P_\nu\psi_t\|^2$ is nearly independent of $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$. In Figure 1, this phenomenon is visible in the proximity of the colored (exact) curves to the black ones. Here, we also provide an improved result about dynamical typicality for random H.

The name "dynamical typicality" was introduced by Bartsch and Gemmer [4] for the following phenomenon: Given an observable A and some value $a \in \mathbb{R}$, there is a function $a(t)$ such that for every $t \in \mathbb{R}$ and most initial wave functions $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle \approx a$, it holds that $\langle\psi_t|A|\psi_t\rangle \approx a(t)$. A rigorous version of this fact was proved by Müller, Gross and Eisert [27] who considered initial wave functions $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle = a$ for a given $a \in \mathbb{R}$. Reimann [33, 32] argued that one also finds for most $\psi_0 \in \mathbb{S}(\mathcal{H})$ with $\langle\psi_0|A|\psi_0\rangle \approx a$ that $\langle\psi_t|B|\psi_t\rangle \approx b(t)$ for another observable B and suitable (ψ_0-independent) $b(t)$. For technical reasons, these results do not cover the case that A is a projection and $a = 1$. In [3], however, a quite general dynamical typicality result was proven that can also be applied to projections, and in [43] we provided a simple proof for this special case. The result for projections can also be obtained as a special case of Eq. (13) of Reimann [29] (applied to a Gaussian distribution with covariance P_μ, $K = 1$, and $A = e^{iHt}P_\nu e^{-iHt}$).

As for generalized normal typicality, we only obtained bounds for the *absolute* errors in [43] (recapitulated as Theorem 3 in Section 2). Unfortunately, we cannot provide here an upper bound on the *relative* errors either; but with the help of our improved version of the no-gaps delocalization result of Rudelson and Vershynin [37] we are able to bound what we call the *comparative error*, which is the absolute error divided by the time average, i.e.,

$$\frac{\big|\|P_\nu\psi_t\|^2 - \mathbb{E}_\mu\|P_\nu\psi_t\|^2\big|}{\overline{\|P_\nu\psi_t\|^2}}, \tag{12}$$

where $\mathbb{E}_\mu$ means the average over initial wave functions from the unit sphere in $\mathcal{H}_\mu$ and the overbar means time average over $[0,\infty)$. In Section 3.2, we formulate a rigorous result about (12) as Theorem 5. It provides an upper bound on (12) valid with high probability for random H (distributed as before) for every t and most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$. If the constants, in particular d_μ, d_ν, and D, are such that this bound is small, then this means that $\|P_\nu\psi_t\|^2$ is nearly deterministic. Moreover, Theorem 5 says that also the whole function $t \mapsto \|P_\nu\psi_t\|^2$ is close to the function $t \mapsto \mathbb{E}_\mu\|P_\nu\psi_t\|^2$ on any interval $[0, T]$ in the L^2 norm.

1.3 Contents

The remainder of this paper is organized as follows: In Section 2, we introduce some notation and recall our previous results from [43]. In Section 3, we formulate our main results. In Section 4, we describe the result of Rudelson and Vershynin [37] that we use (with minor corrections, see Theorem 6) and point out which Gaussian random matrices will satisfy their hypotheses (Theorems 8, 9). In Section 5, we prove our main results after formulating them in a somewhat more general version (Theorems 10, 11) that applies to arbitrary operators B instead of P_ν. In Section 6, we prove with the help of a delocalization result from Ajanki, Erdős, and Krüger [1] (quoted here as Theorem 12) some improved lower bounds for the $M_{\mu\nu}$ that are nontrivial if μ or ν is the thermal equilibrium macro state (Theorem 13). In Section 7 we show that sharper estimates for all $M_{\mu\nu}$ follow from the version of the ETH due to Cipolloni, Erdős, and Henheik [6]. In Appendix A, we include the proof that, with probability 1, a random H with continuous distribution has no degeneracies or resonances. In Appendix B, we give more detail about our numerical examples.

2 Prior Results

We quote here the main results of [43] about the absolute errors as Theorems 1, 2, and 3 alongside some related statements. We start with a few definitions and in particular make precise some notions that appeared in the introduction:

Definition 1. Most ψ, most t, and time-averages.
Suppose that for each $\psi \in \mathbb{S}(\mathcal{H})$, the statement $s(\psi)$ is either true or false, and let $\varepsilon > 0$. We say that $s(\psi)$ is true for $(1-\varepsilon)$-*most* $\psi \in \mathbb{S}(\mathcal{H})$ if and only if

$$u\left(\{\psi \in \mathbb{S}(\mathcal{H}) : s(\psi)\}\right) \geq 1-\varepsilon, \tag{13}$$

where u is the normalized uniform measure over $\mathbb{S}(\mathcal{H})$.[3]

Suppose that for each $t \in [0,\infty)$, the statement $s(t)$ is either true or false, and let $T, \delta > 0$. We say that $s(t)$ is true for $(1-\delta)$-*most* $t \in [0,T]$ if and only if

$$\frac{1}{T}\lambda\left(\{t \in [0,T] : s(t)\}\right) \geq 1-\delta, \tag{14}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$. Moreover, we say that $s(t)$ is true for $(1-\delta)$-*most* $t \in [0,\infty)$ if and only if

$$\liminf_{T\to\infty} \frac{1}{T}\lambda\left(\{t \in [0,T] : s(t)\}\right) \geq 1-\delta. \tag{15}$$

[3]We do not consider here other ensembles of wave functions such as GAP [44] or the class of ensembles considered by Reimann in [29].

For any function $f : [0,\infty) \to \mathbb{C}$, define its *time average* as

$$\overline{f(t)} := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt \tag{16}$$

whenever the limit exists.

In the following we consider Hamiltonians with spectral decomposition

$$H = \sum_{e\in\mathcal{E}} e\, \Pi_e, \tag{17}$$

where $\mathcal{E}$ is the set of distinct eigenvalues of H and Π_e the projection onto the eigenspace of H with eigenvalue e.

Definition 2. Relevant properties of the Hamiltonian.
Let $\kappa > 0$. We define the *maximum degeneracy* of an eigenvalue as

$$D_E := \max_{e\in\mathcal{E}} \operatorname{tr}(\Pi_e) \tag{18}$$

and the *maximal number of gaps in an interval of length* κ as

$$G(\kappa) := \max_{E\in\mathbb{R}} \# \left\{(e,e') \in \mathcal{E}\times\mathcal{E} : e \neq e' \text{ and } e - e' \in [E, E+\kappa)\right\}. \tag{19}$$

Moreover, we define the *maximal gap degeneracy* as

$$D_G := \lim_{\kappa\to 0^+} G(\kappa). \tag{20}$$

Definition 3. Asymptotic superposition weights.
Let $B \in \mathcal{L}(\mathcal{H})$. Given any $\psi \in \mathbb{S}(\mathcal{H})$ and any macro state μ, define

$$M_{\psi B} := \sum_e \operatorname{tr}\left(|\psi\rangle\langle\psi|\Pi_e B \Pi_e\right), \tag{21}$$

$$M_{\mu B} := \frac{1}{d_\mu} \sum_e \operatorname{tr}(P_\mu \Pi_e B \Pi_e). \tag{22}$$

In the special case that $B = P_\nu$ for some macro state ν we define

$$M_{\psi\nu} := M_{\psi P_\nu} \tag{23}$$

and

$$M_{\mu\nu} := M_{\mu P_\nu}\,. \tag{24}$$

In [43] we proved the following theorem concerning the absolute errors:

Theorem 1. *Let $B \in \mathcal{L}(\mathcal{H})$, let $\varepsilon, \delta, \kappa, T > 0$ and let μ be an arbitrary macro state. Then $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,T]$*

$$\big|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}\big| \leq 4\left(\frac{D_E G(\kappa)\|B\|}{\delta\varepsilon d_\mu}\left(1+\frac{8\log_2 d_E}{\kappa T}\right)\min\left\{\|B\|, \frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2}, \tag{25}$$

where D_E is the maximum degeneracy of an eigenvalue and D_G is the maximum degeneracy of an eigenvalue gap of H.

By setting $B = P_\nu$ for some macro state ν, choosing κ small enough such that $G(\kappa) = D_G$ and then taking the limit $T \to \infty$ we immediately obtain:

Theorem 2 (Generalized normal typicality: absolute errors)**.** *Let $\varepsilon, \delta > 0$ and let μ, ν be two arbitrary macro states. Then $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,\infty)$*

$$\big|\|P_\nu\psi_t\|^2 - M_{\mu\nu}\big| \leq 4\left(\frac{D_E D_G}{\delta\varepsilon d_\mu}\min\left\{1, \frac{d_\nu}{d_\mu}\right\}\right)^{1/2}. \tag{26}$$

Theorem 2 tells us, roughly speaking, that as soon as

$$d_\mu \gg D_E D_G, \tag{27}$$

i.e., as soon as the dimension of $\mathcal{H}_\mu$ is huge and no eigenvalues or gaps are macroscopically degenerate, the superposition weight $\|P_\nu\psi_t\|^2$ will be close to the fixed value $M_{\mu\nu}$ for most times t and most initial states $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$.

Of course, the fixed value $M_{\mu\nu}$, or more generally $M_{\mu B}$, can be computed by taking the average of over time and over the sphere in $\mathcal{H}_\mu$. This is the content of the following well known proposition, see also [43].

Proposition 1. *Let $B \in \mathcal{L}(\mathcal{H})$. Then*

$$M_{\psi_0 B} = \overline{\langle\psi_t|B|\psi_t\rangle} \text{ and} \tag{28}$$

$$M_{\mu B} = \int_{\mathbb{S}(\mathcal{H}_\mu)} M_{\psi_0 B}\, u_\mu(d\psi_0) = \mathbb{E}_\mu M_{\psi_0 B}, \tag{29}$$

where u_μ is the normalized uniform measure over $\mathbb{S}(\mathcal{H}_\mu)$.

This motivates us to call $M_{\psi\nu}$ the *time average superposition weight* in $\mathcal{H}_\nu$ and $M_{\mu\nu}$ the *full average superposition weight* in $\mathcal{H}_\nu$.

For a system with N particles or more generally N degrees of freedom the dimension D is of order $\exp(N)$. For any macro state μ we define s_μ, the entropy per particle in the macro state μ, by

$$d_\mu =: \exp(s_\mu N/k_B), \tag{30}$$

where k_B is the Boltzmann constant.

With (30) for the dimensions of the macro spaces, we find the following corollary of Theorem 2 whose proof can also be found in [43]:

Corollary 1. *Assume* (30) *and let* $\varepsilon, \delta > 0$. *Then, for all macro states* μ, ν_-, ν_+ *with*

$$s_{\nu_-} \leq s_\mu \leq s_{\nu_+} \tag{31}$$

it holds for $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ *for* $(1-\delta)$*-most* $t \in [0, \infty)$

$$\left| \|P_{\nu_+}\psi_t\|^2 - M_{\mu\nu_+} \right| \leq \frac{4\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{s_\mu N}{2k_B}\right), \tag{32}$$

$$\left| \|P_{\nu_-}\psi_t\|^2 - M_{\mu\nu_-} \right| \leq \frac{4\sqrt{D_E D_G}}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{\left(s_\mu - \frac{s_{\nu_-}}{2}\right) N}{k_B}\right). \tag{33}$$

Corollary 1 shows that fluctuations of the time-dependent superposition weights around their expected values are exponentially small in the number of particles provided that no eigenvalues and gaps are macroscopically degenerate.

In addition to the theorem concerning generalized normal typicality, we also proved a result concerning dynamical typicality in [43]:

Theorem 3. *Let* μ *be an arbitrary macro state and let* B *be any operator on* $\mathcal{H}$. *Let* $w_{\mu B} : \mathbb{R} \to [0,1]$ *be the function defined by*

$$w_{\mu B}(t) := \frac{1}{d_\mu} \operatorname{tr}\left[P_\mu \exp(iHt) B \exp(-iHt)\right] \tag{34}$$

Then, for every $t \in \mathbb{R}$ *and every* $\varepsilon > 0$, $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ *are such that*

$$\left|\langle\psi_t|B|\psi_t\rangle - w_{\mu B}(t)\right| \leq \min\left\{\frac{\|B\|}{\sqrt{\varepsilon d_\mu}}, \sqrt{\frac{\|B\| \operatorname{tr}(|B|)}{\varepsilon d_\mu^2}}, \sqrt{\frac{18\pi^3 \log(4/\varepsilon)}{d_\mu}}\|B\|\right\}. \tag{35}$$

Moreover, for every μ *and* B, *every* $T > 0$, *and* $(1-\varepsilon)$*-most* $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,

$$\frac{1}{T}\int_0^T \left|\langle\psi_t|B|\psi_t\rangle - w_{\mu B}(t)\right|^2 dt \leq \frac{\|B\|^2}{\varepsilon d_\mu}. \tag{36}$$

We note first that $w_{\mu B}(t)$ is exactly the ensemble average of $\langle\psi_t|B|\psi_t\rangle$, i.e., the average over $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$:

$$w_{\mu B}(t) = \mathbb{E}_\mu \langle\psi_t|B|\psi_t\rangle\,. \tag{37}$$

In particular, by Fubini's theorem,

$$\overline{w_{\mu B}(t)} = M_{\mu B}\,. \tag{38}$$

Eq. (35) offers three estimates that can be useful depending on the sizes of B, ε, and d_μ. Specifically, if $\|B\|$ is of order 1 and $d_\mu \gg 1/\varepsilon$, then the first estimate implies that $\langle\psi_t|B|\psi_t\rangle$ is close to its average $w_{\mu B}(t)$. We note further that the first two estimates in (35) can be obtained by similar arguments as in the proof of Theorem 1 whereas the third estimate is a consequence of Lévy's lemma (see e.g. [42, Sec. II.C]).

In the following we can assume without loss of generality $D_E = D_G = 1$ since we consider random matrices whose joint distribution of their entries is absolutely continuous with respect to the Lebesgue measure. The set of matrices with degenerate eigenvalues or eigenvalue gaps – we also say that the matrix has *resonances* – has Lebesgue measure zero and therefore, with probability 1, $D_E = D_G = 1$. For the convenience of the reader, we give the proof of this statement in Appendix A.

3 Main Results

In this section we present and discuss our main results concerning lower bounds for the $M_{\mu\nu}$ and therefore generalized normal typicality (Theorem 4) as well as a corollary thereof bounding the relative errors (Corollary 2) and a result bounding the comparative error for dynamical typicality (Theorem 5). In all the results presented in this section it is assumed that the Hamiltonian is of a special form and we only consider projections P_ν onto a macro space $\mathcal{H}_\nu$. More general results and the proofs can be found in Section 5.

3.1 Generalized Normal Typicality

We make the following assumption:

Assumption 1. The Hamiltonian H is of the form $H = H_0 + V$, where H_0 is a (deterministic) Hermitian $D \times D$ matrix and V is a Hermitian random Gaussian perturbation, more precisely, $V = \frac{1}{\sqrt{2}}(A + A^*)$, where $A = (a_{ij})$ is a random $D \times D$ matrix with independent Gaussian entries with mean zero, i.e., all random variables $\operatorname{Re} a_{ij}, \operatorname{Im} a_{ij}, i, j \in [D]$, are independent and $\operatorname{Re} a_{ij}, \operatorname{Im} a_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2/2)$ for some $\sigma_{ij} > 0$ that fulfill $\sigma_{ij} = \sigma_{ji}$.

Note that because of Assumption 1 the matrix V is Hermitian and Gaussian, more precisely, $\operatorname{Re} v_{ij}, \operatorname{Im} v_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2)$ for $i \neq j$ and $v_{ii} = \operatorname{Re} v_{ii} \sim \mathcal{N}(0, \sigma_{ii}^2)$, i.e., V is a Hermitian Gaussian Wigner-type matrix.

For a Hamiltonian as in Assumption 1 we define

$$C_{H_0} := D^{-1/2} \max\{\|\operatorname{Re} H_0\|, \|\operatorname{Im} H_0\|\}. \tag{39}$$

Note that for a typical many-body Hamiltonian we expect that $\|\operatorname{Re} H_0\|, \|\operatorname{Im} H_0\| \sim \log D$ and therefore C_{H_0} should be very small for large D.

Theorem 4 (Lower bounds for $M_{\mu\nu}$)**.** *Let $\varepsilon' \in (0, \frac{1}{2})$ and let μ and ν be arbitrary macro states such that $d_\mu, d_\nu > \max\{166, 4|\log_2(\varepsilon'/\sqrt{2})|\}$. Let H satisfy Assumption 1. Let $\sigma_- := \min_{i,j} \sigma_{ij}$ and $\sigma_+ := \max_{i,j} \sigma_{ij}$. Then with probability at least $1 - \varepsilon'$,*

$$M_{\mu\nu} \geq \left(\sqrt{\varepsilon' C_\sigma} \, \frac{\max\{d_\mu, d_\nu\}}{D} \right)^{16} \min\left\{1, \frac{d_\nu}{d_\mu}\right\}, \tag{40}$$

where

$$C_\sigma := \frac{c_- \sigma_-}{c_+ \sigma_+ + C_{H_0}} \tag{41}$$

with C_{H_0} defined in (39) *and absolute constants $c_-, c_+ > 0$.*

The main application of this theorem is provided by our next result: the analogue of Corollary 1 for the relative errors. In addition to (30), define s_{mc} by

$$D = \exp\left(s_{\mathrm{mc}} N / k_B\right). \tag{42}$$

Since we expect that the dimension of the thermal equilibrium macro space d_{eq} is extremely large, it should hold that $s_{\mathrm{mc}} \approx s_{\mathrm{eq}}$.

Corollary 2 (Generalized normal typicality – relative errors)**.** *Let $\varepsilon, \delta > 0$, $\varepsilon' \in (0, \frac{1}{2})$ and let μ and ν be macro states such that $d_\mu, d_\nu > \max\{166, 4|\log_2(\varepsilon'/\sqrt{2})|\}$. Let H be a random Hermitian $D \times D$ matrix as in Theorem 4. Then with probability at least $1 - \varepsilon'$, $(1 - \varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1 - \delta)$-most $t \in [0, \infty)$,*

$$\frac{\left| \|P_\nu \psi_t\|^2 - M_{\mu\nu} \right|}{M_{\mu\nu}} \leq \frac{4}{\sqrt{\varepsilon\delta}} (C_\sigma \varepsilon')^{-8} \exp\left(-\frac{N}{2k_B} \left(\min\{s_\mu, s_\nu\} - 32(s_{\mathrm{mc}} - \max\{s_\mu, s_\nu\}) \right) \right), \tag{43}$$

where C_σ is defined in (41).

The most important aspects of this bound are that it bounds the *relative* error, it shrinks exponentially as N increases, and the rate of shrinking can be given explicitly.

Remark 1. It is sometimes desirable to consider a sequence of systems with increasing dimension $D \to \infty$, corresponding for example to an increasing particle number $N \to \infty$, for which it makes sense to talk of the "same" macro states ν for each member of the sequence. In that case, suppose that C_σ is bounded below away from zero uniformly in D (as C_{H_0} is small in typical models, this is basically a condition on σ_- and σ_+). Then the relative errors in Corollary 2 are small if $s_{\mathrm{mc}} < \max\{s_\mu, s_\nu\} + \min\{s_\mu, s_\nu\}/32$. Since we expect that $s_{\mathrm{mc}} \approx s_{\mathrm{eq}}$, one of the macro spaces $\mathcal{H}_\nu$ or $\mathcal{H}_\mu$ thus needs to have specific entropy not too far from the one of the equilibrium macro state. While this might seem quite restrictive at first sight, observe that even if we assume that the $M_{\mu\nu}$ scale like in the case of normal typicality, i.e.,

$$M_{\mu\nu} \approx \frac{d_\nu}{D} \approx \exp\left(-\frac{s_{\mathrm{mc}} - s_\nu}{k_B} N\right), \tag{44}$$

then the relative errors are small only if $s_{\mathrm{mc}} < \max\{s_\nu, s_\mu\} + \min\{s_\nu, s_\mu\}/2$ (recall Theorem 2 for the absolute errors), see also the discussion in [43].

Remark 2. Suppose again that C_{H_0} is bounded uniformly in D and that $\sigma_\pm = D^{\alpha_\pm}$ with $\alpha_- \leq \alpha_+$. In this case the upper bound for the relative errors becomes

$$\begin{aligned} \frac{\left|\|P_\nu \psi_t\|^2 - M_{\mu\nu}\right|}{M_{\mu\nu}} &\leq \frac{4}{\sqrt{\varepsilon\delta}(c_-\varepsilon')^8} \left(c_+ \exp(\alpha_+ s_{\mathrm{mc}} N/k_B) + C_{H_0}\right)^8 \times \\ &\times \exp\left(-\frac{N}{2k_B}\left(\min\{s_\mu, s_\nu\} - 32\left(\left(1 - \frac{\alpha_-}{2}\right) s_{\mathrm{mc}} - \max\{s_\mu, s_\nu\}\right)\right)\right). \end{aligned} \tag{45}$$

Suppose first that $\alpha_+ < 0$. Then, for large N, the second factor can be bounded by $(2C_{H_0})^8$ and the right-hand side is small if the expression in the exponential function is negative, i.e., if

$$\left(1 - \frac{\alpha_-}{2}\right) s_{\mathrm{mc}} < \max\{s_\mu, s_\nu\} + \frac{1}{32}\min\{s_\mu, s_\nu\}. \tag{46}$$

Since $\alpha_- < 0$ the macro state μ or ν has to be even closer to the equilibrium macro state than in the case that the variances σ_-, σ_+ do not depend on the dimension D. If $\alpha_+ = 0$ then the second factor in (45) can also be bounded by some constant and we obtain again (46) as a condition for the relative errors to be small. If, however, $\alpha_+ > 0$, the second factor can be bounded by a quantity proportional to $\exp(8\alpha_+ s_{\mathrm{mc}} N/k_B)$ and therefore the right-hand side of (45) is small if

$$\left(1 + \frac{\alpha_+ - \alpha_-}{2}\right) s_{\mathrm{mc}} < \max\{s_\mu, s_\nu\} + \frac{1}{32}\min\{s_\mu, s_\nu\}. \tag{47}$$

Thus, if $\alpha_+ > \alpha_-$, then the macro state μ or ν again has to be closer to the equilibrium macro state than in the case discussed in Remark 1. Note that, of course, we can also assume that $\sigma_\pm = c_{\sigma_\pm} D^{\alpha_\pm}$ with constants $c_{\sigma_\pm} > 0$ and the conditions under which the relative errors are small remain the same since the upper bounds only change by multiplicative constants.

3.2 Dynamical Typicality

We now turn to our main result about dynamical typicality. We would like to give again a bound on the *relative error*, but this we can only provide for *most* (rather than *all*) times. But for *most* times, we already know from generalized normal typicality that $\|P_\nu\psi_t\|^2$ is close to $M_{\mu\nu}$ (as visible in Figure 2). On the other hand, for dynamical typicality, we are particularly interested in times before normal equilibration has set in (as in Figure 1). For this situation, we can provide a bound on what we call the *comparative error* and that is given by the quotient of the absolute error (which in this case is time-dependent) and the time average of the quantity considered. In this way, we compare the absolute error to a comparison value that expresses in a way what magnitude to expect of $\|P_\nu\psi_t\|^2$. Smallness of the comparative error at time t means that the absolute error at t is much smaller than the long-time value of $\|P_\nu\psi_t\|^2$, although not necessarily much smaller than the instantaneous ensemble average

$$\mathbb{E}_\mu\|P_\nu\psi_t\|^2 = w_{\mu P_\nu}(t) =: w_{\mu\nu}(t)\,. \tag{48}$$

This statement still gives us a handle on comparatively small $\mathcal{H}_\nu$ for which $\|P_\nu\psi_t\|^2$ is small in absolute terms for most t. Now recall from (28) that

$$M_{\psi_0\nu} = \overline{\|P_\nu\psi_t\|^2}\,. \tag{49}$$

Theorem 5 (Dynamical typicality – comparative errors)**.** *Let $\varepsilon > 0$, $\varepsilon' \in (0, \frac{1}{2})$ and let μ and ν be macro states such that $d_\nu > \max\{166, 4|\log_2\varepsilon'|\}$. Let H be a random Hermitian $D \times D$ matrix as in Theorem 4. Then with probability at least $1-\varepsilon'$, for each $t \in \mathbb{R}$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that*

$$\frac{\big|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)\big|}{M_{\psi_0\nu}} \leq \frac{1}{\sqrt{\varepsilon}}(C_\sigma\varepsilon')^{-8}\exp\left(-\frac{N}{2k_B}\left(2s_\mu - \min\{s_\mu, s_\nu\} - 32(s_{\mathrm{mc}} - s_\nu)\right)\right) \tag{50}$$

where C_σ is defined in (41)*. Moreover, with probability $1-\varepsilon'$, for every $T > 0$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that*

$$\frac{1}{T}\int_0^T \frac{\big|\|P_\nu\psi_t\|^2 - w_{\mu\nu}(t)\big|^2}{M^2_{\psi_0\nu}}\,dt \leq \frac{1}{\varepsilon}(C_\sigma\varepsilon')^{-16}\exp\left(-\frac{N}{k_B}\left(s_\mu - 32(s_{\mathrm{mc}} - s_\nu)\right)\right). \tag{51}$$

Again, key aspects are that the upper bounds in (50) and (51) shrink exponentially with increasing N at explicit rates. The bounds are small if $s_{\mathrm{mc}} < s_\nu + s_\mu/16 - \min\{s_\mu, s_\nu\}/32$ resp. $s_{\mathrm{mc}} < s_\nu + s_\mu/32$, i.e., s_ν must be close to s_{mc} and therefore close to s_{eq}. In that case, (51) shows that the curve $t \mapsto \|P_\nu\psi_t\|^2$ is close to the curve $t \mapsto \mathbb{E}_\mu\|P_\nu\psi_t\|^2$ in the L^2-sense even when compared to the (possibly small) time average $\overline{\|P_\nu\psi_t\|^2}$.

4 No-Gaps Delocalization and Band Matrices

Our main results are based on a "no-gaps delocalization" result of Rudelson and Vershynin (2016) [37, Theorem 1.3]. In this section, we state this result (with certain corrections, Theorem 6 in Section 4.1), provide an extension of it that we will use when proving our main results (Theorem 7 in Section 4.2), and provide examples of random matrices for which the hypotheses of Rudelson and Vershynin are satisfied (Theorems 8 and 9 in Section 4.3).

4.1 Statement of the No-Gaps Delocalization Result of Rudelson and Vershynin

Since explicit exponents and the dependence of constants on the parameters play an important role for our purposes, we have followed the exponents and constants through the proof of Rudelson and Vershynin in [37]. It turned out that in some places we arrived at different values than stated in [37]. In this subsection, we state the relevant result with these values, and provide a discussion of how we arrived at them.

In the following we use the following abbreviation. For a random $D \times D$ matrix H and any number $J > 0$ we introduce the "boundedness event"

$$\mathcal{B}_{H,J} := \left\{ \|H\| \leq J\sqrt{D} \right\}. \tag{52}$$

It will turn out in Section 4.3 that for relevant examples of random matrices H and large D, the event $\mathcal{B}_{H,J}$ occurs with high probability; this conclusion makes use of a result of Latala [22].

So here is our adjusted statement of Theorem 1.3 of [37].

Theorem 6 (No-gaps delocalization; Rudelson, Vershynin (2016))**.** *Let $H = (h_{ij})$ be a $D \times D$ random matrix such that for $i, j \in [D]$ the (continuous) random variable $\operatorname{Re} h_{ij}$ is independent of the other entries of $\operatorname{Re} H$ except possibly $\operatorname{Re} h_{ji}$, the densities $\varrho_{ij}^{\operatorname{Re}}$ of $\operatorname{Re} h_{ij}$ are bounded by some number $K \geq 1$ and the imaginary part is fixed. Choose $J \geq 1$ such that the boundedness event $\mathcal{B}_{H,J}$ holds with probability at least $1/2$. Let $\kappa \in (180/D, 1/2)$ and $s > 0$. Then, conditionally on $\mathcal{B}_{H,J}$, the following holds with probability at least $1 - (Cs)^{\kappa D}$: Every eigenvector v of H satisfies*

$$\|v_I\| \geq (\kappa s)^9 \|v\| \quad \textit{for all } I \subset [D] \textit{ with } |I| \geq \kappa D, \tag{53}$$

where

$$\|v_I\| := \left(\sum_{j \in I} |v_j|^2 \right)^{1/2} \tag{54}$$

and $C = C(K, J) \geq 1$.

Here is how this statement differs from Theorem 1.3 in [37]. The lower bound on ε is changed from $8/D$ to $180/D$ and the exponent of κs in (53) from 6 to 9. Therefore the theorem tells us that subsets $I \subset [D]$ of at least 180 (instead of 8) coordinates of any eigenvector carry a non-negligible part of its mass. Moreover, the lower bound obtained for $\|v_I\|$ is slightly worse since $\kappa < 1$ by assumption and we can assume without loss of generality that also $s < 1$ because otherwise the lower bound on the probability, $1 - (Cs)^{\kappa D}$, is negative.

Here is how we arrived at these exponents and constants. In the rest of this section, all numbers of theorems, subsections etc. refer to the ones in [37] if not stated otherwise. We start from Theorem 5.1 (Section 5.1) to derive Theorem 1.3. Let C_2 be the constant C in Theorem 5.1 and define for $\delta, \kappa > 0$ the *gap event* (called the *localization event* in [37])

$$\mathrm{Gap}(H, \kappa, \delta) := \left\{\exists \text{ eigenvector } v \in \mathbb{C}^D, \|v\| = 1, \exists I \subset [D], |I| = \kappa D : \|v_I\| < \delta\right\}, \tag{55}$$

i.e., the event that in an eigenvector v of H a fraction κ of basis vectors is underrepresented in v.

After an application of Proposition 4.1 one arrives at

$$\mathbb{P}\left(\mathrm{Gap}(H, \kappa, t/8J) \text{ and } \mathcal{B}_{H,J}\right) \leq 5\left(\frac{8J}{t}\right)^2 (e/\kappa)^{\kappa D} p_0 \tag{56}$$

where $t \geq 0$ and $p_0 = (2C_2 K J t^{0.4} \kappa^{-1.4})^{\kappa D/2}$. Set $t = 8J(\kappa s)^\alpha$ for some $\alpha > 0$. (In [37], $\alpha = 6$ was stated, while we arrive at $\alpha = 9$ below, so let us keep the value unspecified for a little while.) What we want to show is that

$$\mathbb{P}\Big(\mathrm{Gap}(H, \kappa, (\kappa s)^\alpha)\Big|\mathcal{B}_{H,J}\Big) \leq (Cs)^{\kappa D}, \tag{57}$$

where C can depend on J and K but not on κ, s, or D. Since $\mathbb{P}(\mathcal{B}_{H,J}) \geq 1/2$, it suffices to bound $\mathbb{P}(\mathrm{Gap}(H, \kappa, (\kappa s)^\alpha) \cap \mathcal{B}_{H,J})$ in a similar way. From (56) we obtain that

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{Gap}(H, \kappa, (\kappa s)^\alpha) \text{ and } \mathcal{B}_{H,J}\right) \\
&\quad\leq 5(\kappa s)^{-2\alpha}(e/\kappa)^{\kappa D}\left(2C_2 K J t^{0.4}\kappa^{-1.4}\right)^{\kappa D/2} && \text{(58a)}\\
&\quad= \left(5^{1/\kappa D}\kappa^{-2\alpha/\kappa D+0.2\alpha-1.7} s^{-2\alpha/\kappa D+0.2\alpha-1} e\sqrt{2C_2KJ}(8J)^{0.2}s\right)^{\kappa D} && \text{(58b)}\\
&\quad\leq \left(\kappa^{-2\alpha/\kappa D+0.2\alpha-1.7} s^{-2\alpha/\kappa D+0.2\alpha-1} 5e\sqrt{2C_2KJ}(8J)^{0.2}s\right)^{\kappa D}. && \text{(58c)}
\end{aligned}$$

So we would like a constant (independent of κ, s, D) as an upper bound for

$$\kappa^{-2\alpha/\kappa D+0.2\alpha-1.7} s^{-2\alpha/\kappa D+0.2\alpha-1}, \tag{59}$$

where $\kappa < 1/2$ and $s < 1$. However, in the limit $\kappa D \to 0$, it follows that $-2\alpha/\kappa D \to -\infty$ and $\kappa^{-2\alpha/\kappa D} \to \infty$, so we need to exclude κ values from a neighborhood of 0 and require, say, $\kappa \geq \ell/D$ for some $\ell \in \mathbb{N}$ (as explicitly done in [37] and in Theorem 6). But then κ can still be quite close to 0, and its exponent in (59) should be ≥ 0 or else (59) will not be bounded as $D \to \infty$, so we want that

$$-\frac{2\alpha}{\kappa D} + 0.2\alpha - 1.7 \geq 0\,. \tag{60}$$

In particular, for $\alpha = 6$ as in [37], this exponent would always be negative, and (59) for $\kappa = \ell/D$ would not be bounded. However, if we suppose that $\ell > 10$ and $\alpha \geq \frac{17\ell}{2(\ell-10)}$, then, for every $\kappa \geq \ell/D$,

$$\begin{aligned} -\frac{2\alpha}{\kappa D} + \frac{\alpha}{5} - \frac{17}{10} &\geq -\frac{2\alpha}{\ell} + \frac{\alpha}{5} - \frac{17}{10} && \text{(61a)} \\ &= \alpha\frac{\ell-10}{5\ell} - \frac{17}{10} && \text{(61b)} \\ &\geq 0. && \text{(61c)} \end{aligned}$$

In this case we obtain that

$$\begin{aligned} \mathbb{P}\left(\mathrm{Gap}\left(H, \kappa, (\kappa s)^{\alpha}\right) \text{ and } \mathcal{B}_{H,J}\right) &\leq \left(5e\sqrt{2C_2KN}(8J)^{0.2}s\right)^{\kappa D} && \text{(62)} \\ &=: \left(\tilde{C}s\right)^{\kappa D} && \text{(63)} \end{aligned}$$

with $\tilde{C} = \tilde{C}(K,J) = 5e\sqrt{2C_2KJ}(8J)^{0.2}$ being the desired constant.[4] Note that $\ell \mapsto \frac{17\ell}{2(\ell-10)}$ for $\ell > 10$ is monotonically decreasing and $\lim_{\ell\to\infty} \frac{17\ell}{2(\ell-10)} = 8.5$. Therefore we can choose $\alpha = 9$ and $\ell = 180$ to get $-2\alpha/\kappa D + 0.2\alpha - 1.7 \geq 0$.

Remark 3 (Constant in Theorem 6)*.* Since the constant $C(K,J)$ in Theorem 6 will appear in the upper bound for the relative errors and we want to allow the case that K and J depend on D, we are interested in the dependence of C on these two parameters. In [37] it is said that all appearing constants (denoted by $C, c, \ldots$) might depend on K and J and therefore we carefully investigated the constants in the theorems, propositions, lemmas etc. needed for the proof of their main theorem (here Theorem 6). To avoid confusion with the constant C in the main theorem, we denote the other constants that are also denoted as C in [37] as $C_1, C_2, \ldots$ in the order of their appearance; other constants that appear multiple times with a different value as well are also numbered in the following in the order of their appearance. An inspection of the proofs reveals that

- $C_1 = e$ in Lemma 3.3,

[4] More precisely, $C = 2\tilde{C}$, where C is the constant in Theorem 6. This is due to the fact that $\mathbb{P}\left(\mathrm{Gap}(H, \kappa, (\kappa s)^{\alpha}\right)$ and $\mathcal{B}_{H,J})$ and not $\mathbb{P}\left(\mathrm{Gap}(H, \kappa, (\kappa s)^{\alpha})|\mathcal{B}_{H,J}\right)$ is considered in the proof of Theorem 1.3.

- $C = 2\tilde{C}$, where $\tilde{C}$ is defined above,
- $C_2 = 2\max\left\{\left(\frac{4}{\tilde{c}_1}\right)^{0.9} C_6, C_4\sqrt{\tilde{c}_2}\right\}$ in Theorem 5.1, where $\tilde{c}_2$ is an absolute constant that bounds the constants $C(p)$ in (5.7) for $p \geq 2$, see also Theorem 3.21 in [41], and $\tilde{c}_1 \in (0, 0.2)$ is another absolute constant that can be chosen arbitrarily, see also Section 5.4.2 and 5.5.3 (note that in (5.9) a square is missing); the values of the other constants are given below,
- $C_3 = \frac{1}{4}C_2$ in (5.1),
- $C_4 = \sqrt{2}\hat{C}\bar{C}$ in Lemma 5.4, where $\hat{C}, \bar{C} > 0$ are absolute constants; more precisely, $\hat{C}$ appears in the upper bound for the density of a random vector PX where $X = (X_1, \ldots, X_D)$ is a vector of real-valued independent random variables whose densities are bounded by some constant a.e. and P is an orthogonal projection onto a subspace of dimension d, see also [36] for details; moreover, the constant $\bar{C}$ appears in the upper bound of the volume of a d-dimensional ball of radius $\tau\sqrt{d}$ for some $\tau > 0$[5],
- $C_{0,1} = 4$ in Lemma 5.5,
- $C_{0,2} = 4e$ in Lemma 5.6,
- $C_5 = 72e$ in Lemma 5.7,
- $C_6 = 72e$ in Lemma 5.8.

Thus, we see that the constant C_2 depends neither on K nor on J, and we conclude from our computation above that $C \sim \sqrt{K}J^{0.7}$.

4.2 An Extension

In Theorem 6, the imaginary part of the random matrix is fixed, and it is assumed that $J \geq 1$. In this section we present and prove a theorem that covers matrices with random imaginary part as well, that also allows $0 < J \leq 1$ and we improve the exponent of κs in the lower bound for $\|v_I\|$ from 9 to 8. More precisely, we prove the following theorem:

[5]The constant $\bar{C}$ can be chosen as $\bar{C} = \sqrt{2\pi e}$ which can be easily seen as follows: For $\tau > 0$, the volume of a d-dimensional ball with radius $\tau\sqrt{d}$ is given by

$$\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\left(\tau\sqrt{d}\right)^d \leq \left(\frac{e}{\frac{d}{2}+1}\right)^{d/2}\left(\sqrt{\pi}\tau\sqrt{d}\right)^d \leq \left(\sqrt{2\pi e}\tau\right)^d = \left(\bar{C}\tau\right)^d \tag{64}$$

Theorem 7. *Let $H = (h_{ij})$ be a $D \times D$ random matrix such that $\operatorname{Re} H$ and $\operatorname{Im} H$ are independent, for $i, j \in [D]$ the (continuous) random variable $\operatorname{Re} h_{ij}$ is independent of the other entries of $\operatorname{Re} H$ except possibly $\operatorname{Re} h_{ji}$, for $i, j \in [D], i \neq j$ the (continuous) random variable $\operatorname{Im} h_{ij}$ is independent of the other entries of $\operatorname{Im} H$ except possibly $\operatorname{Im} h_{ji}$, and the densities $\varrho_{ij}^{\mathrm{Re}}$ are bounded by some number $K > 0$. Choose $J > 0$ such that $JK \geq 1$ and the boundedness events $\mathcal{B}_{\operatorname{Re} H,J}$ and $\mathcal{B}_{\operatorname{Im} H,J}$ hold with probability at least $1 - \eta$ for some $0 < \eta \leq 1/2$. Let $\kappa \in (83/D, 1/2)$ and $0 < s \leq 1$. Then the following holds with probability at least $\left(1 - (c_c\sqrt{KJ}s)^{\kappa D}\right)(1-\eta)^4$, where $c_c \geq 1$ is a universal constant: Every eigenvector v of H satisfies*

$$\|v_I\| \geq (\kappa s)^8 \|v\| \quad \text{for all } I \subset [D] \text{ with } |I| \geq \kappa D\,. \tag{65}$$

Proof. We prove the theorem in three steps. The first step is to establish a modification of Theorem 6 where the imaginary part of the matrix H can also be random. The second step is to relax the condition $J \geq 1$ to $J > 0$ (in this case we have to additionally assume that $JK \geq 1$), which can be done via a simple scaling argument. Finally, in the third step, we show that by a small modification in the proof of Rudelson and Vershynin, the lower bound for $\|v_I\|_2$ can be slightly improved.

1. Step (the complex case): We first assume that $J, K \geq 1$ and that $\kappa \in (180/D, 1/2)$. Let

$$\begin{aligned} S &:= \{\text{Every eigenvector } v \text{ of } H \text{ satisfies } \|v_I\| \geq (\kappa s)^9 \|v\| \\ &\qquad \text{for all } I \subset [D] \text{ with } |I| \geq \kappa D\}, \end{aligned} \tag{66}$$

$$E := \left\{ Q \in \mathbb{R}^{D\times D} : \mathbb{P}\left(\mathcal{B}_{\operatorname{Re} H + iQ, 2J}\right) \geq \frac{1}{2} \right\}. \tag{67}$$

By the law of total expectation and the monotonicity of the conditional expectation we obtain that

$$\begin{aligned} \mathbb{P}(S|\mathcal{B}_{H,2J}) &= \mathbb{E}\left(\mathbb{P}\left(S|\mathcal{B}_{H,2J}, \operatorname{Im} H\right)|\mathcal{B}_{H,2J}\right) && \text{(68a)}\\ &\geq \mathbb{E}\left(\mathbb{1}_{\{\operatorname{Im} H\in E\}}\mathbb{P}\left(S|\mathcal{B}_{H,2J}, \operatorname{Im} H\right)|\mathcal{B}_{H,2J}\right) && \text{(68b)}\\ &\geq \left(1-(Cs)^{\kappa D}\right)\mathbb{E}\left(\mathbb{1}_{\{\operatorname{Im} H\in E\}}|\mathcal{B}_{H,2J}\right) && \text{(68c)}\\ &= \left(1-(Cs)^{\kappa D}\right)\mathbb{P}\left(\operatorname{Im} H \in E|\mathcal{B}_{H,2J}\right) && \text{(68d)}\\ &\geq \left(1-(Cs)^{\kappa D}\right)\mathbb{P}\left(\{\operatorname{Im} H \in E\}\cap\mathcal{B}_{H,2J}\right), && \text{(68e)} \end{aligned}$$

where $C = C(K, 2J) \geq 1$ is the constant in Theorem 6, and Theorem 6 itself was applied in the third line. Next note that

$$\mathcal{B}_{\operatorname{Re} H,J} \cap \mathcal{B}_{\operatorname{Im} H,J} \subset \mathcal{B}_{H,2J}\,, \tag{69}$$

and thus

$$\mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{H,2J}\right) \geq \mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{\operatorname{Re} H,J} \cap \mathcal{B}_{\operatorname{Im} H,J}\right) \tag{70a}$$
$$= \mathbb{P}\left(\mathcal{B}_{\operatorname{Re} H,J}\right) \mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{\operatorname{Im} H,J}\right) \tag{70b}$$
$$\geq (1-\eta)\, \mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{\operatorname{Im} H,J}\right), \tag{70c}$$

where we used that $\operatorname{Re} H$ and $\operatorname{Im} H$ are independent. Next observe that for $Q \in \mathbb{R}^{D\times D}$ such that $\|Q\| \leq J\sqrt{D}$, we have that

$$\mathbb{P}\left(\mathcal{B}_{\operatorname{Re} H+iQ,2J}\right) \geq \mathbb{P}\left(\mathcal{B}_{\operatorname{Re} H,J}\right) \geq 1-\eta \geq \frac{1}{2}. \tag{71}$$

It follows that $\mathcal{B}_{\operatorname{Im} H,J} \subset \{\operatorname{Im} H \in E\}$ and thus

$$\mathbb{P}\left(\{\operatorname{Im} H \in E\} \cap \mathcal{B}_{H,2J}\right) \geq (1-\eta)\, \mathbb{P}\left(\mathcal{B}_{\operatorname{Im} H,J}\right) \geq (1-\eta)^2. \tag{72}$$

Therefore, we finally obtain that

$$\mathbb{P}\left(S|\mathcal{B}_{H,2J}\right) \geq \left(1-(Cs)^{\kappa D}\right)(1-\eta)^2, \tag{73}$$

which implies

$$\mathbb{P}(S) \geq \mathbb{P}\left(S \cap \mathcal{B}_{H,2J}\right) = \mathbb{P}\left(S|\mathcal{B}_{H,2J}\right) \mathbb{P}\left(\mathcal{B}_{H,2J}\right) \tag{74a}$$
$$\geq \mathbb{P}\left(S|\mathcal{B}_{H,2J}\right) \mathbb{P}\left(\mathcal{B}_{\operatorname{Re} H,J}\right) \mathbb{P}\left(\mathcal{B}_{\operatorname{Im} H,J}\right) \tag{74b}$$
$$\geq \left(1-(Cs)^{\kappa D}\right)(1-\eta)^4. \tag{74c}$$

This shows that for complex random $D\times D$ matrices H which satisfy the assumptions of this theorem with $J, K \geq 1$ (instead of $K, J > 0$ such that $JK \geq 1$) and $\kappa \in (180/D, 1/2)$ (instead of $\kappa \in (83/D, 1/2)$), the following holds with probability at least $\left(1-(Cs)^{\kappa D}\right)(1-\eta)^4$: Every eigenvector v of H satisfies

$$\|v_I\| \geq (\kappa s)^9 \|v\| \quad \text{for all } I \subset [D] \text{ with } |I| \geq \kappa D, \tag{75}$$

where $C = C(K, 2J) \geq 1$ and $C(K, 2J) \sim \sqrt{K} J^{0.7}$.

2. Step ($J > 0$): As in the first step of the proof we assume that $\kappa \in (180/D, 1/2)$. Note that if the density of a random variable X is bounded by some constant K, then the density of $J^{-1}X$ is bounded by JK. Therefore, applying the result of the first step to the matrix $\tilde{H} := J^{-1}H$ immediately shows that for complex random $D \times D$ matrices H which satisfy the assumptions of this theorem with $\kappa \in (180/D, 1/2)$ (instead of $\kappa \in (83/D, 1/2)$), the following holds with probability at least $\left(1-(\tilde{c}s)^{\kappa D}\right)(1-\eta)^4$: Every eigenvector v of H satisfies

$$\|v_I\| \geq (\kappa s)^9 \|v\| \quad \text{for all } I \subset [D] \text{ with } |I| \geq \kappa D, \tag{76}$$

where $\tilde{c} = C(KJ, 2) \sim \sqrt{KJ}$.

3. Step (improved bound): We first assume that the imaginary part of H is fixed and that $K, J \geq 1$. In the proof of the Invertibility Theorem 5.1 at the end of Section 5 in [37], they arrive at the upper bound

$$\left[C_6 K J^{0.1} \kappa^{-0.05} \left(\frac{4Jt}{\tilde{c}_1 \tau \kappa^{3/2}} \right)^{0.9} \right]^{\kappa D} + \left(C_4 \sqrt{\tilde{c}_2} K \tau \right)^{\kappa D} \tag{77}$$

for the probability on the left-hand side of (5.18), where $t, \tau > 0$ are arbitrary and where we used the notation for the constants introduced above. In their paper they choose $\tau = \sqrt{t}$ and then they use that they can assume that $t \leq 1$ to bound $t^{0.45}$ and $t^{0.5}$ by $t^{0.4}$. The result can be slightly improved by choosing $\tau = t^{9/19}$ instead. With this choice, the upper bound (77) becomes

$$\left[\left(\frac{4}{\tilde{c}_1} \right)^{0.9} C_6 K J \kappa^{-1.4} t^{9/19} \right]^{\kappa D} + \left(C_4 \sqrt{\tilde{c}_2} K t^{9/19} \right)^{\kappa D} \leq \left(C_2 K J \kappa^{-1.4} t^{9/19} \right)^{\kappa D}, \tag{78}$$

where $C_2 = 2 \max \left\{ \left(\frac{4}{\tilde{c}_1} \right)^{0.9} C_6, C_4 \sqrt{\tilde{c}_2} \right\}$. This shows that the exponent of t in Theorem 5.1 in [37] can be changed from 0.4 to 9/19. This change has implications for the derivation of the no-gaps delocalization theorem from Theorem 5.1: After an application of Proposition 4.1 one now arrives at

$$\mathbb{P} \left(\mathrm{Gap}(H, \kappa, t/8J) \text{ and } \mathcal{B}_{H,J} \right) \leq 5 \left(\frac{8J}{t} \right)^2 (e/\kappa)^{\kappa D} p_0 \tag{79}$$

with $t \geq 0$ and $p_0 = (2 C_2 K J t^{9/19} \kappa^{-1.4})^{\kappa D/2}$. Again we set $t = 8J(\kappa s)^\alpha$ for some $\alpha > 0$. Then a computation similar to the one in the proof of Theorem 6 shows that

$$\begin{aligned} &\mathbb{P} \left(\mathrm{Gap}(H, \kappa, (\kappa s)^\alpha) \text{ and } \mathcal{B}_{H,J} \right) \\ &\qquad \leq \left(\kappa^{-2\alpha/\kappa D + 9\alpha/38 - 1.7} s^{-2\alpha/\kappa D + 9\alpha/38 - 1} 5e \sqrt{2 C_2 K J} (8J)^{9/38} s \right)^{\kappa D}. \end{aligned} \tag{80}$$

In order to obtain an upper bound for $\kappa^{-2\alpha/\kappa D + 9\alpha/38 - 1.7} s^{-2\alpha/\kappa D + 9\alpha/38 - 1}$ we have to determine α such that $-2\alpha/\kappa D + 9\alpha/38 - 1.7 \geq 0$ since $\kappa < 1/2$ and $s \leq 1$. Suppose again that $\kappa \geq \frac{\ell}{D}$ for some $\ell \in \mathbb{N}, \ell > 8$, and $\alpha \geq \frac{323\ell}{5(9\ell - 76)}$. Then

$$-\frac{2\alpha}{\kappa D} + \frac{9\alpha}{38} - \frac{17}{10} \geq -\frac{2\alpha}{\ell} + \frac{9\alpha}{38} - \frac{17}{10} \tag{81a}$$

$$= \alpha \frac{9\ell - 76}{38\ell} - \frac{17}{10} \tag{81b}$$

$$\geq 0. \tag{81c}$$

Thus, we arrive at

$$\mathbb{P}\left(\mathrm{Gap}(H, \kappa, (\kappa s)^{\alpha}) \text{ and } \mathcal{B}_{H,J}\right) \leq \left(5e\sqrt{2C_2KJ}(8J)^{9/38}s\right)^{\kappa D} \tag{82a}$$

$$= \left(\frac{cs}{2}\right)^{\kappa D} \tag{82b}$$

with $c = 10e\sqrt{2C_2KJ}(8J)^{9/38}$, i.e., $c \sim \sqrt{K}J^{14/19}$. Note that $\ell \mapsto \frac{323\ell}{5(9\ell-76)}$ for $\ell > 8$ is monotonically decreasing and $\lim_{\ell\to\infty} \frac{323\ell}{5(9\ell-76)} = \frac{323}{45}$. Therefore we can choose $\alpha = 8$ and $\ell = 83$ to get $-2\alpha/\kappa D + 9\alpha/38 - 1.7 \geq 0$. For matrices whose imaginary part is not fixed, we proceed as in step 1 to obtain the result for complex matrices. In order to relax the condition $J \geq 1$ to $J > 0$ (and also $K \geq 1$ to $K > 0$), one can again apply a scaling result as in step 2, where one replaces K by JK and J by 1, to obtain $c \sim \sqrt{KJ}\,1^{14/19} = \sqrt{KJ}$. We thus arrive at the final result with $c_c = 10e\sqrt{2C_2}8^{9/38}$. Note that it depends on the value of C_2 whether $c_c \geq 1$ or merely $c_c > 0$, however, we can assume without loss of generality that $c_c \geq 1$ since replacing c_c by $\max\{10e\sqrt{2C_2}8^{9/38}, 1\}$ only results in a possibly smaller lower bound for the probability with which no-gaps delocalization at least holds. □

Remark 4.
1. Theorem 7 still holds if some entries of $\operatorname{Im} H$ are fixed. This observation will be important when we construct examples in Section 4.3 since for Hermitian matrices the diagonal entries of $\operatorname{Im} H$ are fixed to zero.
2. In the proof of Theorem 7 the choice $\alpha = 8$ is the smallest one possible if one wants $\alpha \in \mathbb{N}$. Otherwise, any other value $\alpha > \frac{323}{45} \approx 7.18$ is possible, however, at the cost of a larger value for ℓ and therefore κ.

Remark 5 (A different proof strategy). Another possibility (different from the scaling argument) to obtain no-gaps delocalization in the case that $0 < J < 1$ is to find out where the assumption $J \geq 1$ is used in the proof of Rudelson and Vershynin and to suitably modify the assumptions: The assumption $J \geq 1$ is used the first time (except for Section 5.1 to which we will turn at the end) in the proof of Lemma 5.7 (and Lemma 5.8 which follows from an application of Lemma 5.7). If $J \geq 1$, then $J/\sqrt{\kappa} > 1$ and since the constant C on the left-hand side of (5.15) and K are also greater than 1, the upper bound on the probability is only non-trivial if $\theta^{1-2\mu} < 1$. In view of how Lemma 5.7 is applied, we can safely assume that $\mu \leq 1/2$ and thus that $\theta < 1$. In the proof of Lemma 5.7 as well as Lemma 5.8 it is used that $\theta\sqrt{\kappa}/J < 1$ which is obviously fulfilled for this choice of parameter. If we allow that $J < 1$ but require $\kappa < J^2$, then we can draw the same conclusions and argue in the same way to obtain Lemma 5.7 and Lemma 5.8. This means that the upper bound for κ has to be replaced by $\min\{1/2, J^2\}$.

The next (and last) time the assumption $J \geq 1$ is used in the proof of the Invertibility Theorem 5.1 at the end of Section 5. Here the improved estimate that was presented in the third step in the proof of Theorem 7 is necessary.

Note that the assumption $JK \geq 1$ is not needed in this case and we have to assume again that $K \geq 1$. However, in our examples below, we always need that $JK \geq 1$. Moreover, note that we cannot assume anymore that the constant $c \sim \sqrt{K}J^{14/19}$ in the proof of Theorem 7 is larger than one. An adaption of the proof of Theorem 10 shows that with probability at least

$$\left(1 - \min\left\{c_c\sqrt{K}J^{14/19}, \frac{1}{2}\right\}^{\min\{d_\nu, d_\mu, 2DJ^2\}/2}\right)(1-\eta)^4 \tag{83}$$

one finds the lower bound

$$M_{\mu\nu} \geq \left(\min\left\{1, \frac{1}{2c_c\sqrt{K}J^{14/19}}\right\}\min\left\{\frac{d_\mu}{2D}, \frac{d_\nu}{2D}, J^2\right\}\right)^{16}\max\left\{1, \frac{d_\nu}{d_\mu}\right\}. \tag{84}$$

Similarly, with probability at least $\left(1 - \min\{c_c\sqrt{K}J^{14/19}, 1/2\}^{\min\{d_\mu/2, DJ^2\}}\right)(1-\eta)^4$ one has the lower bound

$$\begin{aligned} |M_{\mu B}| \geq &\max\left\{b^{\pm}_{\min}, \left(\min\left\{\frac{d_\mu}{2D}, J^2\right\}\min\left\{1, \frac{1}{2c_c\sqrt{K}J^{14/19}}\right\}\right)^{16}\frac{\mathrm{tr}(B^{\pm})}{d_\mu}\right\} \\ &- \min\left\{b^{\mp}_{\max}, \frac{\mathrm{tr}(B^{\mp})}{d_\mu}\right\}. \end{aligned} \tag{85}$$

This lower bound is sometimes slightly better and sometimes slightly worse than the one we presented in Theorem 4 but yields more complicated expressions e.g. in Corollary 2 and we therefore stick to the simpler lower bound.

4.3 Examples

In this section we give some examples covered by Theorem 7. We will see that among these matrices are matrices with a band structure in a basis that diagonalizes the projections onto the macro spaces to which a "small" Gaussian perturbation is added. As discussed in the introduction, we are interested in such kind of matrices since, in contrast to Wigner matrices or matrices from the GOE/GUE, they can describe systems in which a non-trivial equilibration process passing through multiple macro-states occurs.

Theorem 8 (Gaussian matrices, variances bounded by constants)**.** *Let* $A = (a_{ij})$ *be a* $D \times D$ *random matrix with independent complex Gaussian entries with mean zero, i.e., all random variables* $\mathrm{Re}\, a_{ij}$*,* $\mathrm{Im}\, a_{ij}$*,* $i, j \in [D]$*, are independent and* $\mathrm{Re}\, a_{ij}, \mathrm{Im}\, a_{ij} \sim$

$\mathcal{N}(0, \sigma_{ij}^2/2)$ for some $\sigma_{ij} > 0$. Let $\sigma_{ij} = \sigma_{ji}$, $\sigma_- := \min_{i,j} \sigma_{ij}$ and $\sigma_+ := \max_{i,j} \sigma_{ij}$. Then the Hermitian matrix $H := \frac{1}{\sqrt{2}}(A + A^*)$ is Gaussian, more precisely, its entries h_{ij} satisfy $\operatorname{Re} h_{ij}, \operatorname{Im} h_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2/2)$ for $i \neq j$ and $h_{ii} = \operatorname{Re} h_{ii} \sim \mathcal{N}(0, \sigma_{ii}^2)$, and fulfills the assumptions in Theorem 7 with parameters $K = \frac{1}{\sqrt{2\pi}\sigma_-}$ and $J = \frac{4}{\eta}\hat{C}\sigma_+$ for arbitrary $\eta \in (0, \frac{1}{2})$, where $\hat{C}$ is a certain universal constant with the property (88).

Proof. The matrix H is obviously Hermitian, and since $\operatorname{Re} a_{ij}$ and $\operatorname{Re} a_{ji}$ are independent and normally distributed with zero expectation and variance $\sigma_{ij}^2 = \sigma_{ji}^2$ for $i \neq j$, we have

$$\operatorname{Re} h_{ij} = \frac{1}{\sqrt{2}} (\operatorname{Re} a_{ij} + \operatorname{Re} \overline{a_{ji}}) = \frac{1}{\sqrt{2}} (\operatorname{Re} a_{ij} + \operatorname{Re} a_{ji}) \sim \mathcal{N}(0, \sigma_{ij}^2/2). \tag{86}$$

Similarly, one has $\operatorname{Im} h_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2/2)$ and

$$\operatorname{Re} h_{ii} = \sqrt{2} \operatorname{Re} a_{ii} \sim \mathcal{N}(0, \sigma_{ii}^2). \tag{87}$$

By construction, $\operatorname{Re} h_{ij}$ is independent of the rest of the entries of $\operatorname{Re} H$ except $\operatorname{Re} h_{ji}$, the same holds true for the imaginary parts and obviously $\operatorname{Re} H$ and $\operatorname{Im} H$ are independent.

Latala [22, Theorem 2] showed that

$$\mathbb{E}\|(x_{ij})\| \leq \hat{C} \left(\max_i \sqrt{\sum_j \mathbb{E}x_{ij}^2} + \max_j \sqrt{\sum_i \mathbb{E}x_{ij}^2} + \sqrt[4]{\sum_{i,j} \mathbb{E}x_{ij}^4} \right) \tag{88}$$

for any finite matrix of independent mean zero random variables x_{ij}, where $\hat{C} > 0$ is a universal constant. Without loss of generality we can assume that $\hat{C} \geq 1$.

With the help of (88) the expectation of the norm of $\operatorname{Re} H$ can be bounded in the following way:

$$\begin{aligned}
\mathbb{E}\|\operatorname{Re} H\| &\leq \frac{1}{\sqrt{2}} \mathbb{E}\left(\|\operatorname{Re} A\| + \|\operatorname{Re} A^*\|\right) && \text{(89a)} \\
&\leq \hat{C} \left(\max_i \sqrt{\sum_j \sigma_{ij}^2} + \max_j \sqrt{\sum_i \sigma_{ij}^2} + \sqrt[4]{\sum_{i,j} 3\sigma_{ij}^4} \right) && \text{(89b)} \\
&\leq \hat{C} \left(2\sqrt{D\sigma_+^2} + \sqrt[4]{3D^2\sigma_+^4} \right) && \text{(89c)} \\
&\leq 4\hat{C}\sigma_+\sqrt{D}. && \text{(89d)}
\end{aligned}$$

For $0 < \eta \leq 1/2$ as in Theorem 7 we set $J := \frac{4}{\eta}\hat{C}\sigma_+$ and find with the help of Markov's inequality that

$$\mathbb{P}\left(\mathcal{B}_{\operatorname{Re} H, J}\right) = 1 - \mathbb{P}\left(\|\operatorname{Re} H\| > J\sqrt{D}\right) \geq 1 - \frac{\mathbb{E}\|\operatorname{Re} H\|}{J\sqrt{D}} \geq 1 - \eta, \tag{90}$$

i.e., the boundedness event $\mathcal{B}_{\mathrm{Re}\,H,J}$ holds with probability at least $1-\eta$. In the same way we find that also $\mathcal{B}_{\mathrm{Im}\,H,J}$ holds with probability at least $1-\eta$.

Clearly, the densities of $\mathrm{Re}\,H$ are bounded by $K := \frac{1}{\sqrt{2\pi}\sigma_-}$. This is due to the fact that the density function f of the normal distribution with mean μ and variance σ^2 attains its maximum at $x=\mu$ with $f(\mu) = \frac{1}{\sqrt{2\pi}\sigma}$. and $\frac{1}{\sqrt{2\pi}\sigma_{ij}} \leq \frac{1}{\sqrt{2\pi}\sigma_-}$ for all $i,j \in [D]$. Moreover, the condition $JK \geq 1$ is obviously fulfilled for our choice of the parameters. □

Remark 6. By choosing the variance large close to the diagonal of the matrix and small far away from it, the matrix H in Theorem 8 has some kind of band structure. For example, we can fix a monotonically decreasing function $g : [0,1] \to [\sigma_-, \sigma_+]$, a "variance profile," and define

$$\sigma_{ij} := g\left(\frac{|i-j|}{D}\right) \qquad \forall i,j \in [D]. \tag{91}$$

So far we only considered matrices whose (Gaussian) entries have mean zero. In the following we want to relax this condition and also allow entries with non-zero mean.

Theorem 9 (Gaussian matrices with non-zero mean)**.** *Let $H_0 = (h^0_{ij})$ be a (deterministic) Hermitian matrix with C_{H_0} as in* (39) *and let V be the $D \times D$ random matrix defined in Theorem 8 (there H). Define $H := H_0 + V$. Then H is Hermitian with (non-centered) Gaussian independent (up to conjugate symmetry) entries; more precisely, its entries h_{ij} satisfy $\mathrm{Re}\,h_{ij} \sim \mathcal{N}(\mathrm{Re}\,h^0_{ij}, \sigma^2_{ij}/2)$, $\mathrm{Im}\,h_{ij} \sim \mathcal{N}(\mathrm{Im}\,h^0_{ij}, \sigma^2_{ij}/2)$ for $i \neq j$ and $h_{ii} = \mathrm{Re}\,h_{ii} \sim \mathcal{N}(h^0_{ii}, \sigma^2_{ii})$. Moreover, H satisfies the assumptions in Theorem 7 with $K = \frac{1}{\sqrt{2\pi}\sigma_-}$ and $J = \frac{1}{\eta}(4\hat{C}\sigma_+ + C_{H_0})$ with $\hat{C}$ as before.*

Proof. The claims concerning the distribution of the entries of H are obvious. With the help of the computation in the proof of Theorem 8 we find that

$$\mathbb{E}\|\mathrm{Re}\,H\| \leq \mathbb{E}\|\mathrm{Re}\,H_0\| + \|\mathrm{Re}\,V\| \tag{92a}$$

$$\leq \left(4\hat{C}\sigma_+ + C_{H_0}\right)\sqrt{D} \tag{92b}$$

For $0 < \eta \leq 1/2$ as in Theorem 7 we set $J := \frac{1}{\eta}\left(4\hat{C}\sigma_+ + C_{H_0}\right)$ and obtain

$$\mathbb{P}\left(\mathcal{B}_{\mathrm{Re}\,H,J}\right) \geq 1-\eta, \tag{93}$$

i.e., the boundedness event $\mathcal{B}_{\mathrm{Re}\,H,J}$ holds with probability at least $1-\eta$. The parameter K has already been computed in Theorem 8 and with this choice the condition $KJ \geq 1$ is again automatically fulfilled. This finishes the proof. □

Theorem 9 covers, for example, the case that H_0 is a matrix with a band structure in a basis that diagonalizes the projections P_μ and V is a Gaussian matrix with mean zero and small variances, i.e., a small Gaussian perturbation with entries small in the L^2-sense.

5 More General Results and Proofs

In this section we state and prove a more general version of Theorem 4 (Theorem 10 in Section 5.1), some corollaries thereof and prove Theorem 4. We then state a more general version of Theorem 5 (Theorem 11 in Section 5.2) and prove Theorem 5.

5.1 Generalized Normal Typicality

Let $H = (h_{ij}) \in M_{D\times D}(\mathbb{C})$ be a random Hermitian matrix. Instead of Assumption 1, we now make the following weaker assumption (which follows if Assumption 1 holds):

Assumption 2. The random variables $(\operatorname{Re} h_{ij})_{i\le j}$ and $(\operatorname{Im} h_{ij})_{i<j}$ are mutually independent and continuously distributed. The densities $\varrho_{ij}^{\mathrm{Re}}$ of $\operatorname{Re} h_{ij}$ are bounded.

Let B^+ and B^- denote the positive and negative part of B such that $B = B^+ - B^-$. Recall that d_μ and d_ν denote the dimensions of the macro spaces $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$.

Theorem 10. *Let $\varepsilon' \in (0, \frac{1}{2})$ and let μ be an arbitrary macro state. Let $B \in M_{D\times D}(\mathbb{C})$ be a Hermitian $D\times D$ matrix and let $H = (h_{ij})$ be a random Hermitian $D \times D$ matrix such that Assumption 2 is satisfied. Let K be the least upper bound for the densities $\varrho_{ij}^{\mathrm{Re}}$, i.e.*

$$K := \sup \bigcup_{i\le j} \left\{ \varrho_{ij}^{\mathrm{Re}}(x) : x \in \mathbb{R} \right\} < \infty. \tag{94}$$

Let $d_\mu > \max\{166, 2|\log_2 \varepsilon'|\}$. Moreover, let $\eta \in (0, \frac{1}{2})$ be the unique number that solves

$$1 - \varepsilon' = \left(1 - 2^{-d_\mu/2}\right)(1-\eta)^4 \tag{95}$$

and let $J^{\mathrm{Re}}, J^{\mathrm{Im}} \in (0,\infty)$ be the unique numbers such that

$$\mathbb{P}\left(\|\operatorname{Re} H\| \le J^{\mathrm{Re}} \sqrt{D} \right) = 1 - \eta, \tag{96}$$

$$\mathbb{P}\left(\|\operatorname{Im} H\| \le J^{\mathrm{Im}} \sqrt{D} \right) = 1 - \eta. \tag{97}$$

Set $J := \max\left\{K^{-1}, J^{\mathrm{Re}}, J^{\mathrm{Im}}\right\}$. Then with probability at least $1-\varepsilon'$,

$$|M_{\mu B}| \ge \max\left\{ b_{\min}^+, \left(\frac{d_\mu}{4c_c\sqrt{KJD}}\right)^{16} \frac{\operatorname{tr}(B^+)}{d_\mu} \right\} - \min\left\{ b_{\max}^-, \frac{\operatorname{tr}(B^-)}{d_\mu} \right\}, \tag{98}$$

where $M_{\mu B}$ was defined in (22), $b_{\min}^+$ and $b_{\max}^-$ denote the smallest and largest eigenvalue of B^+ and B^- respectively and $c_c \ge 1$ is the constant in Theorem 7. In particular, if $B = P_\nu$ for some macro state ν, then

$$M_{\mu\nu} \ge \frac{d_\nu}{d_\mu} \left(\frac{d_\mu}{4c_c\sqrt{KJD}} \right)^{16}. \tag{99}$$

Moreover, if $d_\nu \geq 166$, then for any $\eta \in (0, \frac{1}{2})$ (and J chosen as above) it holds with probability at least $\left(1 - 2^{-d_\nu/2}\right)(1-\eta)^4$ that

$$M_{\mu\nu} \geq \left(\frac{d_\nu}{4c_c\sqrt{KJD}}\right)^{16}. \tag{100}$$

The lower bound for $|M_{\mu B}|$ is obviously nontrivial for positive (and negative) operators B but also for operators with positive and negative eigenvalues provided that the spectrum satisfies certain assumptions, e.g. if $b^+_{\min} > b^-_{\max}$.

Note the second lower bound (100) is sharper than (99) if $d_\nu \geq d_\mu$. By combining the lower bounds in Theorem 10 with the upper bound from Theorem 1, keeping in mind that, with probability 1, $D_E = 1$ and $d_E = D$, we immediately obtain the following corollary:

Corollary 3. *Let $\varepsilon, \delta, \kappa, T > 0, \varepsilon' \in (0, \frac{1}{2})$ and let μ be an arbitrary macro state. Let $B \in M_{D\times D}(\mathbb{C})$ be a Hermitian $D \times D$ matrix and let H be a random Hermitian $D \times D$ matrix such that Assumption 2 is satisfied. Let $d_\mu > \max\{166, 2|\log_2 \varepsilon'|\}$ and let $K, J > 0$ and $\eta \in (0, \frac{1}{2})$ be defined as in Theorem 10. Then with probability at least $1 - \varepsilon'$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0, T]$*

$$\frac{\left|\langle\psi_t|B|\psi_t\rangle - M_{\mu B}\right|}{|M_{\mu B}|} \leq \frac{4\left(\frac{G(\kappa)\|B\|}{\delta\varepsilon d_\mu}\left(1 + \frac{8\log_2 D}{\kappa T}\right)\min\left\{\|B\|, \frac{\mathrm{tr}(|B|)}{d_\mu}\right\}\right)^{1/2}}{\max\left\{b^+_{\min}, \left(\frac{d_\mu}{4c_c\sqrt{KJD}}\right)^{16}\frac{\mathrm{tr}(B^+)}{d_\mu}\right\} - \min\left\{b^-_{\max}, \frac{\mathrm{tr}(B^-)}{d_\mu}\right\}}, \tag{101}$$

whenever the denominator of the right-hand side is positive; here $c_c \geq 1$ is the constant in Theorem 7.

The next corollary for the special case that $B = P_\nu$ for some macro state ν follows from combining the lower bounds (99) and (100) for $M_{\mu\nu}$, which yield with probability at least $(1 - 2^{-d_\nu/2} - 2^{-d_\mu/2})(1-\eta)^4$

$$M_{\mu\nu} \geq \frac{d_\nu}{\max\{d_\nu, d_\mu\}}\left(\frac{\max\{d_\nu, d_\mu\}}{4c_c\sqrt{KJD}}\right)^{16} = \left(\frac{\max\{d_\nu, d_\mu\}}{4c_c\sqrt{KJD}}\right)^{16}\min\left\{1, \frac{d_\nu}{d_\mu}\right\}, \tag{102}$$

with the upper bound for the absolute errors from Theorem 2, keeping again in mind that for random matrices with continuously distributed entries, it holds with probability 1 that $D_E = D_G = 1$.

Corollary 4 (Generalized normal typicality: relative errors)**.** *Let $\varepsilon, \delta > 0$, $\varepsilon' \in (0, \frac{1}{2})$ and let μ, ν be two macro states such that $d_\mu, d_\nu > \max\left\{166, 2|\log_2(\varepsilon'/\sqrt{2})|\right\}$. Let H be a random Hermitian $D \times D$ matrix such that Assumption 2 is satisfied and let $K > 0$ be defined as in Theorem 10. Moreover, let $\eta \in (0, \frac{1}{2})$ be the unique number that solves*

$$1 - \varepsilon' = \left(1 - 2^{-d_\mu/2} - 2^{-d_\nu/2}\right)(1-\eta)^4 \tag{103}$$

and let $J > 0$ be defined as in Theorem 10. Then with probability at least $1-\varepsilon'$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,\infty)$

$$\frac{\left|\|P_\nu\psi_t\|^2 - M_{\mu\nu}\right|}{M_{\mu\nu}} \leq \frac{4}{\sqrt{\varepsilon\delta\min\{d_\mu,d_\nu\}}}\left(\frac{4c_c\sqrt{KJ}D}{\max\{d_\mu,d_\nu\}}\right)^{16}, \tag{104}$$

where $c_c \geq 1$ is the constant in Theorem 7.

We now give the proofs of Theorem 10 and of one of our main theorems, Theorem 4.

Proof of Theorem 10. We can write the Hamiltonian as $H = \sum_n E_n|\phi_n\rangle\langle\phi_n|$, where (ϕ_n) is an orthonormal basis of eigenvectors of H with eigenvalues $E_n \in \mathbb{R}$. Moreover, we obtain with the reverse triangle inequality that

$$|M_{\mu B}| \geq \frac{1}{d_\mu}\left(\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|B^+|\phi_n\rangle - \sum_m \langle\phi_m|P_\mu|\phi_m\rangle\langle\phi_m|B^-|\phi_m\rangle\right). \tag{105}$$

Note that we used here the fact that, with probability 1, the eigenvalues of H are distinct. Set $\kappa := \frac{d_\mu}{2D}$ and $s := \frac{1}{2c_c\sqrt{KJ}}$, where $c_c \geq 1$ is the constant in Theorem 7. With these choices all assumptions in Theorem 7 are fulfilled[6] and with

$$I_\mu = \{d_1 + \cdots + d_{\mu-1} + 1, \ldots, d_1 + \cdots + d_\mu\} \tag{106}$$

it follows that with probability at least

$$\left(1 - \left(c_c\sqrt{KJ}s\right)^{\kappa D}\right)(1-\eta)^4 = \left(1 - 2^{-d_\mu/2}\right)(1-\eta)^4 = 1-\varepsilon' \tag{107}$$

it holds that

$$\langle\phi_n|P_\mu|\phi_n\rangle = \sum_{j\in I_\mu}|\phi_n(j)|^2 = \|(\phi_n)_{I_\mu}\|^2 \geq (\kappa s)^{16}. \tag{108}$$

Remember that we assume that the Hamiltonian is written in a basis that diagonalizes the projections onto the macro spaces as in Figure 3. We find the following lower bounds for the first and upper bounds for the second sum in (105):

$$\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|B^+|\phi_n\rangle \geq b^+_{\min}d_\mu, \tag{109a}$$

[6] Note that the factor 1/2 in the definition of κ ensures that $\kappa < 1/2$ since d_μ/D can be arbitrarily close (but not equal to) 1. This is because the trivial case in which there is only one macro space is excluded here since in this case we obviously have normal typicality and there is nothing to show at all.

$$\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|B^+|\phi_n\rangle \geq (\kappa s)^{16}\,\mathrm{tr}(B^+), \tag{109b}$$

$$\sum_m \langle\phi_m|P_\mu|\phi_m\rangle\langle\phi_m|B^-|\phi_m\rangle \leq b^-_{\max} d_\mu, \tag{109c}$$

$$\sum_m \langle\phi_m|P_\mu|\phi_m\rangle\langle\phi_m|B^-|\phi_m\rangle \leq \mathrm{tr}(B^-). \tag{109d}$$

A combination of these bounds yields

$$|M_{\mu B}| \geq \max\left\{b^+_{\min}, (\kappa s)^{16}\frac{\mathrm{tr}(B^+)}{d_\mu}\right\} - \min\left\{b^-_{\max}, \frac{\mathrm{tr}(B^-)}{d_\mu}\right\} \tag{110}$$

$$= \max\left\{b^+_{\min}, \left(\frac{d_\mu}{4c_c\sqrt{KJD}}\right)^{16}\frac{\mathrm{tr}(B^+)}{d_\mu}\right\} - \min\left\{b^-_{\max}, \frac{\mathrm{tr}(B^-)}{d_\mu}\right\}. \tag{111}$$

In particular, if $B = P_\nu$ for some macro state ν, then $B^- = 0$, $b^+_{\min} = 0$, $\mathrm{tr}(B^+) = d_\nu$ and thus

$$M_{\mu\nu} \geq \frac{d_\nu}{d_\mu}\left(\frac{d_\mu}{4c_c\sqrt{KJD}}\right)^{16}. \tag{112}$$

Note that here no absolute value on the left-hand side is needed since it immediately follows from the definition of the $M_{\mu\nu}$ that $M_{\mu\nu} \geq 0$. If $d_\nu \geq 166$, we set $\kappa := \frac{d_\nu}{2D}$ and obtain with Theorem 7 with probability at least $\left(1 - 2^{-d_\nu/2}\right)(1-\eta)^4$

$$\langle\phi_n|P_\nu|\phi_n\rangle \geq (\kappa s)^{16} = \left(\frac{d_\nu}{4c_c\sqrt{KJD}}\right)^{16} \tag{113}$$

and therefore

$$M_{\mu\nu} = \frac{1}{d_\mu}\sum_n \langle\phi_n|P_\nu|\phi_n\rangle\langle\phi_n|P_\mu|\phi_n\rangle \geq \left(\frac{d_\nu}{4c_c\sqrt{KJD}}\right)^{16}. \tag{114}$$

□

Proof of Theorem 4. First note that since H satisfies Assumption 1 it obviously also satisfies Assumption 2 and that $d_\nu, d_\mu > \max\{166, 2|\log_2(\varepsilon'/\sqrt{2})|\}$ by assumption. Let $\eta \in (0, \frac{1}{2})$ be the unique number that solves

$$1 - \varepsilon' = \left(1 - 2^{-2_\mu/2} - 2^{-d_\nu/2}\right)(1-\eta)^4. \tag{115}$$

Then it follows from Theorem 7 as in the proof of Theorem 10 that with probability at least $1 - \varepsilon'$,

$$M_{\mu\nu} \geq \left(\frac{\max\{d_\nu, d_\mu\}}{4c_c\sqrt{KJD}}\right)^{16}\min\left\{1, \frac{d_\nu}{d_\mu}\right\}, \tag{116}$$

see also (102). In order to arrive at the form (40), note first that with C_{H_0} as defined in (39) and with the help of Theorem 9, the factor KJ in (116) is given by

$$KJ = \frac{1}{\sqrt{2\pi}\sigma_-\eta}\left(4\hat{C}\sigma_+ + C_{H_0}\right). \tag{117}$$

Next we derive a lower bound for η in terms of ε' in order to eliminate η from (117). Therefore observe that $d_\nu, d_\mu > 4|\log_2(\varepsilon'/\sqrt{2})| = -4\log_2(\varepsilon'/\sqrt{2})$ and with (115) it follows that

$$1 - \varepsilon' \geq \left(1 - \varepsilon'^2\right)(1-\eta)^4. \tag{118}$$

Solving for η yields

$$\eta \geq 1 - \frac{1}{(1+\varepsilon')^{1/4}} \geq \frac{\varepsilon'}{6}. \tag{119}$$

In the last step it was used that for $f, g : [0, \frac{1}{2}] \to \mathbb{R}$, $f(x) = 1 - \frac{1}{(1+x)^{1/4}}$, $g(x) = \frac{x}{6}$ one has $f \geq g$ which can easily be shown by standard arguments. Thus with (117) we find

$$c_c^2 KJ \leq \frac{6c_c^2}{\sqrt{2\pi}\sigma_-\varepsilon'}\left(4\hat{C}\sigma_+ + C_{H_0}\right) =: \frac{1}{16\varepsilon'}\frac{c_+\sigma_+ + C_{H_0}}{c_-\sigma_-} \tag{120}$$

with $c_- := \frac{\sqrt{2\pi}}{96c_c^2}$ and $c_+ := 4\hat{C}$, i.e., $c \leq 1/(4\sqrt{\varepsilon' C_\sigma})$. Inserting this estimate into (116) finishes the proof. □

5.2 Dynamical Typicality

In order to find an upper bound for the relative error in the dynamical typicality theorem, Theorem 3, as well, we need a lower bound for $|w_{\mu B}(t)| = |\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle|$. Without any further assumptions on the Hamiltonian we immediately find the following proposition:

Proposition 2. *Let $B \in \mathcal{L}(\mathcal{H})$ be Hermitian such that $b := \max\{b^+_{\min} - b^-_{\max}, b^-_{\min} - b^+_{\max}\} > 0$, where $b^\pm_{\min}$ and $b^\pm_{\max}$ are again the smallest and largest eigenvalues of $B^\pm$ (the positive and negative parts of $B = B^+ - B^-$). Let $\varepsilon > 0$, $t \in [0,\infty)$, and let μ be an arbitrary macro state. Then*

$$\left|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right| \geq b, \tag{121}$$

and therefore $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{\left|\langle\psi_t|B|\psi_t\rangle - \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|}{\left|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|} \leq b^{-1}\cdot(\textit{right-hand side of } (35)). \tag{122}$$

Moreover, for every μ and B, every $T > 0$, and $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$,

$$\frac{1}{T}\int_0^T \frac{\big|\langle\psi_t|B|\psi_t\rangle - \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\big|^2}{\big|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\big|^2}\,dt \leq \frac{\|B\|^2}{b^2\varepsilon d_\mu}. \tag{123}$$

Proof. Let (φ_k) be an orthonormal basis of $\mathcal{H}_\mu$ and define $\varphi_{k,t} := e^{-itH}\varphi_k$. Then

$$\begin{aligned}
\big|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\big| &= \frac{1}{d_\mu}\big|\mathrm{tr}(P_\mu e^{itH}Be^{-itH})\big| && \text{(124a)}\\
&= \frac{1}{d_\mu}\Big|\sum_k \langle\varphi_{k,t}|B|\varphi_{k,t}\rangle\Big| && \text{(124b)}\\
&\geq \frac{1}{d_\mu}\max\left\{\sum_k\langle\varphi_{k,t}|B^+ - B^-|\varphi_{k,t}\rangle, \sum_k\langle\varphi_{k,t}|B^- - B^+|\varphi_{k,t}\rangle\right\} && \text{(124c)}\\
&\geq \max\{b^+_{\min} - b^-_{\max}, b^-_{\min} - b^+_{\max}\}, && \text{(124d)}
\end{aligned}$$

i.e., $\big|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\big| \geq b$. The remaining claims now follow immediately from Theorem 3. □

Proposition 2 yields a good lower bound for $\big|\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\big|$ and therefore useful upper bounds for the relative errors if, for example, B is a positive or negative operator (and in this case, $b = b^+_{\min} > 0$ and $b = b^-_{\min} > 0$ respectively). If B has both positive and non-positive eigenvalues, the bounds are only useful if the spectrum of B is rather special since in most cases one has $b \leq 0$. In particular, if $B = P_\nu$ for some macro state ν we have $b = 0$ and Proposition 2 is not applicable. However, with the help of the corrected and improved no gaps delocalization, Theorem 7, we are able to prove an upper bound for the comparative errors which also applies to more general operators and in particular to the case $B = P_\nu$:

Theorem 11. *Let $\varepsilon > 0$, $\varepsilon' \in (0, \frac{1}{2})$ and let μ and ν be macro states such that $d_\mu, d_\nu > \max\{166, 4|\log_2\varepsilon'|\}$. Let B be a Hermitian $D\times D$ matrix and let H be a random Hermitian $D \times D$ matrix as in Theorem 10. Then with probability at least $1-\varepsilon'$, for each $t \in \mathbb{R}$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that*

$$\begin{aligned}
&\frac{\big|\langle\psi_t|B|\psi_t\rangle - \mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\big|}{\big|\overline{\langle\psi_t|B|\psi_t\rangle}\big|}\\
&\qquad\leq \mathrm{LB(B,\psi)}^{-1}\min\left\{\frac{\sqrt{2}\|B\|}{\sqrt{\varepsilon d_\mu}}, \sqrt{\frac{2\|B\|\,\mathrm{tr}(|B|)}{\varepsilon d_\mu^2}}, \sqrt{\frac{18\pi^3\log(8/\varepsilon)}{d_\mu}\|B\|}\right\}, \qquad (125)
\end{aligned}$$

whenever

$$\mathrm{LB}(B,\psi) := \max\left\{b^+_{\min}, \left(\frac{d_\mu}{4c_c\sqrt{KJD}}\right)^{16}\frac{\mathrm{tr}(B^+)}{d_\mu}\right\} - \min\left\{b^-_{\max}, \frac{\mathrm{tr}(B^-)}{d_\mu}\right\}$$

$$-\sqrt{2}\left(\frac{\|B\|}{\varepsilon d_\mu}\min\left\{\|B\|,\frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2} \tag{126}$$

is positive. Moreover, for every μ and B, every $T>0$, and $(1-\varepsilon)$-most $\psi_0\in\mathbb{S}(\mathcal{H}_\mu)$,

$$\frac{1}{T}\int_0^T \frac{\left|\langle\psi_t|B|\psi_t\rangle-\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|^2}{\left|\overline{\langle\psi_t|B|\psi_t\rangle}\right|^2}\,dt\le\frac{2\|B\|^2}{\mathrm{LB}(B,\psi)^2\varepsilon d_\mu}. \tag{127}$$

Proof. It follows from (124) and (125) in [43] that $\left(1-\frac{\varepsilon}{2}\right)$-most $\psi_0\in\mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\left|\overline{\langle\psi_t|B|\psi_t\rangle}-M_{\mu B}\right|\le\sqrt{2}\left(\frac{\|B\|}{\varepsilon d_\mu}\min\left\{\|B\|,\frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2}. \tag{128}$$

Therefore it follows from the triangle inequality and Theorem 10 that with probability at least $1-\varepsilon'$, $\left(1-\frac{\varepsilon}{2}\right)$-most $\psi_0\in\mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\begin{aligned}\left|\overline{\langle\psi_t|B|\psi_t\rangle}\right| &\ge |M_{\mu B}|-\sqrt{2}\left(\frac{\|B\|}{\varepsilon d_\mu}\min\left\{\|B\|,\frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2} && (129)\\ &\ge \max\left\{b^+_{\min},\left(\frac{d_\mu}{4c_c\sqrt{KJD}}\right)^{16}\frac{\operatorname{tr}(B^+)}{d_\mu}\right\}-\min\left\{b^-_{\max},\frac{\operatorname{tr}(B^-)}{d_\mu}\right\}\\ &\quad-\sqrt{2}\left(\frac{\|B\|}{\varepsilon d_\mu}\min\left\{\|B\|,\frac{\operatorname{tr}(|B|)}{d_\mu}\right\}\right)^{1/2} && (130)\\ &=\mathrm{LB}(B,\psi). && (131)\end{aligned}$$

It follows from Theorem 3 that $\left(1-\frac{\varepsilon}{2}\right)$-most $\psi_0\in\mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\left|\langle\psi_t|B|\psi_t\rangle-\mathbb{E}_\mu\langle\psi_t|B|\psi_t\rangle\right|\le\min\left\{\frac{\sqrt{2}\|B\|}{\sqrt{\varepsilon d_\mu}},\sqrt{\frac{2\|B\|\operatorname{tr}(|B|)}{\varepsilon d_\mu^2}},\sqrt{\frac{18\pi^3\log(8/\varepsilon)}{d_\mu}}\|B\|\right\}. \tag{132}$$

A combination of (131) and (132) yields the first claim. Similarly, a combination of (131) and (36) in Theorem 3 gives the second claim. □

Note that for large d_μ (and if $\|B\|$ does not depend on d_μ) the lower bound for $\left|\overline{\langle\psi_t|B|\psi_t\rangle}\right|$, $\mathrm{LB}(B,\psi)$, is up to a small error equal to the lower bound for $|M_{\mu B}|$ that was proved in Theorem 10.

Finally, we give the proof of Theorem 5 which yields a somewhat better bound than the previous theorem in the case that B is a projection P_ν onto some macro space $\mathcal{H}_\nu$:

Proof of Theorem 5. Let (ϕ_n) be an orthonormal basis of eigenvectors of H. With $\psi_0 = \sum_n c_n \phi_n$ we find that

$$\overline{\langle\psi_t|P_\nu|\psi_t\rangle} = \sum_{n,m} c_n^* c_m \overline{e^{it(E_n - E_m)}} \langle\phi_n|P_\nu|\phi_m\rangle = \sum_n |c_n|^2 \langle\phi_n|P_\nu|\phi_n\rangle. \tag{133}$$

Since $\sum_n |c_n|^2 = 1$ and with the help of Theorem 7 we find with probability at least $1 - \varepsilon'$ that

$$\overline{\langle\psi_t|P_\nu|\psi_t\rangle} \geq \left(\frac{d_\nu}{4c_c\sqrt{KJ}D}\right)^{16} \geq \left(\frac{\sqrt{\varepsilon' C_\sigma} d_\nu}{D}\right)^{16} \tag{134}$$

see also the proof of Theorem 10 for the first step. In the second step we used that $c_c\sqrt{KJ} \leq 1/(4\sqrt{\varepsilon' C_\sigma})$, which can be shown similarly as in the proof of Theorem 4. This implies together with Theorem 3 (by using the first two bounds in (35)) that $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{\overline{\left|\|P_\nu\psi_t\|^2 - \mathbb{E}_\mu\|P_\nu\psi_t\|^2\right|}}{\overline{\|P_\nu\psi_t\|^2}} \leq \frac{1}{\sqrt{\varepsilon}}(C_\sigma\varepsilon')^{-8}\frac{1}{d_\mu}\sqrt{\min\{d_\mu, d_\nu\}}\left(\frac{D}{d_\nu}\right)^{16}. \tag{135}$$

Inserting the definition of s_μ, s_ν and s_{mc} thus proves the first claim. For the second claim observe that it follows from Theorem 3 that for every $T > 0$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{1}{T}\int_0^T \left|\|P_\nu\psi_t\|^2 - \mathbb{E}_\mu\|P_\nu\psi_t\|^2\right|^2 dt \leq \frac{1}{\varepsilon d_\mu}. \tag{136}$$

Together with (135) this implies that with probability $1-\varepsilon'$, for every $T > 0$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that

$$\frac{1}{T}\int_0^T \frac{\left|\|P_\nu\psi_t\|^2 - \mathbb{E}_\mu\|P_\nu\psi_t\|^2\right|^2}{\overline{\|P_\nu\psi_t\|^2}} dt \leq \frac{1}{\varepsilon}(C_\sigma\varepsilon')^{-16}\frac{1}{d_\mu}\left(\frac{D}{d_\nu}\right)^{32}. \tag{137}$$

Now also the second claim follows immediately from the definition of s_μ, s_ν and s_{mc}. □

6 An Improved Lower Bound for $M_{\mu\,\mathrm{eq}}$, $M_{\mathrm{eq}\,\nu}$ and $M_{\mathrm{eq\,eq}}$ for Wigner-Type Matrices

We expect that in many cases a significantly stronger lower bound for the $M_{\mu\nu}$ than the one obtained with the help of our improved version of the no gaps delocalization result from Rudelson and Vershynin [37], Theorem 4, should hold. More precisely, it is expected that for band matrices H with sufficiently large band width, the eigenvectors are delocalized, and in that situation we expect that $M_{\mu\nu} \approx \frac{d_\nu}{D}$.

In this section, we show with the help of a result from Ajanki, Erdős and Krüger [1] concerning the delocalization of eigenvectors that for a special class of random matrices a much better lower bound for $M_{\mu\,\mathrm{eq}}$, $M_{\mathrm{eq}\,\nu}$ and $M_{\mathrm{eq}\,\mathrm{eq}}$ can be obtained provided that the equilibrium macro space is very dominant (in a sense made precise below). These matrices do not have a band structure. However, the results strengthen the expectation that often the lower bound in Theorem 4 can be significantly improved. We quote the relevant statement from [1] as Theorem 12 and obtain from it lower bounds for $M_{\mu\nu}$ in Theorem 13.

For using the result of [1], we need a certain shift of perspective. So far in this paper, we always regarded D as fixed and considered a single, randomly chosen $D\times D$ matrix H. In contrast, in this section and the next, we will consider a *sequence* of random matrices $H^{(D)}$, one for every $D \in \mathbb{N}$. In fact, our reasoning also applies if we merely allow an infinite set of D values (say, all powers of 2), and a random matrix $H^{(D)}$ for every D from that set; but for simplicity, we will pretend we have an $H^{(D)}$ for every $D \in \mathbb{N}$. More precisely, we assume that, for every $D \in \mathbb{N}$, we are given a probability distribution $\mathbb{P}^{(D)}$ over the Hermitian $D \times D$ matrices. We will show that, under suitable assumptions on $\mathbb{P}^{(D)}$, certain estimates hold for *sufficiently large* D, but we are not necessarily able to make explicit how large D has to be.

In order to state the result of [1], we need their notion of *stochastic domination*:

Definition 4 (Stochastic domination)**.** For two sequences $X = (X^{(D)})_D$ and $Y = (Y^{(D)})_D$ of non-negative random variables we say that X is stochastically dominated by Y if there exists a function $D_0 : (0,\infty)^2 \to \mathbb{N}$ such that for all $\tau > 0$ and $\alpha > 0$,

$$\mathbb{P}\left(X^{(D)} > D^\tau Y^{(D)}\right) \le D^{-\alpha} \qquad \forall D \ge D_0(\tau,\alpha). \tag{138}$$

In this case we write $X \prec Y$.

That is, $X \prec Y$ means that for large D it has high probability that X is not much larger than Y. The key fact for our purposes will be that under certain assumptions on $\mathbb{P}^{(D)}$, every eigenvector ϕ_n of $H = H^{(D)} \sim \mathbb{P}^{(D)}$ satisfies

$$\|\phi_n\|_\infty \prec \frac{1}{\sqrt{D}} \tag{139}$$

(more precise statement around (147) below). That is, each component of ϕ_n is not much larger than $1/\sqrt{D}$ (the magnitude that a component would have to have if all components had the same magnitude). Since if some components were much smaller than $1/\sqrt{D}$, others would have to be larger, this also entails that not a large fraction of the components can be much smaller than $1/\sqrt{D}$. But this still allows that a *small* fraction of the components could be arbitrarily small; for example, if d_ν (despite being a large number) is a small fraction of D, then all components of ϕ_n in $\mathcal{H}_\nu$ could be arbitrarily small, so $\langle\phi_n|P_\nu|\phi_n\rangle$ could be arbitrarily small, and then the expression (5) suggests that $M_{\mu\nu}$ could be arbitrarily small, and we would not obtain a useful lower bound for $M_{\mu\nu}$. In fact, (139) provides such a bound only if either d_μ or d_ν is sufficiently close to D, as the detailed analysis below confirms.

We now turn to describing the matrices considered in [1] and in this section.

Assumption 3. For every D, $H^{(D)} = (h_{ij}) \sim \mathbb{P}^{(D)}$ is a Hermitian $D \times D$ matrix of *Wigner type*, i.e., its entries h_{ij} are centered random variables and the entries $(h_{ij})_{i\leq j}$ are independent. The matrix of variances $S = (\sigma_{ij}^2)$, defined by

$$\sigma_{ij}^2 := \mathbb{E}|h_{ij}|^2\,, \tag{140}$$

is *flat*, i.e.,

$$\sigma_{ij}^2 \leq \frac{1}{D}, \qquad i,j = 1,\dots,D. \tag{141}$$

Let $p, P > 0$ and $L \in \mathbb{N}$ be parameters independent of D and let $\mu = (\mu_1, \mu_2, \dots)$ be a sequence of non-negative real numbers.

Assumption 4. The matrix S is *uniformly primitive*, i.e.

$$(S^L)_{ij} \geq \frac{p}{D}, \qquad i,j = 1,\dots,D. \tag{142}$$

It can be shown that the corresponding *vector Dyson equation*

$$-\frac{1}{m_i(z)} = z + \sum_{j=1}^{D} \sigma_{ij}^2 m_j(z), \qquad \text{for all } i = 1,\dots,D \text{ and } z \in \mathbb{H} \tag{143}$$

for a function $m = (m_1,\dots,m_D) : \mathbb{H} \to \mathbb{H}^D$ on the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ has a unique solution [2].

Assumption 5. The matrix S induces a bounded solution of the vector Dyson equation, i.e., the unique solution of (143) corresponding to S is bounded,

$$|m_i(z)| \leq P, \qquad i = 1,\dots,D, \quad z \in \mathbb{H}. \tag{144}$$

Assumption 6. The entries h_{ij} of the random matrix H have *bounded moments*, i.e.

$$\mathbb{E}|h_{ij}|^k \leq \mu_k \sigma_{ij}^k, \qquad k \in \mathbb{N}, \quad i,j = 1,\dots,D. \tag{145}$$

Sufficient conditions for Assumption 5 are given in [2] Theorem 6.1. The reason we make these assumptions is that under these conditions, Ajanki, Erdős and Krüger (2017) [1] proved the following theorem concerning the delocalization of the eigenvectors of H:

Theorem 12 ([1] Corollary 1.14)**.** *Consider a sequence $(\mathbb{P}^{(D)})_{D\in\mathbb{N}}$ of probability distributions over the Hermitian $D\times D$ matrices with $H^{(D)}\sim\mathbb{P}^{(D)}$ satisfying Assumptions 3-6. Let $E_1^{(D)}\leq\ldots\leq E_D^{(D)}$ be the eigenvalues of the random matrix $H^{(D)}$ and $\phi_n^{(D)}\in\mathbb{C}^D$ a normalized eigenvector of $H^{(D)}$ with eigenvalue $E_n^{(D)}$. Then for every sequence of unit vectors $b^{(D)}\in\mathbb{C}^D$ and every sequence $n^{(D)}\in\{1,\ldots,D\}$,*

$$\left|\left\langle b^{(D)}\middle|\phi^{(D)}_{n^{(D)}}\right\rangle\right| \prec \frac{1}{\sqrt{D}}\,. \tag{146}$$

In particular, the eigenvectors are completely delocalized, i.e.,

$$\|\phi^{(D)}_{n^{(D)}}\|_\infty \prec \frac{1}{\sqrt{D}}\,. \tag{147}$$

The function $D_0(\tau,\alpha)$ implicit in the $\prec$ symbol in (147) *depends only on the constants $p,P,L,(\mu_k)_{k\in\mathbb{N}}$ from Assumptions 3-6.*

With the help of Theorem 12 we find the following lower bounds for the $M_{\mu B}$:

Theorem 13 (Lower bounds for $|M_{\mu B}|$)**.** *Consider a sequence $(\mathbb{P}^{(D)})_{D\in\mathbb{N}}$ of probability distributions over the Hermitian $D\times D$ matrices with $H^{(D)}\sim\mathbb{P}^{(D)}$ satisfying Assumptions 3-6 and let $D_0:(0,\infty)^2\to\mathbb{N}$ be the function provided by Theorem 12 for* (147)*. Let $\tau>0,\alpha>1$, $D\geq D_0(\tau,\alpha)$, and let B be a Hermitian $D\times D$ matrix. Then with probability at least $1-D^{-\alpha+1}$ it holds for every macro state μ that*

$$|M_{\mu B}| \geq \max\left\{b^+_{\min},\frac{\operatorname{tr}(B^+)}{d_\mu}\left(1-\frac{D-d_\mu}{D^{1-2\tau}}\right)\right\} - \min\left\{b^-_{\max},\frac{\operatorname{tr}(B^-)}{d_\mu}\right\}. \tag{148}$$

In particular, if $B=P_\nu$ for some macro state ν, then

$$M_{\mu\nu} \geq \frac{d_\nu}{d_\mu}\left(1-\frac{D-d_\mu}{D^{1-2\tau}}\right), \tag{149}$$

and, moreover,

$$M_{\mu\nu} \geq 1-\frac{D-d_\nu}{D^{1-2\tau}}\,. \tag{150}$$

Proof. Let $\tau>0,\alpha>0,D\geq D_0(\tau,\alpha)$. Since D_0 does not depend on the sequence $n^{(D)}$,

$$\mathbb{P}^{(D)}\left(\|\phi_n^{(D)}\|_\infty > D^\tau D^{-1/2}\right)\leq D^{-\alpha} \tag{151}$$

for all $n=1,\ldots,D$. Writing $\mathbb{P}$ for $\mathbb{P}^{(D)}$ and ϕ_n for $\phi_n^{(D)}$, we obtain that

$$\mathbb{P}\left(\forall n:\|\phi_n\|_\infty\leq D^{-1/2+\tau}\right) = 1-\mathbb{P}\left(\exists n:\|\phi_n\|_\infty > D^{-1/2+\tau}\right) \tag{152a}$$

$$\geq 1 - \sum_{n=1}^{D} \mathbb{P}\left(\|\phi_n\|_\infty > D^{-1/2+\tau}\right) \tag{152b}$$
$$\geq 1 - D^{-\alpha+1}. \tag{152c}$$

Now assume that $\|\phi_n\|_\infty \leq D^{-1/2+\tau}$ for all $n = 1, \ldots, D$. Then

$$\langle\phi_n|P_\mu|\phi_n\rangle = 1 - \sum_{\mu'\neq\mu} \langle\phi_n|P_{\mu'}|\phi_n\rangle \tag{153a}$$
$$= 1 - \sum_{l\in[D]\setminus I_\mu} |\phi_n(l)|^2 \tag{153b}$$
$$\geq 1 - \frac{D - d_\mu}{D^{1-2\tau}} \tag{153c}$$

where $I_\mu = \{d_1 + \cdots + d_{\mu-1} + 1, \ldots, d_1 + \cdots + d_\mu\}$.

As in the proof of Theorem 4, we have that

$$|M_{\mu B}| \geq \frac{1}{d_\mu}\left(\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|B^+|\phi_n\rangle - \sum_m \langle\phi_m|P_\mu|\phi_m\rangle\langle\phi_m|B^-|\phi_m\rangle\right), \tag{154}$$

where B^+ and B^- denote the positive and negative part of B, respectively.

We find that

$$\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|B^+|\phi_n\rangle \geq \mathrm{tr}(B^+)\left(1 - \frac{D - d_\mu}{D^{1-2\tau}}\right), \tag{155}$$

and together with (109a), (109c) and (109d) we obtain that

$$|M_{\mu B}| \geq \max\left\{b^+_{\min}, \frac{\mathrm{tr}(B^+)}{d_\mu}\left(1 - \frac{D - d_\mu}{D^{1-2\tau}}\right)\right\} - \min\left\{b^-_{\max}, \frac{\mathrm{tr}(B^-)}{d_\mu}\right\}. \tag{156}$$

In particular, if $B = P_\nu$ for some macro state ν, then $B^+ = P_\nu$, $B^- = 0$, $\mathrm{tr}(B^+) = d_\nu$, $b^\pm_{\min} = b^-_{\max} = 0$, $b^+_{\max} = 1$, and thus

$$M_{\mu\nu} \geq \frac{d_\nu}{d_\mu}\left(1 - \frac{D - d_\mu}{D^{1-2\tau}}\right). \tag{157}$$

Alternatively, in the case that $B = P_\nu$ we can apply the estimate in (153c) to $\langle\phi_n|P_\nu|\phi_n\rangle$, which immediately yields

$$M_{\mu\nu} = \frac{1}{d_\mu}\sum_n \langle\phi_n|P_\mu|\phi_n\rangle\langle\phi_n|P_\nu|\phi_n\rangle \geq 1 - \frac{D - d_\nu}{D^{1-2\tau}}. \tag{158}$$

□

If the macro state μ resp. ν is such that $(D - d_\mu)D^{-1+2\tau} > 1$ resp. $(D - d_\nu)D^{-1+2\tau} > 1$, then the lower bound in (149) resp. (150) becomes negative and therefore useless since we always have the trivial bound $M_{\mu\nu} \geq 0$. However, if μ or ν is the equilibrium macro space in the sense that the corresponding macro space is extremely dominant, more precisely, if $d_{\text{eq}} = D - o(D^{1-2\tau})$, then the lower bounds for the $M_{\mu\nu}$ are nontrivial. If $\nu = \text{eq}$, then (150) implies that $M_{\mu\nu} \gtrsim 1 \approx \frac{d_{\text{eq}}}{D}$ and if $\mu = \text{eq}$, then (149) shows that $M_{\mu\nu} \gtrsim \frac{d_\nu}{d_\mu} \approx \frac{d_\nu}{D}$, in agreement with our expectations.

7 Consequences of the Eigenstate Thermalization Hypothesis

Another result, due to Cipolloni, Erdős and Henheik (CEH) [6], shows that Wigner matrices (i.e., $H = H^*$ such that h_{ij} for $i \leq j$ are centered, independent random variables with bounded moments, the h_{ij} for $i < j$ are identically distributed, and the h_{ii} are identically distributed) satisfy a version of the eigenstate thermalization hypothesis (ETH) that implies that the eigenvectors are delocalized. Although the matrices we are most interested in, the band matrices, are not Wigner matrices, we make explicit in this section which lower bounds on $M_{\mu\nu}$ (essentially versions of $M_{\mu\nu} \approx d_\nu/D$) would follow from the ETH in the version formulated by CEH (Proposition 3) and what they would imply about the relative error of generalized normal typicality (Corollary 5). After all, as mentioned in the beginning of Section 6, it is believed that for band matrices with sufficiently wide band, all eigenvectors are delocalized.

We begin by formulating the precise condition:

Definition 5 (ETH according to CEH [6])**.** We say that a sequence $(\mathbb{P}^{(D)})_{D\in\mathbb{N}}$ of probability distributions over the Hermitian $D\times D$ matrices satisfies the CEH-version of ETH if for every sequence $(B^{(D)})_{D\in\mathbb{N}}$ of $D\times D$ matrices with $\|B^{(D)}\| \leq 1$ and every $\gamma > 0$ and $\xi > 0$, there is $\tilde{D}_0 \in \mathbb{N}$ such that for $D \geq \tilde{D}_0$, it has probability at least $1 - D^{-\gamma}$ that

$$\max_{i,j\in[D]} \left| \langle \phi_i^{(D)} | B^{(D)} | \phi_j^{(D)} \rangle - \frac{\operatorname{tr}(B^{(D)})}{D} \delta_{ij} \right| \leq \frac{D^\xi}{D} \operatorname{tr}(|\mathring{B}^{(D)}|^2)^{1/2}, \tag{159}$$

where $\phi_1^{(D)}, \ldots, \phi_D^{(D)}$ is any orthonormal eigenbasis of $H^{(D)} \sim \mathbb{P}^{(D)}$ and $\mathring{B}^{(D)} = B^{(D)} - \operatorname{tr}(B^{(D)})/D$ denotes the traceless part of B.

In that case, we obtain in particular that for any sequence $(B^{(D)})$ of Hermitian $D \times D$ matrices, any $\xi > 0$ and $\gamma > 0$, sufficiently large D, $B = B^{(D)}$ and every orthonormal eigenbasis $\phi_1, \ldots, \phi_D$ of $H = H^{(D)}$,

$$\langle \phi_n | B | \phi_n \rangle \geq \frac{\operatorname{tr}(B)}{D} - \frac{D^\xi}{D} \operatorname{tr}(|\mathring{B}|^2)^{1/2} \tag{160}$$

for all $n = 1, \ldots, D$ simultaneously with probability at least $1 - D^{-\gamma}$. Now recall that, if H is non-degenerate, then

$$M_{\mu B} = \frac{1}{d_\mu} \sum_n \langle \phi_n | P_\mu | \phi_n \rangle \langle \phi_n | B | \phi_n \rangle \,. \tag{161}$$

Proposition 3 (Lower bound for $|M_{\mu B}|$)**.** *Let $\xi > 0$, let H be a Hermitian $D \times D$ matrix with orthonormal eigenbasis $\{\phi_1, \ldots, \phi_D\}$, and let B be a Hermitian $D \times D$ matrix such that* (160) *is satisfied for all $n = 1, \ldots, D$. Then, for every macro state μ,*

$$|M_{\mu B}| \geq \frac{|\operatorname{tr}(B)|}{D} - \frac{D^\xi}{D} \operatorname{tr}(|\mathring{B}|^2)^{1/2}. \tag{162}$$

In particular,

$$M_{\mu\nu} \geq \frac{d_\nu}{D} \left(1 - \frac{D^\xi}{\sqrt{d_\nu}} \right). \tag{163}$$

for every macro state ν. Moreover,

$$M_{\mu\nu} \geq \frac{d_\nu}{D} \left(1 - \frac{D^\xi}{\sqrt{d_\mu}} \right). \tag{164}$$

Proof. From (160) we obtain that

$$M_{\mu B} = \frac{1}{d_\mu} \sum_n \langle \phi_n | P_\mu | \phi_n \rangle \langle \phi_n | B | \phi_n \rangle \geq \frac{\operatorname{tr}(B)}{D} - \frac{D^\xi}{D} \operatorname{tr}(|\mathring{B}|^2)^{1/2}. \tag{165}$$

Similarly one finds that

$$M_{\mu B} \leq \frac{\operatorname{tr}(B)}{D} + \frac{D^\xi}{D} \operatorname{tr}(|\mathring{B}|^2)^{1/2} \tag{166}$$

and therefore

$$\left| M_{\mu B} - \frac{\operatorname{tr}(B)}{D} \right| \leq \frac{D^\xi}{D} \operatorname{tr}(|\mathring{B}|^2)^{1/2}. \tag{167}$$

With the help of the reverse triangle inequality we obtain (162). If $B = P_\nu$ for some macro state ν we have $\operatorname{tr}(B) = d_\nu$, $|\mathring{B}|^2 = P_\nu - 2P_\nu d_\nu / D + d_\nu^2 / D^2$, therefore $\operatorname{tr}(|\mathring{B}|^2) = d_\nu - 2d_\nu^2/D + d_\nu^2/D^2 \leq d_\nu$ and (163) thus follows immediately from (162).

Concerning the last bound, observe that (160) for $B = P_\mu$ implies

$$M_{\mu\nu} = \frac{1}{d_\mu} \sum_n \langle \phi_n | P_\mu | \phi_n \rangle \langle \phi_n | P_\nu | \phi_n \rangle \geq \frac{d_\nu}{d_\mu} \left(\frac{d_\mu}{D} - \frac{D^\xi \sqrt{d_\mu}}{D} \right). \tag{168}$$

□

For large dimensions D we find for any macro states μ and ν that $M_{\mu\nu} \gtrsim \frac{d_\nu}{D}$ (provided that d_ν is large enough such that $\frac{d_\nu}{D} \gg D^{\xi-1}\sqrt{d_\nu}$), in agreement with our expectations. For the relative errors of the $\|P_\nu\psi_t\|^2$, we obtain the following.

Corollary 5. *Suppose that $(\mathbb{P}^{(D)})$ satisfies the CEH-version of ETH, and that it is a continuous distribution for every D. Let $\varepsilon, \delta, \xi, \gamma > 0$, and let μ and ν be arbitrary macro states such that $\sqrt{d_\nu} \geq 2D^\xi$ or $\sqrt{d_\mu} \geq 2D^\xi$, i.e.,*

$$s_\nu \geq 2\xi s_{\mathrm{mc}} + \frac{2k_B}{N}\ln 2 \quad \text{or} \quad s_\mu \geq 2\xi s_{\mathrm{mc}} + \frac{2k_B}{N}\ln 2. \tag{169}$$

Then, for sufficiently large D and with probability at least $1 - D^{-\gamma}$, $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,\infty)$,

$$\frac{\left|\|P_\nu\psi_t\|^2 - M_{\mu\nu}\right|}{M_{\mu\nu}} \leq \frac{8}{\sqrt{\varepsilon\delta}} \exp\left(-\frac{N}{k_B}\left(\max\{s_\mu, s_\nu\} - s_{\mathrm{mc}} + \frac{1}{2}\min\{s_\nu, s_\mu\}\right)\right). \tag{170}$$

Proof. It follows immediately from (163) that

$$M_{\mu\nu} \geq \frac{d_\nu}{2D}. \tag{171}$$

Together with the upper bound for the absolute error from Theorem 2, we obtain that $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ are such that for $(1-\delta)$-most $t \in [0,\infty)$,

$$\frac{\left|\|P_\nu\psi_t\|^2 - M_{\mu\nu}\right|}{M_{\mu\nu}} \leq \frac{8}{\sqrt{\varepsilon\delta}} \frac{D}{d_\nu\sqrt{d_\mu}} \left(\min\left\{1, \frac{d_\nu}{d_\mu}\right\}\right)^{1/2} \tag{172}$$

The claim now follows immediately from the definition of s_μ, s_ν and s_{mc}. $\square$

The relative errors in Corollary 5 are small for large N if $s_{\mathrm{mc}} < \max\{s_\mu, s_\nu\} + \min\{s_\mu, s_\nu\}/2$, i.e., we recover the condition we found in the case of normal typicality, see also Remark 1.

A Appendix - No Resonances

In this appendix we provide a proof of the fact that Hermitian matrices whose joint distribution of their entries is absolutely continuous with respect to the Lebesgue measure have, with probability 1, no degeneracies and no resonances (i.e., also the eigenvalue gaps are non-degenerate). This fact is widely known, but we could not find a proof in the literature.

Proposition 4 (No resonances). *Let H be a random Hermitian $D \times D$ matrix with eigenvalues $\lambda_1, \dots, \lambda_D$ such that the joint distribution of its entries is absolutely continuous with respect to the Lebesgue measure. Then,*

$$\mathbb{P}\left(\lambda_i - \lambda_j = \lambda_k - \lambda_l \text{ for some } (i \neq j \text{ or } k \neq l) \text{ and } (i \neq k \text{ or } j \neq l)\right) = 0. \tag{173}$$

We begin with some preparations for the proof. Let $S(D)$ denote the set of Hermitian $D \times D$ matrices and $U(D)$ the unitary group of $D \times D$ matrices, here regarded as orthonormal bases of the underlying Hilbert space. We define $\chi : \mathbb{R}^D \times U(D) \to S(D)$ by

$$\chi(\lambda_1, \ldots, \lambda_D, \psi_1, \ldots, \psi_D) = \sum_{j=1}^{D} \lambda_j |\psi_j\rangle\langle\psi_j|, \tag{174}$$

where ψ_j denotes the jth column of the unitary matrix $[\psi_1, \ldots, \psi_D]$. Since the matrix $\chi(\lambda_1, \ldots, \lambda_D, \psi_1, \ldots, \psi_D)$ remains constant when the phase of the ψ_i is changed, χ defines a mapping $\varphi : \mathbb{R}^D \times U(D)/U(1)^D \to S(D)$.

Lemma 1. *The map φ defined above is smooth.*

Proof. First note that the function χ in (174) defined on $\mathbb{R}^D \times \mathbb{C}^{D\times D} \to \mathbb{C}^{D\times D}$ (extended in the obvious way) is smooth since all components of $\chi(\lambda_1, \ldots, \lambda_D, \psi_1, \ldots, \psi_D)$ are polynomials in the λ_j and the $(\psi_j)_k$. Since $U(D)$ is a (embedded) submanifold of $\mathbb{C}^{D\times D}$, the restriction of χ to $\mathbb{R}^D \times U(D)$ is also smooth [24, Prop. 5.27]. It remains to show that also the replacing of $U(D)$ by the quotient $U(D)/U(1)^D$ does not destroy the smoothness. To this end, consider the map

$$f : \mathbb{C}^D \backslash \{0\} \to \mathbb{C}^{D\times D},\ \psi \mapsto f(\psi) = |\psi\rangle\langle\psi|. \tag{175}$$

This map is obviously smooth and remains smooth when considered as a map on $(\mathbb{C}^D \backslash \{0\})/U(1)$. Thus, we can conclude that φ is smooth. □

Proof of Proposition 4. Since $U(1)^D$ is a closed subset of $U(D)$ (because limits of diagonal matrices are diagonal), and since $U(1)^D$ is a Lie subgroup of $U(D)$, it follows that the set $U(D)/U(1)^D$ of left cosets is a smooth manifold [24, Thm. 21.17]. It is well known that $\dim U(D) = D^2$, $\dim U(D)/U(1)^D = D^2 - D$, $\dim S(D) = D^2$ and thus $\dim\left(\mathbb{R}^D \times (U(D)/U(1)^D)\right) = \dim S(D)$. The spaces $U := \mathbb{R}^D \times (U(D)/U(1)^D)$ and $V := S(D)$ are therefore smooth manifolds of the same dimension and $\varphi\big|_{\mathbb{R}^D \times (U(D)/U(1)^D)} : U \to V$ is smooth by Lemma 1. Since the set

$$N_{\mathrm{res}} := \Big\{(x_1, \ldots, x_D) \in \mathbb{R}^D : \\ x_i - x_j = x_k - x_l \text{ for some } (i \neq j \text{ or } k \neq l) \text{ and } (i \neq k \text{ or } j \neq l)\Big\} \tag{176}$$

is a null set in $\mathbb{R}^D$, it follows that $N_{\mathrm{res}} \times (U(D)/U(1)^D)$ is a null set in $\mathbb{R}^D \times (U(D)/U(1)^D)$. Generally, if $f : U \to V$ is a smooth mapping between smooth manifolds U, V of equal dimension and $N \subset U$ is a null set, then $f(N)$ is a null set in V [24, Thm. 6.9]. Thus,

$$\varphi(N_{\mathrm{res}} \times (U(D)/U(1)^D)) = \chi(N_{\mathrm{res}} \times U(D)) \tag{177}$$

is a null set in $S(D)$. Altogether we have shown that the set of Hermitian matrices that have resonances has measure zero. Finally, the absolute continuity of the joint distribution of the entries of H with respect to the Lebesgue measure immediately proves the claim. □

B Appendix - Numerical Examples

We briefly describe the numerical simulations we used to create Figures 1 and 2. These simulations serve for illustrating our results and stating some further conjectures. We partition the D-dimensional Hilbert space $\mathcal{H} := \mathbb{C}^D = \bigoplus_{\nu=1}^4 \mathbb{C}^{d_\nu}$ into four macro spaces $\mathcal{H}_\nu$ of dimension d_ν with $d_\nu \ll d_{\nu+1}$ for $\nu = 1, 2, 3$; more precisely, we choose $d_\nu = 2 \times 10^{\nu-1}$. Then $\mathcal{H}_1$ is spanned by the first d_1 canonical basis vectors, $\mathcal{H}_2$ by the next d_2 canonical basis vectors and so on, and $\mathcal{H}_4$, the largest macro space in the decomposition, corresponds to the "equilibrium space." The Hamiltonian $H = (h_{jk})$ is modelled by a Hermitian random $D \times D$ matrix with a band structure which means that it couples neighboring macro spaces more strongly than distant ones, see also Figure 3. The entries of H satisfy $h_{jj} \sim \mathcal{N}(0, \sigma_{jj}^2)$ and $h_{jk} \sim \mathcal{N}(0, \sigma_{jk}^2/2) + i\mathcal{N}(0, \sigma_{jk}^2/2)$ for $j \neq k$, where

$$\sigma_{jk}^2 := \exp(-s|j-k|) \tag{178}$$

with some parameter $s > 0$ that controls the bandwidth of the random matrix, i.e., the variances decrease exponentially in the distance from the diagonal. Note that this model is not covered by our examples in Section 4.3 since the σ_{jk} cannot be bounded by a positive D-independent constant or by D^α for some $\alpha \in \mathbb{R}$ from below. However, it also suggests that similar results should hold in more general situations than in the ones we were able to study.

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B. Submitted Manuscripts

B.1. Macroscopic Thermalization for Highly Degenerate Hamiltonians After Slight Perturbation

Macroscopic Thermalization for Highly Degenerate Hamiltonians After Slight Perturbation

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Abstract

We say of an isolated macroscopic quantum system in a pure state ψ that it is in macroscopic thermal equilibrium if ψ lies in or close to a suitable subspace $\mathcal{H}_{\mathrm{eq}}$ of Hilbert space. It is known that every initial state ψ_0 will eventually reach macroscopic thermal equilibrium and stay there most of the time ("thermalize") if the Hamiltonian is non-degenerate and satisfies the appropriate version of the eigenstate thermalization hypothesis (ETH), i.e., that every eigenvector is in macroscopic thermal equilibrium. Shiraishi and Tasaki recently proved the ETH for a certain perturbation H_θ^{fF} of the Hamiltonian H_0^{fF} of $N \gg 1$ free fermions on a one-dimensional lattice. The perturbation is needed to remove the high degeneracies of H_0^{fF}. Here, we point out that also for degenerate Hamiltonians, all ψ_0 thermalize if the ETH holds for *every* eigenbasis, and we prove that this is the case for H_0^{fF}. On top of that and more generally, we develop another strategy of proving thermalization, inspired by the fact that there is *one* eigenbasis of H_0^{fF} for which ETH can be proven more easily and with smaller error bounds than for the others. This strategy applies to arbitrarily small *generic* perturbations H^{fF} of H_0^{fF} and to arbitrary spatial dimensions. In fact, we consider any given H_0, suppose that the ETH holds for some but not necessarily every eigenbasis of H_0, and add a small generic perturbation, $H = H_0 + \lambda V$ with $\lambda \ll 1$. Then, although H (which is non-degenerate) may still not satisfy the ETH, we show that nevertheless (i) every ψ_0 thermalizes

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for most perturbations V, and more generally, (ii) for any subspace $\mathcal{H}_\nu$ (such as corresponding to a non-equilibrium macro state), most perturbations V are such that most ψ_0 from $\mathcal{H}_\nu$ thermalize.

Key words: eigenstate thermalization hypothesis (ETH); generic perturbation; thermal equilibrium subspace.

1 Introduction

We consider an isolated macroscopic quantum system S in a pure state $\psi \in \mathcal{H}$ evolving unitarily, $\psi_t = e^{-iHt}\psi_0$, for simplicity with ψ_0 (and thus ψ_t) in a micro-canonical "energy shell" $\mathcal{H}_{\mathrm{mc}}$, the spectral subspace of H corresponding to energies in a small interval $[E - \Delta E, E]$. There are different concepts of thermal equilibrium of quantum systems [24, 22, 19, 12, 31, 9, 17, 20, 18, 14, 30, 15] (for a comparison of some, see [10, 11]), and one important concept for an isolated system S is that its state ψ lies, at least approximately, in a certain subspace $\mathcal{H}_{\mathrm{eq}} \subseteq \mathcal{H}_{\mathrm{mc}}$ containing the pure states that "look macroscopically like thermal equilibrium states." Following [10, 11], we call this concept "macroscopic thermal equilibrium" (MATE) [12, 31, 20, 13, 29] and speak of "macroscopic thermalization" if ψ_t reaches MATE sooner or later (even though ψ_0 may be far from MATE) and stays there for most of the time. For brevity, we will drop the adjective "macroscopic" and just speak of "thermal equilibrium" and "thermalization" in the following.

To take a concrete example, consider a gas of many but finitely many particles in a box. In the classical case it is well known, going back to Boltzmann, that the majority of microstates on some energy shell look macroscopically like a gas in thermal equilibrium, i.e., have uniform empirical position distribution and Maxwellian empirical momentum distribution. Moreover, it is expected (but very hard to prove) that the majority of those fewer microstates that start in some non-equilibrium macrostate (say all particles start in the same half of the box) will also thermalize after some time, i.e., look macroscopically like a gas in thermal equilibrium for most later times. Note that this concept of thermalization does not require a heat bath, nor infinite system size, nor randomness in the evolution. The framework sketched above captures the analogous question for quantum systems, and our explicit example is indeed the (perturbed) free Fermi gas in a box.

A natural question is under which conditions on $\mathcal{H}_{\mathrm{eq}}$, H, and ψ_0 the system will thermalize. An observation made by Goldstein et al. [12] and Tasaki [31] is that if H has non-degenerate spectrum and satisfies the appropriate version of the eigenstate thermalization hypothesis (ETH) [5, 28], i.e., if

$$\text{every eigenvector of } H \text{ is in MATE,} \tag{1}$$

then *every* ψ_0 thermalizes. Goldstein et al. [12] further proved that if $\dim \mathcal{H}_{\mathrm{eq}} / \dim \mathcal{H}_{\mathrm{mc}}$ is close to 1 (which is usually satisfied in practice), and if we take H to be a random matrix with unitarily invariant distribution (or, equivalently, with an eigenbasis that is uniformly (Haar) distributed over all orthonormal bases and independent of the eigenvalues) and non-degenerate eigenvalues, then it satisfies the ETH (1) with probability close to 1. (A similar result was obtained by Reimann [23].) An observation that we add in Proposition 1 below is that (1) alone guarantees that every ψ_0 thermalizes, even for Hamiltonians with highly degenerate spectra.

The present paper is inspired particularly by recent works of Tasaki [33, 32] in which he focused on specifying concrete $\mathcal{H}_{\text{eq}}$ and H and proving for them that every ψ_0 thermalizes. The goal of proving thermalization for specific Hamiltonians brings into focus difficulties arising from highly degenerate eigenvalues; this paper is mainly about ways to deal with these difficulties.

Concretely, Tasaki [33, 32] (and earlier Shiraishi and Tasaki [26]) considered $N \gg 1$ free non-relativistic fermions ("the free Fermi gas") on the 1d lattice $\Lambda := \mathbb{Z}/L\mathbb{Z}$ with $L > N$ sites, which defines a Hilbert space $\mathcal{H}$ and a Hamiltonian H_0^{fF}, and took $\mathcal{H}_{\text{eq}}$, as a simple model, to comprise the states for which the number of particles in a subinterval $\Gamma \subset \Lambda$ of the lattice lies within a suitable tolerance of $N|\Gamma|/L$.[1] Since for this simple model, the choice of $\mathcal{H}_{\text{eq}}$ involves no conditions on the energy, the restriction to $[E - \Delta E, E]$ plays no role, so we can simply take $\mathcal{H}_{\text{mc}} = \mathcal{H}$. It is easy to see that $\dim \mathcal{H}_{\text{eq}} / \dim \mathcal{H}$ is indeed close to 1. However, H_0^{fF} has highly degenerate eigenvalues, and for this reason Shiraishi and Tasaki considered a perturbation H_θ^{fF} of H_0^{fF} by a small magnetic flux θ through the ring, which removes all degeneracies if $L \geq 3$ is prime. Correspondingly, they proved thermalization of any initial state ψ_0 under the evolution generated by H_θ^{fF}. The question whether a similar statement holds also for H_0^{fF} was left open.

One of our main results (Theorem 2 in Section 3.2) proves the ETH (1) relative to this $\mathcal{H}_{\text{eq}}$ also for the unperturbed free fermion Hamiltonian H_0^{fF} in one spatial dimension. As a corollary we conclude, using the observation explained above, that also under the evolution of H_0^{fF} every initial state ψ_0 thermalizes.

We also present results in another direction: Consider for an orthonormal basis (ONB) B the condition that

$$\text{every } \phi \in B \text{ lies in MATE.} \tag{2}$$

[1] A more realistic model of $\mathcal{H}_{\text{eq}}$ would involve (a) not only one region Γ but every (suitably coarse-grained) macroscopic region in space, (b) the (coarse-grained) distribution of momenta, and (c) other macroscopic observables such as total spin. Item (a) can be implemented rather easily. Indeed, in [33, 32], Tasaki partitioned the lattice into a (not too large) number of subintervals Γ_i (thought of as macroscopic regions) and required, as the definition of $\mathcal{H}_{\text{eq}}$, that the number of particles in each Γ_i lies within suitable tolerances of $N|\Gamma_i|/L$, so the coarse grained empirical distribution of particles is approximately uniform in 1d physical space. Since this setup can be dealt with mathematically in much the same way as just considering a single Γ (see Remark 1), we will stick here with the simpler model. Item (b), on the other hand, is pointless for the free Fermi gas, since individual momenta are conserved, and the treatment of the interacting Fermi gas is far beyond the scope of this paper.

It turns out that there is *one* eigen-ONB B_1 of H_0^{fF} that is particularly good in several ways (see Proposition 3 in Section 3):

(i) B_1 satisfies (2) with much smaller error bounds (i.e., smaller deviations from $\mathcal{H}_{\mathrm{eq}}$) than other eigen-ONBs, in fact like e^{-N} instead of a negative power of N;

(ii) it is easier to prove (2) for B_1 than for other eigen-ONBs;

(iii) in higher dimensions (i.e., considering a lattice $\mathbb{Z}^d/L\mathbb{Z}^d$ with $d > 1$), we could find a proof of (2) for (the analog of) B_1, but not for other eigen-ONBs of (the analog of) H_0^{fF}.

This situation motivates us to propose a strategy for dealing with degenerate Hamiltonians for which some, but not every, eigen-ONB B satisfies the ETH (2). Let us call such a general Hamiltonian H_0. (Note that violations of (1) imply the existence of some ψ_0 that will not thermalize: for example, eigenstates that are not initially in MATE will never reach MATE because they are stationary.) If H_0 is only moderately degenerate, then the ETH for one eigenbasis would enforce the ETH for every other eigenbasis with moderately worse error bounds (see Corollary 1 in Section 2), but in the example of the free Fermi gas in $d \geq 1$ space dimensions, the degeneracy is around 2^{Nd} and thus too high. For this reason, we also propose to consider a perturbation of H_0,[2] but a *random* perturbation

$$H = H_0 + \lambda V\,, \tag{3}$$

thought of as a *generic* perturbation. We argue in Section 2.4 that it is physically appropriate to consider a generic perturbation. In fact, as soon as V has continuous probability distribution, H has non-degenerate eigenvalues and eigenvalue gaps with probability 1 for every $0 < \lambda < \lambda_0$ for suitable λ_0.[3] So, generic (arbitrarily small) perturbations remove the degeneracy of H.

Moreover, and this is the content of our other main result (Theorem 1 in Section 3), under the assumption that the distribution of V is invariant under all unitaries or at least those commuting with H_0, most non-equilibrium ψ_0 will thermalize under H, although H may fail to satisfy the ETH (1). Theorem 1 is formulated, not just for the specific H_0^{fF} of free fermions on a lattice, but for general H_0, and can be applied also when H_0 is highly degenerate, in fact whenever H_0 possesses one eigen-ONB satisfying (2).

Let us give some details. We use the notation P_{eq} for the projection to $\mathcal{H}_{\mathrm{eq}}$ and

$$\mathbb{S}(\mathcal{H}) = \big\{\psi \in \mathcal{H} : \|\psi\| = 1\big\} \tag{4}$$

[2]Still, the considerations stay fully rigorous. In particular, they do not involve neglecting higher-order terms in a series expansion (as the word "perturbation" might suggest in some contexts).

[3]Since we could not find a good reference for this fact, we have formulated this fact as Lemma 2 in Section 2 and included a proof in Section 4.

for the unit sphere in any given Hilbert space $\mathcal{H}$. If a Hamiltonian H_0 possesses an eigen-ONB satisfying (2), then some eigen-ONBs may still violate it substantially (see Section 2.2), and in that case there is reason to expect (see Section 3.3) that also $H = H_0 + \lambda V$ with $\lambda \ll 1$ violates (1) with high probability,[4] and thus not *all* ψ_0 thermalize under H. Now it becomes natural to ask whether *most* ψ_0 will thermalize. But it is well known [12, 10, 11] that $\dim \mathcal{H}_{\mathrm{eq}} / \dim \mathcal{H}_{\mathrm{mc}} \approx 1$ implies that most $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{mc}})$ are in MATE already at $t = 0$, and it is also always true, regardless of properties of H_0, that for most ψ_0, ψ_t is in MATE for most t (because the time average of the ensemble average of $\|P_{\mathrm{eq}}\phi\|^2$ is close to 1). That is why we want to consider "non-equilibrium" ψ_0, which we can take to mean that $\psi_0 \in \mathcal{H}_{\mathrm{neq}} := \mathcal{H}_{\mathrm{eq}}^{\perp}$; it is a non-trivial statement that most $\psi_0 \in \mathbb{S}(\mathcal{H}_{\mathrm{neq}})$ thermalize, and Theorem 1 says this is the case. Theorem 1 says even more: We can assume a specific non-equilibrium macro state (e.g., characterized by the condition that between 36% and 37% of all particles are in the left half of the lattice); let $\mathcal{H}_\nu \subseteq \mathcal{H}_{\mathrm{mc}}$ be the subspace of pure states compatible with this macro state. Most V are such that most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ thermalize for sufficiently small λ. We also obtain the following converse statement: for every $\psi_0 \in \mathbb{S}(\mathcal{H})$, most V are such that ψ_0 thermalizes; but keep in mind that once we think of V as fixed (since one usually thinks of the Hamiltonian as fixed), not every ψ_0 thermalizes.

Apart from the general, abstract argument expressed in Theorem 1, we also include the application to the free Fermi gas (Proposition 3 and Corollary 4 in Section 3.2): the existence of an eigen-ONB B_1 of H_0^{fF} satisfying the ETH (2) (with a tolerance ε that is exponentially small in N) can be established in any dimension d of physical space; the thermalization of most non-equilibrium ψ_0 for $H = H_0^{\mathrm{fF}} + \lambda V$ then follows from Theorem 1.

Note that analogous results for the classical free gas of N particles have been obtained in [1, 4, 2]. There, no perturbation is needed to prove thermalization of the gas for long times in any spatial dimension, but only for most and not all initial states.[5] For the one-dimensional quantum mechanical free Fermi gas we obtain an even stronger result, namely thermalization for all initial data. For higher dimensions, however, we need small generic perturbations of H_0^{fF} to infer thermalization for most initial data.

It should be noted that neither [26, 33, 32] nor our results provide meaningful estimates of the time required to reach thermal equilibrium in the (perturbed) free Fermi gas. For the classical gas such estimates have been obtained in [2, 4]. A more quantitative understanding of the non-equilibrium dynamics of one-dimensional inte-

[4] This is presumably different if λV is not small but its columns have magnitude of order 1. Specifically, for $\lambda = 1$ and V from the Gaussian unitary ensemble (GUE), it follows from [3, Thm. 2.7] that $H = H_0 + V$ satisfies a version of ETH, which suggests that it also satisfies our ETH (1), at least up to few exceptions.

[5] It is obvious that for the classical free gas not all initial states can thermalize, not even in the sense of becoming spatially homogeneous.

grable quantum gases is provided by so called Quantum Generalized Hydrodynamics, see, e.g., [6, 25]. However, this theory is not rigorous yet and applies only to rather special initial states.

Finally, let us mention another non-trivial application of Theorem 1. Shortly after the first version of our paper appeared as a preprint, Hal Tasaki [34] realized that Theorem 1 can be applied to the Ising model in two dimensions below the critical temperature. Roughly speaking, he proves that any initial state in a given highly degenerate eigenspace of the Hamiltonian thermalizes under most slightly perturbed dynamics in the sense that the macroscopic magnetization approaches and remains very close to the corresponding microcanonical expectation value. In this system it is indeed the case that some eigenbases of the unperturbed Hamiltonian satisfy the ETH and others do not.

This paper is structured as follows. In Section 2, we provide an overview of the background and provide further motivation. In Section 3, we state our main results. In Section 4, we give the proofs. In Section 5, we conclude.

2 Motivation

In this section, we give some more details about the considerations outlined in the introduction.

2.1 MATE and ETH

We will only operate within one energy shell $\mathcal{H}_{\mathrm{mc}}$ and pretend that this subspace remains unchanged even when we vary the Hamiltonian; we take $\mathcal{H}_{\mathrm{mc}}$ to be "the" Hilbert space of the system S and simply write $\mathcal{H}$ for it. We take for granted that $\mathcal{H}$ has finite dimension. Following von Neumann [37], we regard macroscopic observables as given which are suitably coarse grained so that they commute with each other and their eigenvalues are rounded to the macroscopic resolution. Then $\mathcal{H}_{\mathrm{eq}}$ can be thought of as one of their simultaneous eigenspaces, with eigenvalues given by the thermal equilibrium values [12, 13]. In our mathematical result, $\mathcal{H}_{\mathrm{eq}}$ could be any subspace, although it will play a role that $\mathcal{H}_{\mathrm{eq}}$ has most of the dimensions of $\mathcal{H}$.

Let u be the uniform (normalized surface area) measure on $\mathbb{S}(\mathcal{H})$; a u-distributed vector will also be said to be "purely random." When saying that "the statement $S(\psi)$ is true for $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$," we mean that

$$u\{\psi \in \mathbb{S}(\mathcal{H}) : S(\psi) \text{ holds}\} \geq 1-\varepsilon\,. \tag{5}$$

Analogously, when saying that "the statement $S(t)$ is true for $(1-\delta)$-most $t \in [0,\infty)$," we mean that

$$\liminf_{T\to\infty} \frac{1}{T}\big|\{t \in [0,T] : S(t) \text{ holds}\}\big| \geq 1-\delta\,, \tag{6}$$

where $|\{\cdot\}|$ means the length (Lebesgue measure) of the set $\{\cdot\}$.

Since $\mathbb{S}(\mathcal{H}_{\mathrm{eq}})$ is a null set in $\mathbb{S}(\mathcal{H})$ relative to u, we regard a $\psi \in \mathbb{S}(\mathcal{H})$ as being in thermal equilibrium whenever it lies in the set

$$\mathrm{MATE}_\varepsilon = \left\{\psi \in \mathbb{S}(\mathcal{H}) : \|P_{\mathrm{eq}}\psi\|^2 \geq 1-\varepsilon\right\}, \tag{7}$$

once we have chosen the desired tolerance $\varepsilon > 0$. The statement that the ETH (1) implies that every ψ_0 thermalizes can be formulated rigorously as follows.

Proposition 1. *Suppose* $\dim \mathcal{H} =: D < \infty$*,* $\mathcal{H}_{\mathrm{eq}}$ *is a subspace and* P_{eq} *the projection to it, the operator* H *on* $\mathcal{H}$ *is self-adjoint,* $\varepsilon, \delta > 0$*, and (ETH)*

$$\forall \text{ normalized eigenvector } \phi \text{ of } H\text{:} \quad \phi \in \mathrm{MATE}_{\varepsilon\delta}. \tag{8}$$

Then for every $\psi_0 \in \mathbb{S}(\mathcal{H})$ *and* $(1-\delta)$*-most* $t \in [0,\infty)$*,*

$$\psi_t \in \mathrm{MATE}_\varepsilon\,. \tag{9}$$

All proofs are given in Section 4. Note that the inaccuracy assumed in the ETH ($\varepsilon\delta$) must be smaller than the inaccuracy desired for the thermalization of ψ_t (ε) by a factor given by the tolerance (δ) desired for the notion of "most t." Note that for non-degenerate H the statement of Proposition 1 was contained in [12] as a step in a proof and in [31] as Theorem 7.1. However, to our knowledge the observation that the ETH can also be used for degenerate Hamiltonians in the form (8) to derive thermalization of every initial state seems not to have been mentioned before in the literature. It entails in particular for the Hamiltonian H_0^{fF} of the one-dimensional free Fermi gas, which satisfies (8) by Theorem 2, that *all* ψ_0 thermalize (rather than merely *most* non-equilibrium ψ_0, as shown by Theorem 1).

2.2 The Problem About High Degeneracy

Suppose that a Hamiltonian H_0 is highly degenerate and possesses an eigen-ONB $B = (\phi_k)_k$ that satisfies the ETH in the sense (2) or more precisely

$$\forall k: \quad \phi_k \in \mathrm{MATE}_\varepsilon\,. \tag{10}$$

The question is how much the tolerance ε has to be increased to ensure that all eigenvectors are in MATE (or, equivalently, all orthonormal eigenbases satisfy ETH). The following elementary result for matrices helps us answer this question.

Lemma 1. *Let* $\varepsilon > 0$*. If a positive semi-definite* $D \times D$ *matrix* M *has all diagonal entries* $\leq \varepsilon$*, then*

$$\|M\| \leq \varepsilon D\,, \tag{11}$$

where $\|\cdot\|$ *denotes the operator norm. The bound is sharp, i.e., there exists* M *for which equality holds.*

With this, we can guarantee ETH (10) with a larger error instead of ε:

Corollary 1. *If P_{eq} is any projection in the D-dimensional Hilbert space $\mathcal{H}$ and H_0 a Hamiltonian with maximal degeneracy D_E that has an eigen-ONB $(\phi_k)_k$ satisfying* (10)*, then for any normalized eigenvector ϕ of H_0,*

$$\phi \in \mathrm{MATE}_{\varepsilon D_E}\,. \tag{12}$$

The bound is sharp in the sense that for any D_E and any $\varepsilon = 1/n$ for some natural number n there are H_0, P_{eq}, and ϕ such that

$$\|P_{\mathrm{eq}}\phi\|^2 = \max\{1 - \varepsilon D_E, 0\}\,. \tag{13}$$

This means that the error bound we can guarantee in (8) is εD_E instead of ε, but for our example of H_0 (the free Fermi gas) in $d \geq 1$ dimensions,

$$D_E \geq 2^{Nd} \tag{14}$$

(see Proposition 3), which is so large that the bound εD_E is no longer small and thus becomes useless.

2.3 Consequences of Generic Perturbations

The following lemma provides a precise formulation of the intuitively rather obvious statement that a generic perturbation will lift the degeneracy of eigenvalues and eigenvalue gaps. (In this paper, it will play no role that the eigenvalue gaps are non-degenerate, but we take note of this fact for the sake of completeness, as the degeneracy of the eigenvalue gaps is expected to determine the time scales on which thermalization takes place.)

Lemma 2. *Let V be a random matrix whose distribution is continuous in the space of Hermitian $D \times D$ matrices. Then with probability 1, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, the Hamiltonian $H = H_0 + \lambda V$ has non-degenerate eigenvalues and eigenvalue gaps.*

Now let P_{eq} be any projection, $P_{\mathrm{neq}} = I - P_{\mathrm{eq}}$, and let H_0 be a degenerate Hamiltonian for which one eigen-ONB satisfies (10) but others do not necessarily. By Corollary 1, some eigenvectors can deviate from $\mathcal{H}_{\mathrm{eq}}$ by εD_E, which need not be small as D_E can be exponentially large in the particle number N. On the other hand, even if D_E is that large, a *typical* unit vector is not that bad:

Proposition 2. *Let $\varepsilon > 0$, $\mathcal{H}_e$ be an eigenspace of H_0 with dimension D_e, and suppose that one eigen-ONB of H_0 satisfies* (10)*. Then for $\delta = 2\exp(-C\varepsilon^2 D_e)$ with $C = 2/9\pi^3$, $(1-\delta)$-most $\phi \in \mathbb{S}(\mathcal{H}_e)$ lie in $\mathrm{MATE}_{2\varepsilon}$.*

Now we consider $H = H_0 + \lambda V$, where the distribution of the random matrix V is continuous and invariant under all unitaries commuting with H_0. Then the eigen-ONB of H (which is unique up to phase factors because H is non-degenerate) will be arbitrarily close, for sufficiently small λ, to some eigen-ONB of H_0. In fact, the eigen-ONB of H (with suitably chosen phase factors) converges as $\lambda \to 0$ (for fixed V) to an eigen-ONB $(\chi_k)_k$ of H_0.[6] Which one? That depends on V. Due to the unitary invariance, the ONB $(\chi_k)_k$ is, in each eigenspace $\mathcal{H}_e$ of H_0, uniformly distributed among the ONBs of $\mathcal{H}_e$. This brings us to the question, will a purely random ONB of $\mathcal{H}_e$ satisfy the ETH? As we discuss in detail in Section 3.3, the answer can be shown to be yes if $\dim \mathcal{H}_e$ is smaller than a certain critical size (using Corollary 1) or if $\dim \mathcal{H}_e$ is larger than another critical size (using Proposition 2), but in between there remains a regime for which we do not know the answer. That is why it is relevant to have a result, like our Theorem 1 below, guaranteeing that even if ETH might be violated, still most non-equilibrium ψ_0 will thermalize.

2.4 Physical Relevance of Generic Perturbations

In physics, we sometimes make models (e.g., write down a formula for the Hamiltonian) and sometimes consider generic situations (e.g., consider a random Hamiltonian). When we have proved that most Hamiltonians relative to a particular distribution have a certain property P, then this still leaves open whether the true Hamiltonian has this property. On the other hand, models involve idealizations and simplifications, and therefore are not necessarily realistic. So, when we have proved that a model has property P, this also leaves open whether the true Hamiltonian has this property.

In order to increase the reliability of mathematical results about P, one can try to add more realism. For a model, this might mean to add corrections, such as relativistic corrections, previously neglected interaction terms between the particles of the system, or interactions with the outside that the system is not perfectly shielded from (such as gravitational interactions). Of course, this will often make the model intractable. For a random Hamiltonian, on the other hand, increasing realism may mean making the distribution narrower, either by conditioning on properties P' that we believe the true Hamiltonian has (e.g., symmetries) or by choosing a distribution near some H_0 that we believe the true Hamiltonian is close to. The latter strategy is, in fact, a kind of combination of the two strategies of considering a model H_0 and considering a random Hamiltonian.

Our assumption that the distribution of the perturbation V is invariant under

[6] This fact also entails that if we subdivide the energy axis into "micro-canonical" intervals, then for sufficiently small λ, each $\mathcal{H}_{\mathrm{mc}}$ obtained from H_0 stays invariant during any time interval $[0, T]$ to an arbitrary degree of precision under the time evolution generated by H, with the consequence that each micro-canonical subspace can be treated separately, and our simplifying assumption that $\mathcal{H}_{\mathrm{mc}}$ is invariant caused no harm.

unitaries (at least those commuting with H_0) is motivated (i) by the thought that, since we are considering very small perturbations, many different kinds of interaction with the environment may contribute to V and (ii) by the facts that this would be the case for the simplest distributions of V, such as the Gaussian unitary ensemble GUE, and that this allows us to answer the question how many ψ_0 will thermalize. The fact that a random V with unitarily invariant distribution involves super-long-range super-multi-body interactions makes it seem unrealistic. On the other hand, when we consider *very* weak perturbations, then already the fact that no system is *exactly* closed becomes relevant—that every system is slightly interacting with an environment such as a gas of photons (or of gravitational waves etc.). The reason we are considering closed systems (that evolve in a Hamiltonian rather than Lindbladian way) is that being open is unnecessary for thermalization. And yet, when it comes to arbitrarily weak perturbations, a weak interaction with an environment may be expected to have a similar effect as a weak generic perturbation of H. After all, if the particles of the environment (say, photons) are entangled with each other, so that distant parts of the system will effectively interact with each other by interacting with different entangled photons. Thus, the system's evolution seems quite similar to a unitary model in which every part of the system is weakly interacting with every other.

A relevant trait of our results in this paper is that they apply to random perturbations of H_0 that are *arbitrarily weak*. Such results can be regarded as stating an *instability* of a property P: For example, being degenerate is an unstable property in the sense that every degenerate H_0 possesses a neighborhood in the space of self-adjoint operators in which the degenerate operators form a null set. It is therefore not believable that the true Hamiltonian is degenerate, given that it is close to H_0. Also deterministic corrections to H_0 may be expected to break the degeneracy, but again it may be intractable to prove this.

The upshot is that assuming an arbitrarily small generic perturbation may be quite realistic after all. The typical behavior of such a perturbation may be a pretty good prediction of the behavior of the true Hamiltonian.

3 Main Results

In this section, we present and discuss our main results. In Section 3.1, we state our results for general Hamiltonians and in Section 3.2 we apply them to the free Fermi gas. We end this section with a discussion of whether H should be expected to satisfy the ETH in Section 3.3.

3.1 For General Hamiltonians

Theorem 1. *Let $\mathcal{H}$ be a Hilbert space with $D := \dim \mathcal{H} < \infty$, $\mathcal{H}_{\mathrm{eq}}$ and $\mathcal{H}_\nu$ any two subspaces, P_{eq} and P_ν the associated projections, and $P_{\mathrm{neq}} := I - P_{\mathrm{eq}}$.*

Let $H_0 \in \mathcal{L}(\mathcal{H})$ be self-adjoint and assume that H_0 has an orthonormal eigenbasis $(\phi_k)_{k\in\{1,\dots,D\}}$ satisfying the ETH (10) for some $\varepsilon > 0$ (i.e., $\|P_{\mathrm{neq}}\phi_k\|^2 < \varepsilon$ for all k). For $\lambda \in \mathbb{R}$ let $H := H_0 + \lambda V$, where V is a self-adjoint operator drawn randomly from a continuous distribution invariant under conjugation with all unitaries commuting with H_0. Finally, let ψ_0 be uniformly distributed in the unit sphere of $\mathcal{H}_\nu$ and $\psi_t := e^{-iHt}\psi_0$.

Then

$$\lim_{\lambda\to 0} \mathbb{E}_V \lim_{T\to\infty} \mathbb{E}_{\psi_0} \frac{1}{T}\int_0^T \|P_{\mathrm{neq}}\psi_t\|^2 dt < 2\varepsilon. \tag{15}$$

As a consequence, for all $\delta, \delta', \delta'' > 0$ there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, for $(1-\delta)$-most V, $(1-\delta')$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ are such that for $(1-\delta'')$-most $t \in [0,\infty)$,

$$\|P_{\mathrm{neq}}\psi_t\|^2 < \frac{3\varepsilon}{\delta\delta'\delta''}, \quad \textit{i.e.,} \quad \psi_t \in \mathrm{MATE}_{3\varepsilon/\delta\delta'\delta''} . \tag{16}$$

While the condition of invariance under the unitaries commuting with H_0 is the minimal condition we need to mathematically prove the consequence, the most relevant cases in practice are perhaps those in which the distribution is invariant under *all* unitaries, as is the case for the Gaussian unitary ensemble (GUE) in which the entries of V are (up to Hermitian symmetry) i.i.d. complex Gaussian random variables.

Although $\mathcal{H}_{\mathrm{eq}}$ and $\mathcal{H}_\nu$ are arbitrary subspaces in the theorem, they are intended to physically mean the macro spaces of thermal equilibrium and of some non-equilibrium macro state ν. Then Theorem 1 establishes thermalization for most initial states with macro state ν as discussed in Section 1. Alternatively, we can consider an arbitrary fixed (non-equilibrium) initial state ψ_0 and consider the 1d subspace spanned by ψ_0 in the role of $\mathcal{H}_\nu$: then for this ψ_0, most perturbations V lead to thermalization; this is expressed by the following corollary.

Corollary 2. *Let $\varepsilon, \mathcal{H}, \mathcal{H}_{\mathrm{eq}}, H_0, \phi_k, V, H$ be as in Theorem 1, and let $\psi_0 \in \mathbb{S}(\mathcal{H})$ be arbitrary. Then for all $\delta, \delta' > 0$ there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and all $\psi_0 \in \mathbb{S}(\mathcal{H})$, for $(1-\delta)$-most V, $(1-\delta')$-most $t \in [0,\infty)$,*

$$\|P_{\mathrm{neq}}\psi_t\|^2 < \frac{3\varepsilon}{\delta\delta'} \quad \textit{i.e.,} \quad \psi_t \in \mathrm{MATE}_{3\varepsilon/\delta\delta'} . \tag{17}$$

3.2 For the Free Fermi Gas

In this subsection we discuss the concrete example of the free, non-relativistic Fermi gas of N particles on a d-dimensional lattice $\Lambda := \{1, \dots, L\}^d$ with periodic boundary conditions, where $L \in \mathbb{N}$ and $d \geq 1$. We will in particular show for $d = 1$ that all eigen-ONBs satisfy the ETH, so Proposition 1 applies, and for $d \geq 1$ that Theorem 1 applies.

The Hamiltonian is given by

$$H_0^{\mathrm{fF}} := - \sum_{\substack{x,y \in \Lambda \\ \mathrm{dist}(x,y)=1}} c_x^\dagger c_y, \tag{18}$$

where c_x and $c_x^\dagger$ denote the annihilation and creation operators of a fermion at site $x \in \Lambda$. ($H_0 + 2NdI$ is the negative discrete Laplacian.) The relevant Hilbert space is the N-particle sector of fermionic Fock space, i.e., $\mathcal{H} \simeq \mathbb{C}^D$ with $D = \binom{L^d}{N}$.

Like Shiraishi and Tasaki [26, 33, 32], we will use a highly simplified model of "thermal equilibrium" defined only in terms of the spatial distribution of particles. In particular, the restriction to a micro-canonical energy shell is not relevant for us in the following. We will discuss extensions to more realistic models of thermal equilibrium in Remark 1. Choose any spatial region $\Gamma \subseteq \Lambda$, let

$$\mu := |\Gamma|/|\Lambda| \tag{19}$$

be its relative size, and let

$$N_\Gamma := \sum_{x \in \Gamma} c_x^\dagger c_x \tag{20}$$

be the number operator of the particles in Γ. Throughout this section we define the equilibrium subspace $\mathcal{H}_{\mathrm{eq},\eta}$ for a given threshold $\eta > 0$ as the spectral subspace of N_Γ specified by the condition $\left|\frac{N_\Gamma}{N} - \mu\right| \leq \eta$, i.e.,

$$P_{\mathrm{eq},\eta} := \mathbf{1}_{[N(\mu-\eta),N(\mu+\eta)]}(N_\Gamma)\,, \tag{21}$$

and set $P_{\mathrm{neq},\eta} := I - P_{\mathrm{eq},\eta}$. Thus $\mathcal{H}_{\mathrm{eq},\eta}$ contains those states ψ for which the Born distribution of N_Γ/N is supported in an η neighborhood around μ. Note that this is a much stronger condition than just requiring that the expectation value of N_Γ/N in a state ψ lies in this interval. In Remark 1 below we discuss a more realistic definition of $P_{\mathrm{eq},\eta}$ that takes into account not only the number of particles in one region Γ, but a coarse-grained density all over Λ.

For $d = 1$ we can show that H_0^{fF} satisfies the version (1) of the ETH, i.e., that every eigenvector of H_0^{fF} is close to $\mathcal{H}_{\mathrm{eq},\eta}$ for large N, assuming that Γ is an interval.

Theorem 2 (ETH for the free Fermi gas in 1d)**.** *Let $d = 1$, L prime, $46 \leq N < L/4$, $\Gamma \subset \Lambda$ an interval, $\eta > \frac{2(\ln N+1)}{N}$, and $\mathcal{H}_{\mathrm{eq},\eta}$ and $P_{\mathrm{neq},\eta}$ as above. Then every normalized eigenstate ϕ of H_0^{fF} satisfies*

$$\left\|P_{\mathrm{neq},\eta}\phi\right\|^2 \leq \frac{32 \ln N}{\eta^2 N}. \tag{22}$$

The condition in Theorem 2 that L must be prime guarantees that the eigenvalues of the one-body Hamiltonian on such a chain are rationally independent (see [26] and similar arguments in [16]). As a consequence, the only degeneracies in the many-body spectrum arise from the fact that one-body eigenstates with momentum k and $-k$ have the same energy. The latter degeneracy was shown by Shiraishi and Tasaki to be removed by piercing the ring with a small magnetic flux. Theorem 2 shows that even if these degeneracies are not removed, H_0^{fF} still satisfies the ETH.

Together with Proposition 1, Theorem 2 implies that all initial states reach a small neighborhood of $\mathcal{H}_{\mathrm{eq},\eta}$:

Corollary 3 (Thermalization of the free Fermi gas in 1d)**.** *Let $d = 1$, $N \geq 46$, and let L, Γ, η, $\mathcal{H}_{\mathrm{eq},\eta}$ and $P_{\mathrm{eq},\eta}$ be as in Theorem 2. Let $\varepsilon, \delta > 0$ be such that $\varepsilon\delta \geq \frac{32 \ln N}{\eta^2 N}$. Then for every $\psi_0 \in \mathbb{S}(\mathcal{H})$ and $(1-\delta)$-most $t \in [0, \infty)$, $\psi_t := e^{-iH_0^{\mathrm{fF}}t}\psi_0$ satisfies*

$$\|P_{\mathrm{neq},\eta}\psi_t\|^2 < \varepsilon\,. \tag{23}$$

For general $d \geq 1$, we can still prove the ETH for one eigenbasis B_1 of H_0^{fF}, in fact with better bounds. To define this eigenbasis, we need to introduce some notation first.

For odd L let

$$\mathcal{K} := \left\{ \frac{2\pi}{L}\nu \,\middle|\, \nu \in \left\{0, \pm 1, \ldots, \pm\frac{L-1}{2}\right\}^d \right\}, \tag{24}$$

and for even L let

$$\mathcal{K} := \left\{ \frac{2\pi}{L}\nu \,\middle|\, \nu \in \left\{0, \pm 1, \ldots, \pm\Big(\frac{L}{2}-1\Big), \frac{L}{2}\right\}^d \right\}. \tag{25}$$

For $k \in \mathcal{K}$ we define

$$a_k^\dagger := \frac{1}{L^{d/2}} \sum_{x\in\Lambda} e^{ik\cdot x} c_x^\dagger. \tag{26}$$

Let $\mathcal{K}_{\neq}^N$ be the set of $k = (k_1, \ldots, k_N) \in \mathcal{K}^N$ such that $k_i \neq k_j$ for all $i \neq j$. The permutation group S_N acts on $\mathcal{K}_{\neq}^N$ via $(k_1, \ldots, k_N) \mapsto (k_{\pi(1)}, \ldots, k_{\pi(N)})$ for any $\pi \in S_N$. Let $\tilde{\mathcal{K}}^N \subset \mathcal{K}_{\neq}^N$ contain exactly one representative from each orbit, i.e., from each permutation class. We define

$$B_1 := \big\{|\Psi_k\rangle : k \in \tilde{\mathcal{K}}^N\big\}, \text{ where} \tag{27}$$

$$|\Psi_k\rangle := a_{k_1}^\dagger a_{k_2}^\dagger \ldots a_{k_N}^\dagger |\Phi_{\mathrm{vac}}\rangle \tag{28}$$

with $|\Phi_{\mathrm{vac}}\rangle$ the vacuum vector in Fock space. The states $|\Psi_k\rangle$ are $\binom{L^d}{N}$ different eigenfunctions of the unperturbed Hamiltonian H_0^{fF} and therefore form an orthonormal basis of $\mathcal{H}$.

For this eigenbasis B_1 of H_0^{fF} we can prove the ETH using similar methods as Tasaki [33] used in the case of the free fermion chain in one dimension.

Proposition 3 (ETH for one eigenbasis of the free Fermi gas, any d)**.** *Let $d \geq 1$, $\Gamma \subset \Lambda$ arbitrary, $0 < \eta < \frac{3}{2}\mu(1-\mu)$, and $\mathcal{H}_{\mathrm{eq},\eta}$ and $P_{\mathrm{neq},\eta}$ as in (21). Then every eigenstate $\Psi_k \in B_1$ of H_0^{fF} given by (18) satisfies*

$$\left\|P_{\mathrm{neq},\eta}\Psi_k\right\|^2 < 2e^{-\frac{\eta^2}{3\mu(1-\mu)}N}. \tag{29}$$

Furthermore, if $N < L/2d$ then the maximal degree of degeneracy D_E is at least 2^{Nd}.

As an immediate consequence of Proposition 3 and Theorem 1 we obtain the following corollary:

Corollary 4 (Thermalization of the perturbed free Fermi gas in any dimension)**.** *Let $d \geq 1$ and Γ, η, $\mathcal{H}_{\mathrm{eq},\eta}$, and $P_{\mathrm{neq},\eta}$ be as in Proposition 3. For $\lambda \in \mathbb{R}$ let $H := H_0^{\mathrm{fF}} + \lambda V$, where V is drawn randomly from a continuous distribution invariant under conjugation with all unitaries commuting with H_0^{fF}. Let $\mathcal{H}_\nu \subseteq \mathcal{H}$ be any subspace and for $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ let $\psi_t := e^{-iHt}\psi_0$.*

Then for all $\delta, \delta', \delta'' > 0$ there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, for $(1-\delta)$-most V, $(1-\delta')$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ are such that for $(1-\delta'')$-most $t \in [0, \infty)$

$$\|P_{\mathrm{neq},\eta}\psi_t\|^2 < \frac{6}{\delta\delta'\delta''}e^{-\frac{\eta^2}{3\mu(1-\mu)}N}. \tag{30}$$

Remark 1. In a similar way as in [33], Theorem 2 and Proposition 3 (and therefore also Corollary 3 and Corollary 4) can easily be generalized to the situation in which we define equilibrium by requiring that in $m \in \mathbb{N}$ subsets of $\Gamma_i \subset \Lambda$ the fraction of particles is close to $\mu_i := |\Gamma_i|/|\Lambda|$. To this end, note that

$$P_{\mathrm{neq},\eta} := P\left(\exists\, i = 1, \dots, m : \left|\frac{N_{\Gamma_i}}{N} - \mu_i\right| > \eta\right) \leq \sum_{i=1}^m P\left(\left|\frac{N_{\Gamma_i}}{N} - \mu_i\right| > \eta\right), \tag{31}$$

where $P(\dots)$ denotes the orthogonal projection onto the specified subspace. Let $\mu_* := \arg\max_{\mu_1,\dots,\mu_m}(\mu_i(1-\mu_i))$. Then it follows from Proposition 3 that

$$\|P_{\mathrm{neq},\eta}\Psi_k\|^2 \leq 2\sum_{i=1}^m e^{-\frac{\eta^2}{3\mu_i(1-\mu_i)}N} \leq 2m\, e^{-\frac{\eta^2}{3\mu_*(1-\mu_*)}N}. \tag{32}$$

Thus, as long as m is not too large and N is large, the right-hand side in (32) is small.

3.3 ETH for a Random Eigenbasis of H_0?

Consider again a general Hamiltonian H_0 with degenerate eigenvalues for which one eigenbasis satisfies the ETH (10). How likely and how strongly will a purely random eigenbasis of H_0 violate the ETH (10)?

Consider an eigenspace $\mathcal{H}_e$ of dimension D_e. The probability that any member of a purely random ONB $(\phi_k)_{k=1}^{D_e}$ in $\mathcal{H}_e$ lies outside $\mathrm{MATE}_{2\varepsilon}$ can be bounded according to

$$\mathbb{P}(\exists k \le D_e : \phi_k \notin \mathrm{MATE}_{2\varepsilon}) \le \sum_{k=1}^{D_e} \mathbb{P}(\phi_k \notin \mathrm{MATE}_{2\varepsilon}) \tag{33a}$$

$$\le 2D_e \exp(-C\varepsilon^2 D_e) \tag{33b}$$

by Proposition 2. Additionally, by Corollary 1, *every* ONB satisfies ETH with tolerance εD_e instead of ε. Could these bounds be used to prove (e.g., for the free Fermi gas) that H satisfies ETH (with some small tolerance)?

It seems that the answer is no in general. Consider ε of order $\exp(-c_1 N)$ and D_e of order $\exp(c_2 N)$ with $c_1, c_2 > 0$ for large number of particles N. There are three regimes. If $c_2 > 2c_1$, then (33b) is small for large N, and thus most eigenbases of the eigenspace satisfy ETH with tolerance exponentially small in N. Similarly, for $c_2 < c_1$, every ONB satisfies ETH with tolerance εD_e, which is exponentially small in N. However, for the range in between, $c_2 \in [c_1, 2c_1]$, it is unclear whether a typical basis of $\mathcal{H}_e$ will obey the ETH with small tolerance. For the free Fermi gas in $d > 1$ space dimensions, Proposition 3 provides a suitable ε and corresponding c_1. Furthermore, the maximal degeneracy is larger than 2^{Nd} (corresponding to $c_2 \ge d \ln 2 > 2c_1$). However, the proof of Proposition 3 suggests that also smaller eigenspaces exist, potentially with c_2 falling into $[c_1, 2c_1]$.

The upshot is that it is quite possible that most eigen-ONBs of H_0 (and thus the eigenbasis of H as $\lambda \to 0$) violate the ETH. While in the light of Corollary 1, one might have thought that the largest D_e create the biggest obstacle, it is actually the D_e's of an intermediate size, roughly between $1/\varepsilon$ and $1/\varepsilon^2$. At this point, our Theorem 1 provides an alternative, as it shows that the ETH is not necessary for the thermalization of most non-equilibrium ψ_0.

4 Proofs

4.1 Proof of Proposition 1

Let $P_{\mathrm{neq}} := I - P_{\mathrm{eq}}$, let $\mathcal{E}$ be the spectrum of H, and Π_e the projection to the eigenspace of H with eigenvalue e; consider the time average

$$\overline{\langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle} = \sum_{e,e' \in \mathcal{E}} \overline{e^{i(e-e')t}} \, \langle \psi_0, \Pi_e P_{\mathrm{neq}} \Pi_{e'} \psi_0 \rangle \tag{34a}$$

$$= \sum_{e \in \mathcal{E}} \underbrace{\langle \psi_0, \Pi_e P_{\mathrm{neq}} \Pi_e \psi_0 \rangle}_{\le \varepsilon\delta \|\Pi_e \psi_0\|^2 \text{ by } (8)} \tag{34b}$$

$$\leq \varepsilon\delta \sum_{e\in\mathcal{E}} \langle\psi_0, \Pi_e\psi_0\rangle = \varepsilon\delta \,. \tag{34c}$$

Thus, for every $\eta > 0$, there is $T_0 > 0$ such that for every $T > T_0$,

$$\frac{1}{T}\int_0^T dt\, \langle\psi_t, P_{\text{neq}}\psi_t\rangle < \varepsilon\delta + \eta \,. \tag{35}$$

By the Markov inequality,

$$\frac{1}{T}\Big|\big\{t \in [0,T] : \langle\psi_t, P_{\text{neq}}\psi_t\rangle > \varepsilon\big\}\Big| \leq \frac{\varepsilon\delta + \eta}{\varepsilon} = \delta + \frac{\eta}{\varepsilon} \,. \tag{36}$$

Taking the limes superior as $T \to \infty$, we find that

$$\limsup_{T\to\infty} \frac{1}{T}\Big|\big\{t \in [0,T] : \langle\psi_t, P_{\text{neq}}\psi_t\rangle > \varepsilon\big\}\Big| \leq \delta + \frac{\eta}{\varepsilon} \,. \tag{37}$$

Since $\eta > 0$ was arbitrary, we must have that

$$\limsup_{T\to\infty} \frac{1}{T}\Big|\big\{t \in [0,T] : \langle\psi_t, P_{\text{neq}}\psi_t\rangle > \varepsilon\big\}\Big| \leq \delta \tag{38}$$

as claimed.

4.2 Proof of Lemma 1

The sum of eigenvalues of M equals $\operatorname{tr} M \leq \varepsilon D$. Since all the eigenvalues are non-negative, the maximal eigenvalue is bounded by εD. The maximal eigenvalue also equals the norm $\|M\|$.

To see that the bound is sharp, consider the matrix with all entries equal to ε, which has an eigenvector with all entries 1 and eigenvalue εD.

4.3 Proof of Corollary 1

Corollary 1 is a consequence of Lemma 1. Let $P_{\text{neq}} = I - P_{\text{eq}}$. For all k we have $\|P_{\text{eq}}\phi_k\|^2 \geq 1 - \varepsilon$ and equivalently $\langle\phi_k, P_{\text{neq}}\phi_k\rangle \leq \varepsilon$. Let ϕ be an eigenvector of H_0 with eigenvalue e. Denote by Π_e the orthogonal projection onto the eigenspace of H_0 corresponding to e. In the basis (ϕ_k) restricted to in the eigenspace corresponding to eigenvalue e, the matrix corresponding to $\Pi_e P_{\text{neq}} \Pi_e$ has all diagonal entries $\leq \varepsilon$. Thus by Lemma 1

$$\langle\phi, P_{\text{neq}}\phi\rangle = \langle\phi, \Pi_e P_{\text{neq}}\Pi_e\phi\rangle \leq \varepsilon D_e \leq \varepsilon D_E$$

and equivalently $\phi \in \text{MATE}_{\varepsilon D_E}$.

For the second part, consider a Hilbert space of dimension $D = \max\{n, D_E\}$ with an orthonormal basis $(\phi_k)_{k=1}^D$. Choose H_0 diagonal in this basis, with the first D_E diagonal entries equal to zero and the other entries all nonzero and pairwise distinct. Let $P_{\text{neq}} = I - P_{\text{eq}}$ be the projection which in this basis corresponds to the matrix with all entries equal to $1/D$. In particular the diagonal entries of P_{eq} are $1 - 1/D \geq 1 - 1/n = 1 - \varepsilon$ and thus all $\phi_k \in \text{MATE}_\varepsilon$. Let Π_0 be the projection onto the eigenspace corresponding to eigenvalue zero. Then $\|\Pi_0 P_{\text{neq}} \Pi_0\| = D_E/D = \min\{\varepsilon D_E, 1\}$. Choosing ϕ to be an eigenvector to the maximal eigenvalue of $\Pi_0 P_{\text{neq}} \Pi_0$ we obtain that

$$\|P_{\text{eq}}\phi\|^2 = \max\{1 - \varepsilon D_E, 0\}\,. \tag{39}$$

4.4 Proof of Lemma 2

We cite a key fact from Appendix A in [36]:

Lemma 3. *If H has continuous distribution in the Hermitian $n \times n$ matrices, then with probability 1 it has non-degenerate eigenvalues and eigenvalue gaps.*

Lemma 2 says more in that there is a whole interval $(0, \lambda_0)$ of λ values for which $H_0 + \lambda V$ will have non-degenerate eigenvalues and eigenvalue gaps. As a preparation for the proof, we establish the following lemma:

Lemma 4. *The set of Hermitian $n \times n$ matrices with degenerate eigenvalues can be written as the zero set of a polynomial in the matrix entries. Likewise, the set of Hermitian $n \times n$ matrices with distinct eigenvalues but with degenerate eigenvalue gaps can be written as the zero set of a polynomial in the matrix entries.*

Proof. Let A be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The matrix A has degenerate eigenvalues if and only if its discriminant

$$\operatorname{disc}(A) = \prod_{i<j} (\lambda_i - \lambda_j)^2 \tag{40}$$

vanishes. Since the discriminant of a matrix can be written as a polynomial in the matrix entries, see, e.g., Lemma 1 in [21], the first claim follows.

For the second claim, we follow the proof strategy of Lemma 1 in [21] and adapt it to our situation. Let A be a Hermitian $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Then A has degenerate eigenvalue gaps if and only if

$$\prod_{(i,j,k,l) \in I} (\lambda_i - \lambda_j - (\lambda_k - \lambda_l)) = 0, \tag{41}$$

where

$$I := \big\{(i,j,k,l) \in [n]^4 : (i \neq k \text{ or } j \neq l) \text{ and } (i \neq j \text{ or } k \neq l)\big\} \tag{42}$$

and $[n] := \{1, \dots, n\}$. Since the tuples $(i,j,k,l) \in I$ with $i = j$ and $k \neq l$ (or $k = l$ and $i \neq j$) lead to non-zero factors in (41) (due to the non-degeneracy of the eigenvalues), we can replace the set I in (41) by the set

$$I' := \left\{(i,j,k,l) \in [n]^4 : (i \neq k \text{ or } j \neq l) \text{ and } (i \neq j \text{ and } k \neq l)\right\}. \tag{43}$$

We enumerate the eigenvalue differences $(\lambda_i - \lambda_j)_{i\neq j}$ as $x_1 := \lambda_1 - \lambda_2, x_2 := \lambda_1 - \lambda_3, \dots, x_{n-1} := \lambda_1 - \lambda_n, x_n := \lambda_2 - \lambda_1, x_{n+1} := \lambda_2 - \lambda_3, \dots, x_M := \lambda_n - \lambda_{n-1}$, where $M := n(n-1)$, and consider the Vandermonde matrix $V = V(x_1, \dots, x_M)$ which is defined as

$$V(x_1,\dots,x_M) = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{M-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{M-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_M & x_M^2 & \dots & x_M^{M-1} \end{pmatrix}. \tag{44}$$

It is well known that

$$\det V = \prod_{1\leq i<j\leq M} (x_j - x_i) \tag{45}$$

and thus

$$(\det V)^2 = \prod_{1\leq i<j\leq M} (x_j - x_i)^2 = (-1)^{M(M-1)/2} \prod_{i\neq j} (x_i - x_j) \tag{46a}$$

$$= (-1)^{M(M-1)/2} \prod_{(i,j,k,l)\in I'} (\lambda_i - \lambda_j - (\lambda_k - \lambda_l)). \tag{46b}$$

Therefore A has degenerate eigenvalue gaps if and only if $(\det V)^2 = 0$.

We define the $M \times M$ matrix $B = (B_{ij})$ in the following way:

$$B_{ij} := \sum_{k=1}^{M} x_k^{i+j-2}. \tag{47}$$

One immediately sees that $B = V^T V$ which implies $\det B = (\det V)^2$. Obviously, $B_{11} = M$ and for $(i,j) \neq (1,1)$ we have that

$$B_{ij} = \sum_{k,l=1}^{n} (\lambda_k - \lambda_l)^{i+j-2} \tag{48a}$$

$$= \sum_{k,l=1}^{n} \sum_{p=0}^{i+j-2} \binom{i+j-2}{p} \lambda_k^{i+j-2-p} \lambda_l^p (-1)^p \tag{48b}$$

$$= \sum_{p=0}^{i+j-2} \binom{i+j-2}{p} (-1)^p \operatorname{tr}(A^{i+j-2-p}) \operatorname{tr}(A^p). \tag{48c}$$

We conclude that the entries of B are polynomials in the entries of A and thus that also $\det B = (\det V)^2$ is a polynomial in the entries of A. This proves the second claim. $\square$

Lemma 5. *Suppose there is $\tilde{\lambda} > 0$ such that $H_0 + \tilde{\lambda} V$ has non-degenerate eigenvalues and eigenvalue gaps. Then there is $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, $H = H_0 + \lambda V$ has non-degenerate eigenvalues and eigenvalue gaps.*

Proof. We consider H as a function of λ. By Lemma 4 there is a polynomial P_1 in the entries of H and therefore a polynomial $\tilde{P}_1$ in λ such that its zeros are exactly the matrices with degenerate eigenvalues. By assumption, $\tilde{P}_1(\tilde{\lambda}) \neq 0$. Thus, $\tilde{P}_1$ does not vanish identically and therefore $\tilde{P}_1$ has only finitely many zeros. We know that $\lambda = 0$ is one of the zeros and therefore there exists a $\tilde{\lambda}_0$ such that $H = H_0 + \lambda V$ has non-degenerate eigenvalues for all $\lambda \in (0, \tilde{\lambda}_0)$.[7]

Again it follows from Lemma 4 that there is a polynomial P_2 in the entries of H and therefore a polynomial $\tilde{P}_2$ in $\lambda \in (0, \tilde{\lambda}_0)$ such that its zeros are exactly the matrices with degenerate eigenvalue gaps (but non-degenerate eigenvalues). Note that $\tilde{P}_2$ can be considered as a polynomial on $[0, \infty)$ that vanishes in the (finitely many) zeros of $\tilde{P}_1$. Because of $\tilde{P}_2(\tilde{\lambda}) \neq 0$, $\tilde{P}_2$ does not vanish identically and has therefore only finitely many zeros. Its zeros in $(0, \tilde{\lambda}_0)$ are matrices with non-degenerate eigenvalues but degenerate eigenvalue gaps. Since there are only finitely many such zeros, it follows that there exists a $0 < \lambda_0 \leq \tilde{\lambda}_0$ such that $H = H_0 + \lambda V$ has non-degenerate eigenvalues and non-degenerate eigenvalue gaps for all $\lambda \in (0, \lambda_0)$. $\square$

Now Lemma 2 follows from Lemma 3 and Lemma 5 by fixing $\tilde{\lambda} = 1$ and choosing a V for which $H_0 + V$ has non-degenerate eigenvalues and eigenvalue gaps.

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[7] Here is an alternative proof of the eigenvalue statement of Lemma 5: Consider the polynomial $P(\lambda, E) = \det(H_0 + \lambda V - EI)$ in 2 variables, which vanishes if and only if E is an eigenvalue of $H_0 + \lambda V$. P has degree $\leq D$ because det is a degree-D polynomial in the matrix entries, and each entry is a degree-1 polynomial in (λ, E). Since for $\lambda = \tilde{\lambda}$, P has D distinct zeros, P has degree D and is square-free. Then the number of singular points in the plane is finite (e.g., [7, Bemerkung 3.2]). The zero set S of P is known to consist of finitely many smooth curves that are either closed or tend to infinity in both directions, and can intersect themselves or each other only in singular points. There are only finitely many points p on the smooth curves where the tangent is vertical, i.e., parallel to the E axis, because at such points p, $\partial P/\partial E(p) = 0$, so p is a joint zero of P and $\partial P/\partial E$; since these two polynomials have no common prime factor (see the proof of [7, Bemerkung 3.2]), they intersect only in finitely many points by Bézout's theorem. Now along any vertical line L, points p in $L \cap S$ are simple roots unless either p is a singular point or a curve has a vertical tangent at p. Thus, for every λ except finitely many exceptions, every zero of P in E is simple, so H is non-degenerate. Let λ_0 be the smallest positive exception.

4.5 Proof of Proposition 2

As shown by Reimann [23, Eq. (7)], Lévy's lemma implies that for a uniformly distributed unit vector ϕ in a D-dimensional Hilbert space $\mathcal{H}$ and a self-adjoint operator A (which is not a multiple of the identity),

$$\mathbb{P}\Big(\big|\langle\phi, A\phi\rangle - \mathrm{tr}(A)/D\big| \geq \varepsilon\Big) \leq 2\exp\Big(-\frac{C\varepsilon^2 D}{\Delta_A^2}\Big) \tag{49}$$

with C as in Proposition 2 and $\Delta_A > 0$ the difference between the largest and the smallest eigenvalue of A. Now we specialize to our case with $\mathcal{H} = \mathcal{H}_e$, $D = D_e$, and $A = \Pi_e P_{\mathrm{neq}} \Pi_e$ with Π_e the projection to $\mathcal{H}_e$. Since $0 \leq A \leq I$, we have that $\Delta_A \leq 1$ (so the right-hand side of (49) can only become larger if we replace Δ_A by 1). By assumption, there is a basis (ϕ_k) for which every vector lies in $\mathrm{MATE}_\varepsilon$; evaluating the trace in this basis, we find that $\mathrm{tr}(A) \leq \varepsilon D_e$. Thus,

$$\mathbb{P}\Big(\phi \notin \mathrm{MATE}_{2\varepsilon}\Big) = \mathbb{P}\Big(\|P_{\mathrm{neq}}\phi\|^2 > 2\varepsilon\Big) \tag{50a}$$

$$\leq \mathbb{P}\Big(\|P_{\mathrm{neq}}\phi\|^2 \geq \mathrm{tr}(A)/D_e + \varepsilon\Big) \tag{50b}$$

$$\leq \mathbb{P}\Big(\big|\|P_{\mathrm{neq}}\phi\|^2 - \mathrm{tr}(A)/D_e\big| \geq \varepsilon\Big) \tag{50c}$$

$$\leq 2\exp\big(-C\varepsilon^2 D_e\big) \tag{50d}$$

by (49), which completes the proof.

Remark 2. An alternative argument based on the Markov inequality instead of Lévy's lemma (and the fact that the average of $\|P_{\mathrm{neq}}\phi\|^2$ over $\mathbb{S}(\mathcal{H}_e)$ is $\leq \varepsilon$, still assuming that one eigen-ONB of H_0 satisfies (10)) yields for any $\delta > 0$ that $(1-\delta)$-most $\psi \in \mathbb{S}(\mathcal{H}_e)$ lie in $\mathrm{MATE}_{\varepsilon/\delta}$.

4.6 Proof of Theorem 1

We shall first prove that

$$\lim_{\lambda\to 0} \mathbb{E}_V \lim_{T\to\infty} \mathbb{E}_{\psi_0} \frac{1}{T}\int_0^T \langle\psi_t, P_{\mathrm{neq}}\psi_t\rangle dt = \frac{1}{d_\nu}\sum_e d_e\, \mathbb{E}_{\psi_e}\left[\langle\psi_e, P_\nu\psi_e\rangle\langle\psi_e, P_{\mathrm{neq}}\psi_e\rangle\right]. \tag{51}$$

where $d_\nu = \dim \mathcal{H}_\nu$, e are the eigenvalues of H_0, d_e is the dimension of the eigenspace corresponding to e, and the ψ_e are uniformly distributed on the unit sphere in this eigenspace.

Pick a V such that for λ small enough the eigenvalues of $H = H_0 + \lambda V$ are non-degenerate. Let $e_j(\lambda)$ denote the eigenvalues of H, sorted from smallest to largest.

Let $\psi_j(\lambda)$ denote the corresponding eigenvectors of H. With $\psi_t = e^{-iHt}\psi_0$ and the non-degeneracy of the eigenvalues, we have that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}}\psi_t \rangle \, dt = \sum_{j=1}^{D} \langle \psi_0, \psi_j(\lambda)\rangle \langle \psi_j(\lambda), P_{\mathrm{neq}}\psi_j(\lambda)\rangle \langle \psi_j(\lambda), \psi_0\rangle. \tag{52}$$

Next, we shall take the average over $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$. Since for any operator B on a d-dimensional Hilbert space, the average over uniformly distributed normalized ψ equals

$$\mathbb{E}_\psi \langle \psi, B\psi\rangle = \frac{1}{d} \operatorname{tr} B, \tag{53}$$

see, e.g., [8, App. C], we have that

$$\mathbb{E}_{\psi_0} \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}}\psi_t \rangle \, dt = \frac{1}{d_\nu} \sum_{j=1}^{D} \langle \psi_j(\lambda), P_\nu \psi_j(\lambda)\rangle \langle \psi_j(\lambda), P_{\mathrm{neq}}\psi_j(\lambda)\rangle. \tag{54}$$

Note also that, by the dominated convergence theorem, we may interchange $\mathbb{E}_{\psi_0}$ and the limit $T \to \infty$.

For suitable choice of phases, the set of $\psi_j(\lambda)$ converges to an eigenbasis $\psi_j(0)$ of H_0 as $\lambda \to 0$ (see, e.g., [27, Chapter XII, Problem 17]). For an eigenvalue e of H_0 and d_e the dimension of the corresponding eigenspace, if $e_{j+1}(\lambda), \dots e_{j+d_e}(\lambda)$ are all the eigenvalues converging to e, we shall denote the corresponding eigenvectors $\psi_{j+1}(0), \dots, \psi_{j+d_e}(0)$ by $\chi_{e,1}(V), \dots \chi_{e,d_e}(V)$. With this notation,

$$\begin{aligned} \lim_{\lambda\to 0} \lim_{T\to\infty} \frac{1}{T} \, \mathbb{E}_{\psi_0} \int_0^T \langle \psi_t, P_{\mathrm{neq}}\psi_t\rangle \, dt \\ = \frac{1}{d_\nu} \sum_e \sum_{j=1}^{d_e} \langle \chi_{e,j}(V), P_\nu \chi_{e,j}(V)\rangle \langle \chi_{e,j}(V), P_{\mathrm{neq}}\chi_{e,j}(V)\rangle. \end{aligned} \tag{55}$$

Let U be a unitary commuting with H_0. Note that the eigenvectors of $UHU^\dagger = H_0 + \lambda UVU^\dagger$ are exactly $U\psi_j(\lambda)$. In the limit $\lambda \to 0$, we obtain

$$\chi_{e,j}(UVU^\dagger) = U\chi_{e,j}(V).$$

Now we are going to average over V. The construction above works for all V such that the eigenvalues of H are non-degenerate for small enough λ. The set of V, for which this fails has measure zero by Lemma 2. Since the distribution of V is invariant under conjugation with U, it holds that

$$\begin{aligned} \mathbb{E}_V \Big(\langle \chi_{e,j}(V), P_\nu \chi_{e,j}(V)\rangle \langle \chi_{e,j}(V), P_{\mathrm{neq}}\chi_{e,j}(V)\rangle \Big) = \\ \mathbb{E}_V \Big(\langle \chi_{e,j}(UVU^\dagger), P_\nu \chi_{e,j}(UVU^\dagger)\rangle \langle \chi_{e,j}(UVU^\dagger), P_{\mathrm{neq}}\chi_{e,j}(UVU^\dagger)\rangle \Big). \end{aligned} \tag{56}$$

We now average this equality over random unitaries U that are block diagonal relative to the eigenspace $\mathcal{H}_e$ with eigenvalue e, act as the identity on the orthogonal complement of $\mathcal{H}_e$, and whose block in $\mathcal{H}_e$ is uniformly distributed in the unitary group of $\mathcal{H}_e$. This gives

$$\begin{aligned}\mathbb{E}_V\Big(\langle\chi_{e,j}(V), P_\nu\chi_{e,j}(V)\rangle\langle\chi_{e,j}(V), P_{\mathrm{neq}}\chi_{e,j}(V)\rangle\Big) = \\ \mathbb{E}_V\Big(\mathbb{E}_U\Big(\langle U\chi_{e,j}(V), P_\nu U\chi_{e,j}(V)\rangle\langle U\chi_{e,j}(V), P_{\mathrm{neq}}U\chi_{e,j}(V)\rangle\Big)\Big) \\ = \mathbb{E}_{\psi_e}\Big(\langle\psi_e, P_\nu\psi_e\rangle\langle\psi_e, P_{\mathrm{neq}}\psi_e\rangle\Big).\end{aligned} \tag{57}$$

Combining this with (55), we obtain that

$$\mathbb{E}_V \lim_{\lambda\to 0}\lim_{T\to\infty}\mathbb{E}_{\psi_0}\frac{1}{T}\int_0^T\langle\psi_t, P_{\mathrm{neq}}\psi_t\rangle dt = \frac{1}{d_\nu}\sum_e d_e\,\mathbb{E}_{\psi_e}\Big[\langle\psi_e, P_\nu\psi_e\rangle\langle\psi_e, P_{\mathrm{neq}}\psi_e\rangle\Big]. \tag{58}$$

By dominated convergence, we can also swap the limit $\lambda \to 0$ and the expectation over V, and we have proved (51).

To evaluate the right hand side of (51), we use that for ψ distributed uniformly in a space of dimension d and operators B, C it holds that

$$\mathbb{E}_\psi[\langle\psi|B|\psi\rangle^*\langle\psi|C|\psi\rangle] = \frac{\mathrm{tr}(B^\dagger)\,\mathrm{tr}(C) + \mathrm{tr}(B^\dagger C)}{d(d+1)}, \tag{59}$$

see [35, Eq. (40)]. Applying this in $\mathcal{H}_e$ to $B = \Pi_e P_\nu \Pi_e$ and $C = \Pi_e P_{\mathrm{neq}}\Pi_e$ with Π_e the projection to $\mathcal{H}_e$, we obtain that the right-hand side of (51) equals

$$\frac{1}{d_\nu}\sum_e \frac{\mathrm{tr}(P_{\mathrm{neq}}\Pi_e)\,\mathrm{tr}(\Pi_e P_\nu) + \mathrm{tr}(\Pi_e P_{\mathrm{neq}}\Pi_e P_\nu)}{d_e + 1}. \tag{60}$$

By computing $\mathrm{tr}(P_{\mathrm{neq}}\Pi_e)$ in the basis ϕ_k, we obtain $\mathrm{tr}(P_{\mathrm{neq}}\Pi_e) < \varepsilon d_e$. Furthermore, Lemma 1 implies that

$$\|\Pi_e P_{\mathrm{neq}}\Pi_e\| \le \varepsilon d_e. \tag{61}$$

With this, $\mathrm{tr}(\Pi_e P_{\mathrm{neq}}\Pi_e P_\nu) \le d_e\varepsilon\,\mathrm{tr}(\Pi_e P_\nu)$. In total, we can bound (60) by

$$\frac{1}{d_\nu}\sum_e \frac{2\varepsilon d_e\,\mathrm{tr}(\Pi_e P_\nu)}{d_e + 1} < 2\varepsilon. \tag{62}$$

For the second part of the proof first note that there exists a $\lambda_0 > 0$ such that

$$\mathbb{E}_V \lim_{T\to\infty}\mathbb{E}_{\psi_0}\frac{1}{T}\int_0^T\langle\psi_t, P_{\mathrm{neq}}\psi_t\rangle\,dt < 3\varepsilon \tag{63}$$

for all $\lambda \in (0, \lambda_0)$. Markov's inequality implies that for all $\lambda \in (0, \lambda_0)$, $(1-\delta)$-most V are such that

$$\mathbb{E}_{\psi_0} \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle \, dt < \frac{3\varepsilon}{\delta}, \tag{64}$$

where we used again that, by dominated convergence, the limit $T \to \infty$ and $\mathbb{E}_{\psi_0}$ can be interchanged. Applying Markov's inequality again shows that for all $\lambda \in (0, \lambda_0)$, $(1-\delta)$-most V are such that for $(1-\delta')$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle \, dt < \frac{3\varepsilon}{\delta\delta'}. \tag{65}$$

For every $T > 0$ we find that

$$\frac{1}{T} \Big| \big\{ t \in [0,T] : \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle > \delta'' \big\} \Big| \leq \frac{1}{\delta''} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle \, dt \,. \tag{66}$$

Taking the limes superior on both sides, we obtain

$$\limsup_{T\to\infty} \frac{1}{T} \Big| \big\{ t \in [0,T] : \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle > \delta'' \big\} \Big| \leq \frac{3\varepsilon}{\delta\delta'\delta''}. \tag{67}$$

Substituting $\delta'' \to \frac{3\varepsilon}{\delta\delta'\delta''}$ yields the claim.

4.7 Proof of Corollary 2

With $P_\nu = |\psi_0\rangle\langle\psi_0|$, Theorem 1 yields

$$\lim_{\lambda\to 0} \mathbb{E}_V \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle dt < 2\varepsilon. \tag{68}$$

Note that the expression

$$\mathbb{E}_V \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle \, dt$$

depends continuously on λ and ψ_0, which follows from (52) using the continuity of $\psi_j(\lambda)$ in λ [27, Chapter XII, Problem 17] and dominated convergence. Since $\mathbb{S}(\mathcal{H})$ is compact, there exists a $\lambda_0 > 0$ such that for all $\psi_0 \in \mathbb{S}(\mathcal{H})$

$$\mathbb{E}_V \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle \, dt < 3\varepsilon \tag{69}$$

for all $\lambda \in (0, \lambda_0)$. Markov's inequality implies that for all $\lambda \in (0, \lambda_0)$, $(1-\delta)$-most V are such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t, P_{\mathrm{neq}} \psi_t \rangle \, dt < \frac{3\varepsilon}{\delta}. \tag{70}$$

For every $T > 0$ we find that

$$\frac{1}{T}\Big|\big\{t \in [0,T] : \langle \psi_t, P_{\mathrm{neq}}\psi_t\rangle > \delta''\big\}\Big| \leq \frac{1}{\delta''}\frac{1}{T}\int_0^T \langle \psi_t, P_{\mathrm{neq}}\psi_t\rangle \, dt\,. \tag{71}$$

Taking the limes superior on both sides, we obtain

$$\limsup_{T\to\infty} \frac{1}{T}\Big|\big\{t \in [0,T] : \langle \psi_t, P_{\mathrm{neq}}\psi_t\rangle > \delta''\big\}\Big| \leq \frac{3\varepsilon}{\delta\delta''}. \tag{72}$$

Substituting $\delta' = \frac{3\varepsilon}{\delta\delta''}$ yields the claim.

4.8 Proof of Theorem 2

In dimension one, the eigenstates Ψ_k defined in (28) have energies $E_k := -2\sum_{i=1}^N \cos k_i$. We first remark that the assumption that L is a prime number ensures that all degeneracies in the spectrum of H_0^{fF} are trivial, i.e., only due to changing the signs of the k_i. This is shown in [26] at the beginning of the proof of Theorem 3.2. Note that the model considered there agrees with the model in the present paper in the case that $\theta = 0$. (This parameter is introduced in [26] to remove the degeneracies and to this end has to be chosen to be small but non-zero.)

The proof of Theorem 2 will make use of several propositions that we formulate now and prove in the subsequent subsections. The first one states that the expectation of N_Γ in an arbitrary eigenstate of H_0^{fF} is close to $N|\Gamma|/|\Lambda|$, provided that N is sufficiently large.

Proposition 4 (Expectation of N_Γ in arbitrary eigenstates)**.** *Let $d = 1$ and let $k = (k_1, \ldots, k_N) \in \tilde{\mathcal{K}}^N$. Let $E_k := -2\sum_{i=1}^N \cos k_i$ be the corresponding eigenvalue and $\mathcal{H}_{E_k} = \mathrm{span}\{\Psi_{k'} : k'_j = \pm k_j \text{ for all } j\}$ the corresponding eigenspace. Let $\Gamma \subset \Lambda$ be an interval, and let $\phi \in \mathbb{S}(\mathcal{H}_{E_k})$. Then*

$$\left|\langle \phi, N_\Gamma \phi\rangle - N\frac{|\Gamma|}{|\Lambda|}\right| \leq \ln N + 1. \tag{73}$$

For the proof of Proposition 4 we need the following proposition concerning the expectation of N_Γ in the eigenstates Ψ_k of H_0^{fF}.

Proposition 5 (Expectation of N_Γ in eigenstates Ψ_k)**.** *Let $d \geq 1$, let $k, k' \in \tilde{\mathcal{K}}^N$ and $x \in \Gamma$. Then*

$$\langle \Psi_k, N_\Gamma \Psi_k\rangle = N\frac{|\Gamma|}{|\Lambda|} \tag{74}$$

and

$$\langle \Psi_k, c_x^\dagger c_x \Psi_{k'}\rangle = \frac{1}{|\Lambda|}\mathrm{sgn}(\tilde{\sigma})e^{i(k_{\tilde{\sigma}^{-1}(l)} - k_l)\cdot x} \tag{75}$$

if only k'_l does not appear in k and $\tilde{\sigma} \in \mathcal{S}_N$ is the permutation such that $k'_{\tilde{\sigma}^{-1}(m)} = k_m$ for all $m \neq l$. Thus, if exactly one component of k' does not appear in k, then

$$|\langle \Psi_k, N_\Gamma \Psi_{k'} \rangle| \leq \frac{|\Gamma|}{|\Lambda|}. \tag{76}$$

If more than one component of k' does not appear in k, then

$$\langle \Psi_k, N_\Gamma \Psi_{k'} \rangle = 0\,. \tag{77}$$

For the proof of Proposition 5 we need the following lemma:

Lemma 6. *Let $d \geq 1$, $x_1, \ldots, x_N \in \{1, \ldots, L\}$, and let $k \in \mathcal{K}^N$ such that $\tau(k) := (k_{\tau(1)}, \ldots, k_{\tau(N)}) \in \tilde{\mathcal{K}}^N$ for some $\tau \in \mathcal{S}_N$, where $\mathcal{S}_N$ denotes the symmetric group. Then*

$$\langle \Phi_{\mathrm{vac}}, c_{x_N} \ldots c_{x_1} a^\dagger_{k_1} \ldots a^\dagger_{k_N} \Phi_{\mathrm{vac}} \rangle = \frac{1}{L^{Nd/2}} \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \prod_{j=1}^N e^{ik_j \cdot x_{\sigma(j)}}. \tag{78}$$

This formula is well known; it was stated, e.g., in [26] in (C3) in the case that $d = 1$. For convenience of the reader, we include the proof in the next subsection.

The next proposition shows that the variance of N_Γ in an arbitrary eigenstate of H_0^{fF} is small provided that N is sufficiently large.

Proposition 6 (Variance of N_Γ in arbitrary eigenstates)**.** *Let $d = 1$ and let $k \in \tilde{\mathcal{K}}^N$, $\mathcal{H}_{E_k}$, Γ and ϕ be as in Proposition 4. Then,*

$$\begin{aligned} \langle \phi, N_\Gamma^2 \phi \rangle - (\langle \phi, N_\Gamma \phi \rangle)^2 &\leq 4N \ln N + 13N + 3(\ln N)^2 + 13 \ln N + 10 \qquad (79)\\ &\overset{N \geq 46}{\leq} 8N \ln N\,. \qquad (80) \end{aligned}$$

Before proving Proposition 6, we show a similar statement for the variance of N_Γ in the eigenstates Ψ_k of H_0^{fF}.

Proposition 7 (Variance of N_Γ in eigenstates Ψ_k)**.** *Let $d \geq 1$ and $k \in \tilde{\mathcal{K}}^N$. Then*

$$\langle \Psi_k, N_\Gamma^2 \Psi_k \rangle - (\langle \Psi_k, N_\Gamma \Psi_k \rangle)^2 \leq N \frac{|\Gamma|}{|\Lambda|} \left(1 - \frac{|\Gamma|}{|\Lambda|} \right). \tag{81}$$

Proof of Theorem 2. Let $\phi \in \mathbb{S}(\mathcal{H})$ be any eigenvector of H_0^{fF}. The Born distribution $\mathbb{P}$ associated with ϕ and the observable N_Γ has expectation $E := \langle \phi, N_\Gamma \phi \rangle$ and variance $V := \langle \phi, N_\Gamma^2 \phi \rangle - \langle \phi, N_\Gamma \phi \rangle^2$. Writing $\overline{B}_r(x)$ for $[x - r, x + r]$, we can express Chebyshev's inequality as

$$\mathbb{P}\Big(\overline{B}_{\alpha\sqrt{V}}(E) \Big) \geq 1 - \frac{1}{\alpha^2} \tag{82}$$

for any $\alpha > 0$. By Propositions 4 and 6 for $N \geq 46$,

$$\overline{B}_{\alpha\sqrt{V}}(E) \subseteq \overline{B}_{\ln N+1+\alpha\sqrt{8N\ln N}}\left(N\frac{|\Gamma|}{|\Lambda|}\right) \subseteq \overline{B}_{N\eta}\left(N\frac{|\Gamma|}{|\Lambda|}\right) \tag{83}$$

for $\alpha = \frac{N\eta}{2\sqrt{8N\ln N}}$ using $N\eta/2 > \ln N + 1$, so

$$\mathbb{P}\left(\overline{B}_{N\eta}\left(N\frac{|\Gamma|}{|\Lambda|}\right)\right) \geq 1 - \frac{32N\ln N}{N^2\eta^2}\,, \tag{84}$$

which is equivalent to (22). □

4.9 Proof of Lemma 6

Without loss of generality assume that $k \in \tilde{\mathcal{K}}^N$. We prove (78) by induction. First note that as a consequence of the canonical anticommutation relations we immediately see from the definition of the $a_k^\dagger$ that $\{c_x, a_k^\dagger\} = e^{ikx}/\sqrt{L}$ and $\{a_k^\dagger, a_{k'}^\dagger\} = 0$. Now (78) can be shown by induction. For $N = 1$ the equation holds because

$$\langle \Phi_{\text{vac}}, c_{x_1} a_{k_1}^\dagger \Phi_{\text{vac}}\rangle = \frac{e^{ik_1\cdot x_1}}{\sqrt{L}}\langle \Phi_{\text{vac}}, \Phi_{\text{vac}}\rangle - \langle \Phi_{\text{vac}}, a_{k_1}^\dagger c_{x_1} \Phi_{\text{vac}}\rangle = \frac{e^{ik_1\cdot x_1}}{\sqrt{L}}. \tag{85}$$

Now suppose that (78) holds for some $N \in \mathbb{N}$. Then we have that

$$\begin{aligned}
&\langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_1} a_{k_1}^\dagger \dots a_{k_{N+1}}^\dagger \Phi_{\text{vac}}\rangle \\
&= (-1)^N \langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_1} a_{k_{N+1}}^\dagger a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\text{vac}}\rangle && \text{(86a)}\\
&= (-1)^N \left(\frac{e^{ik_{N+1}\cdot x_1}}{\sqrt{L}}\langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_2} a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\text{vac}}\rangle\right.\\
&\qquad\left. - \langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_2} a_{k_{N+1}}^\dagger c_{x_1} a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\text{vac}}\rangle\right) && \text{(86b)}\\
&= (-1)^N \left(\frac{e^{ik_{N+1}\cdot x_1}}{\sqrt{L}}\langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_2} a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\text{vac}}\rangle\right.\\
&\qquad - \frac{e^{ik_{N+1}\cdot x_2}}{\sqrt{L}}\langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_3} c_{x_1} a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\text{vac}}\rangle\\
&\qquad\left. + \langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_3} a_{k_{N+1}}^\dagger c_{x_2} c_{x_1} a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\text{vac}}\rangle\right) && \text{(86c)}\\
&= \dots && \text{(86d)}\\
&= \frac{(-1)^N}{\sqrt{L}} \sum_{l=1}^{N+1} (-1)^{l+1} e^{ik_{N+1}\cdot x_l} \langle \Phi_{\text{vac}}, c_{x_{N+1}} \dots c_{x_{l+1}} c_{x_{l-1}} \dots c_{x_1} a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\text{vac}}\rangle && \text{(86e)}\\
&= \frac{1}{L^{(N+1)/2}} \sum_{l=1}^{N+1} (-1)^{N+l+1} e^{ik_{N+1}\cdot x_l} \sum_{\sigma\in\mathcal{S}_{N,l}} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} e^{ik_j\cdot x_{\sigma(j)}}, && \text{(86f)}
\end{aligned}$$

where $\mathcal{S}_{N,l}$ denotes the set of permutations $\sigma : \{1, \ldots, N\} \to \{1, \ldots, l-1, l+1, \ldots, N+1\}$ of the set $\{1, \ldots, l-1, l+1, \ldots, N+1\}$. Note that we used the induction hypothesis in the last step.

Any $\sigma \in \mathcal{S}_{N,l}$ is related to a permutation $\tau \in \mathcal{S}_{N+1}$ with $\tau(N+1) = l$ via $N+1-l$ transpositions and vice versa. Therefore, we obtain

$$\begin{aligned}
&\langle \Phi_{\text{vac}}, c_{x_{N+1}} \ldots c_{x_1} a^\dagger_{k_1} \ldots a^\dagger_{k_{N+1}} \Phi_{\text{vac}} \rangle \\
&= \frac{1}{L^{(N+1)/2}} \sum_{l=1}^{N+1} \sum_{\substack{\tau \in \mathcal{S}_{N+1} \\ \tau(N+1)=l}} (-1)^{N+l+1} \text{sgn}(\tau) (-1)^{N+1-l} \prod_{j=1}^{N+1} e^{ik_j \cdot x_{\tau(j)}} && \text{(87a)} \\
&= \frac{1}{L^{(N+1)/2}} \sum_{\tau \in \mathcal{S}_{N+1}} \text{sgn}(\tau) \prod_{j=1}^{N+1} e^{ik_j \cdot x_{\tau(j)}}, && \text{(87b)}
\end{aligned}$$

which finishes the proof of (78).

4.10 Proof of Proposition 5

Let $x \in \Gamma$. With the help of Lemma 6 we find that

$$\begin{aligned}
&\langle \Psi_k, c^\dagger_x c_x \Psi_{k'} \rangle \\
&= \langle \Phi_{\text{vac}}, a_{k_N} \ldots a_{k_1} c^\dagger_x c_x a^\dagger_{k'_1} \ldots a^\dagger_{k'_N} \Phi_{\text{vac}} \rangle && \text{(88a)} \\
&= \frac{1}{L^{Nd/2}} \sum_{x_1, \ldots, x_N \in \Lambda} e^{-ik_1 \cdot x_1} \ldots e^{-ik_N \cdot x_N} \langle \Phi_{\text{vac}}, c_{x_N} \ldots c_{x_1} c^\dagger_x c_x a^\dagger_{k'_1} \ldots a^\dagger_{k'_N} \Phi_{\text{vac}} \rangle && \text{(88b)} \\
&= \frac{1}{L^{Nd/2}} \sum_{x_1, \ldots, x_N \in \Lambda} e^{-ik_1 \cdot x_1} \ldots e^{-ik_N \cdot x_N} \chi_{\{x \in \{x_1, \ldots, x_N\}\}} \langle \Phi_{\text{vac}}, c_{x_N} \ldots c_{x_1} a^\dagger_{k'_1} \ldots a^\dagger_{k'_N} \Phi_{\text{vac}} \rangle && \text{(88c)} \\
&= \frac{1}{L^{Nd}} \sum_{x_1, \ldots, x_N \in \Lambda} e^{-ik_1 \cdot x_1} \ldots e^{-ik_N \cdot x_N} \chi_{\{x \in \{x_1, \ldots, x_N\}\}} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) \prod_{j=1}^{N} e^{ik'_j \cdot x_{\sigma(j)}} && \text{(88d)} \\
&= \frac{1}{L^{Nd}} \sum_{l=1}^{N} e^{-ik_l \cdot x} \sum_{\substack{x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_N \in \Lambda, \\ x_l = x}} \left(\prod_{\substack{m=1 \\ m \neq l}}^{N} e^{-ik_m \cdot x_m} \right) \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) \prod_{j=1}^{N} e^{ik'_j \cdot x_{\sigma(j)}} && \text{(88e)} \\
&= \frac{1}{L^{Nd}} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) \sum_{l=1}^{N} e^{i(k'_{\sigma^{-1}(l)} - k_l) \cdot x} \prod_{\substack{m=1 \\ m \neq l}}^{N} \left(\sum_{x_m \in \Lambda} e^{i(k'_{\sigma^{-1}(m)} - k_m) \cdot x_m} \right). && \text{(88f)}
\end{aligned}$$

For $k, k' \in \mathcal{K}$ we have

$$\sum_{y \in \Lambda} e^{i(k'-k) \cdot y} = \sum_{y_1, \ldots, y_d = 1}^{L} e^{i(k'_1 - k_1) y_1} \ldots e^{i(k'_d - k_d) y_d} \tag{89a}$$

$$= \prod_{m=1}^{d} \sum_{y_m=1}^{L} e^{i(k'_m - k_m)y_m} \tag{89b}$$

$$= \prod_{m=1}^{d} \left(L\delta_{k'_m k_m} + \chi_{\{k'_m \neq k_m\}} \frac{e^{i(k'_m-k_m)} - e^{i(k'_m-k_m)(L+1)}}{1 - e^{i(k'_m-k_m)}} \right) \tag{89c}$$

$$= L^d \delta_{k'k}, , \tag{89d}$$

where we used that $(k'_m - k_m)L$ is a multiple of 2π and therefore $e^{i(k'_m-k_m)L} = 1$.

Thus we see that if $k = k'$, only the permutation $\sigma = \mathrm{id}$ gives a non-vanishing contribution in (88f) and we obtain

$$\langle \Psi_k, c_x^\dagger c_x \Psi_k \rangle = \frac{N}{L^d} \tag{90}$$

independently of $x \in \Gamma$. This implies

$$\langle \Psi_k, N_\Gamma \Psi_k \rangle = \frac{|\Gamma| N}{L^d}. \tag{91}$$

If $k \neq k'$, then (88f) only does not vanish if exactly one component of k and k' is different. Assume that k'_l for some $1 \leq l \leq N$ does not appear in k and let $\tilde{\sigma} \in \mathcal{S}_N$ be the permutation such that $k'_{\tilde{\sigma}^{-1}(m)} = k_m$ for all $m \neq l$. Then we get

$$\langle \Psi_k, c_x^\dagger c_x \Psi_{k'} \rangle = \frac{1}{L^d} \mathrm{sgn}(\tilde{\sigma}) e^{i(k_{\tilde{\sigma}^{-1}(l)} - k_l)\cdot x} \tag{92}$$

and therefore

$$\Big| \langle \Psi_k, N_\Gamma \Psi_{k'} \rangle \Big| \leq \frac{|\Gamma|}{L^d}. \tag{93}$$

Combining (91) and (93) and using that $|\Lambda| = L^d$ finishes the proof.

4.11 Proof of Proposition 4

Since H_0^{fF} is invariant under cyclic permutations of Λ, there is no loss of generality in assuming $\Gamma = \{1, \ldots, |\Gamma|\}$.

We first consider the case that there are no k_l, k_m such that $k_l = -k_m$. In this case, we can without loss of generality assume that $k_j \geq 0$ for all j. We express ϕ in the basis of the $\Psi_{k'}$ with $k'_j = \pm k_j$ for all j, i.e., we write

$$|\phi\rangle = \sum_{k'} \alpha_{k'} |\Psi_{k'}\rangle, \tag{94}$$

where $\alpha_{k'} = \langle \Psi_{k'} | \phi \rangle$. We compute

$$\langle \phi, N_\Gamma \phi \rangle = \sum_{k',k''} \alpha^*_{k'} \alpha_{k''} \langle \Psi_{k'}, N_\Gamma \Psi_{k''} \rangle \tag{95a}$$

$$= \sum_{k'} |\alpha_{k'}|^2 N \frac{|\Gamma|}{|\Lambda|} + \sum_{k' \neq k''} \alpha^*_{k'} \alpha_{k''} \langle \Psi_{k'}, N_\Gamma \Psi_{k''} \rangle \tag{95b}$$

$$= N \frac{|\Gamma|}{|\Lambda|} + \sum_{x=1}^{|\Gamma|} \sum_{k' \neq k''} \alpha^*_{k'} \alpha_{k''} \langle \Psi_{k'}, c^\dagger_x c_x \Psi_{k''} \rangle. \tag{95c}$$

Because of Proposition 5 we see that $\langle \Psi_{k'}, c^\dagger_x c_x \Psi_{k''} \rangle$ with $k' \neq k''$ only does not vanish if k' and k'' differ in exactly one component. First suppose that $0 < k_j < \pi$ for all j. With Proposition 5 we find that

$$\sum_{x=1}^{|\Gamma|} \sum_{k' \neq k''} \alpha^*_{k'} \alpha_{k''} \langle \Psi_{k'}, c^\dagger_x c_x \Psi_{k''} \rangle$$

$$= \frac{2}{L} \operatorname{Re} \left(\sum_{j=1}^{N} \sum_{x=1}^{|\Gamma|} e^{-2ik_j x} \sum_{k' : k'_j > 0} \chi_{\{k''_j = -k'_j ; k''_l = k'_l, l \neq j\}} \alpha^*_{k'} \alpha_{k''} \right) \tag{96a}$$

$$= \frac{2}{L} \operatorname{Re} \left(\sum_{j=1}^{N} \frac{e^{-2ik_j} - e^{-2ik_j(|\Gamma|+1)}}{1 - e^{-2ik_j}} \sum_{k' : k'_j > 0} \chi_{\{k''_j = -k'_j ; k''_l = k'_l, l \neq j\}} \alpha^*_{k'} \alpha_{k''} \right). \tag{96b}$$

Next note that with the Cauchy-Schwarz inequality we get

$$\left| \sum_{k' : k'_j > 0} \chi_{\{k''_j = -k'_j ; k''_l = k'_l, l \neq j\}} \alpha^*_{k'} \alpha_{k''} \right| \leq \left(\sum_{k'} |\alpha_{k'}|^2 \sum_{k''} |\alpha_{k''}|^2 \right)^{1/2} = 1. \tag{97}$$

Moreover, the inequality $|1 - e^{ix}| \geq 2|x|/\pi$ for $x \in [-\pi, \pi]$ implies

$$\left| 1 - e^{-2ik_j} \right| \geq \frac{4k_j}{\pi} = \frac{8}{L} \nu_j \tag{98}$$

if $-2k_j \in [-\pi, \pi]$ where we used that $k_j = \frac{2\pi}{L} \nu_j$ for some $\nu_j \in \{1, \ldots, (L-1)/2\}$ if L is odd and $\nu_j \in \{1, \ldots, L/2 - 1\}$ if L is even. If $-2k_j < -\pi$ we obtain

$$\left| 1 - e^{-2ik_j} \right| = \left| 1 - e^{-2ik_j + 2\pi i} \right| \geq \frac{4|\pi - k_j|}{\pi} = 4 \left(1 - \frac{2}{L} \nu_j \right). \tag{99}$$

We get

$$\left| \sum_{x=1}^{|\Gamma|} \sum_{k' \neq k''} \alpha^*_{k'} \alpha_{k''} \langle \Psi_{k'}, c^\dagger_x c_x \Psi_{k''} \rangle \right| \leq \frac{2}{L} \sum_{j=1}^{N} \frac{2}{|1 - e^{-2ik_j}|} \tag{100a}$$

$$\leq \frac{4}{L} \sum_{j=1}^{N} \frac{L}{4j} \tag{100b}$$

$$\leq \ln N + 1. \tag{100c}$$

Altogether we therefore obtain

$$\left| \langle \phi, N_\Gamma \phi \rangle - N \frac{|\Gamma|}{|\Lambda|} \right| \leq \ln N + 1. \tag{101}$$

If $k_{j_0} = 0$ or $k_{j_0} = \pi$ for one j_0, the computation is basically the same; the only difference is that this index does not appear in the sum over j (which therefore consists only of $N-1$ terms). Moreover, if there are k_l, k_m such that $k_l = -k_m$, then again this only leads to less terms in the sums over j. The upper bound $\ln N + 1$ thus remains valid also in these cases.

4.12 Proof of Proposition 7

We start by computing $\langle \Psi_k, N_\Gamma^2 \Psi_k \rangle$. To this end we first note that

$$\langle \Psi_k, N_\Gamma^2 \Psi_k \rangle = \sum_{x,y \in \Gamma} \langle \Psi_k, c_x^\dagger c_x c_y^\dagger c_y \Psi_k \rangle. \tag{102}$$

If $x = y \in \Gamma$ then

$$\langle \Psi_k, c_x^\dagger c_x c_x^\dagger c_x \Psi_k \rangle = \langle \Psi_k, c_x^\dagger c_x \Psi_k \rangle = \frac{N}{L^d}, \tag{103}$$

see (90). Now suppose that $x \neq y$. Then we find with the help of Lemma 6 that

$$\langle \Psi_k, c_x^\dagger c_x c_y^\dagger c_y \Psi_k \rangle$$

$$= \langle \Phi_{\mathrm{vac}}, a_{k_N} \dots a_{k_1} c_x^\dagger c_x c_y^\dagger c_y a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\mathrm{vac}} \rangle \tag{104a}$$

$$= \frac{1}{L^{Nd/2}} \sum_{x_1,\dots,x_N \in \Lambda} e^{-ik_1 \cdot x_1} \dots e^{-ik_N \cdot x_N} \langle \Phi_{\mathrm{vac}}, c_{x_N} \dots c_{x_1} c_x^\dagger c_x c_y^\dagger c_y a_{k_1}^\dagger \dots a_{k_N}^\dagger \Phi_{\mathrm{vac}} \rangle \tag{104b}$$

$$= \frac{1}{L^{Nd/2}} \sum_{x_1,\dots,x_N \in \Lambda} e^{-ik_1 \cdot x_1} \dots e^{-ik_N \cdot x_N} \chi_{\{x,y \in \{x_1,\dots,x_N\}\}} \langle \Phi_{\mathrm{vac}}, c_{x_N} \dots c_{x_1} a_{k_1}^\dagger \dots a_{k_N}^\dagger | \Phi_{\mathrm{vac}} \rangle \tag{104c}$$

$$= \frac{1}{L^{Nd}} \sum_{x_1,\dots x_N \in \Lambda} e^{-ik_1 \cdot x_1} \dots e^{-ik_N \cdot x_N} \chi_{\{x,y \in \{x_1,\dots,x_N\}\}} \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} e^{ik_j \cdot x_{\sigma(j)}} \tag{104d}$$

$$= \frac{1}{L^{Nd}} \sum_{\substack{l,m=1 \\ l \neq m}}^{N} e^{-ik_l \cdot x} e^{-ik_m \cdot y} \sum_{\substack{x_1,\dots,x_N \in \Lambda \\ x_l = x, x_m = y}} \left(\prod_{\substack{n=1 \\ n \neq l,m}}^{N} e^{-ik_n \cdot x_n} \right) \sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} e^{ik_{\sigma^{-1}(j)} \cdot x_j} \tag{104e}$$

$$= \frac{1}{L^{Nd}} \sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn}(\sigma) \sum_{\substack{l,m=1 \\ l \neq m}}^{N} e^{i(k_{\sigma^{-1}(l)} - k_l) \cdot x} e^{i(k_{\sigma^{-1}(m)} - k_m) \cdot y} \prod_{\substack{n=1 \\ n \neq l,m}}^{N} \left(\sum_{x_n \in \Lambda} e^{i(k_{\sigma^{-1}(n)} - k_n) \cdot x_n} \right). \tag{104f}$$

Because $\sum_{y \in \Lambda} e^{i(k'-k)\cdot y} = L\delta_{k,k'}$, see (89d), we only get contributions from the permutations $\sigma = \mathrm{id}$ and transpositions τ_{pq} with $p, q \in \{1, \dots N\}$. From $\sigma = \mathrm{id}$ we get the contribution

$$\frac{1}{L^{Nd}} N(N-1) L^{(N-2)d} = \frac{N(N-1)}{L^{2d}} \tag{105}$$

and any transposition τ_{pq} contributes the term

$$- \frac{1}{L^{Nd}} \left(e^{i(k_p - k_q)\cdot x} e^{i(k_q - k_p)\cdot y} + e^{i(k_q - k_p)\cdot x} e^{i(k_p - k_q)\cdot y} \right) L^{(N-2)d} = -\frac{2}{L^{2d}} \operatorname{Re} \left(e^{i(k_p - k_q)\cdot x} e^{i(k_q - k_p)\cdot y} \right). \tag{106}$$

Therefore the overall contribution of transpositions becomes

$$-\frac{2}{L^{2d}} \sum_{\substack{p,q=1 \\ p<q}}^{N} \operatorname{Re} \left(e^{i(k_p - k_q)\cdot x} e^{i(k_q - k_p)\cdot y} \right). \tag{107}$$

Altogether we obtain

$$\langle \Psi_k, c_x^\dagger c_x c_y^\dagger c_y \Psi_k \rangle = \frac{N(N-1)}{L^{2d}} - \frac{2}{L^{2d}} \sum_{\substack{p,q=1 \\ p<q}}^{N} \operatorname{Re} \left(e^{i(k_p - k_q)\cdot x} e^{i(k_q - k_p)\cdot y} \right). \tag{108}$$

Summing over $x, y \in \Gamma$ we arrive at

$$\langle \Psi_k, N_\Gamma^2 \Psi_k \rangle$$

$$= \frac{N}{L^d} |\Gamma| + \frac{N(N-1)}{L^{2d}} |\Gamma|(|\Gamma| - 1) - \frac{2}{L^{2d}} \sum_{\substack{x,y \in \Gamma \\ x \neq y}} \sum_{\substack{p,q=1 \\ p<q}}^{N} \operatorname{Re} \left(e^{i(k_p - k_q)\cdot x} e^{i(k_q - k_p)\cdot y} \right) \tag{109a}$$

$$= \frac{N}{L^d} |\Gamma| + \frac{N(N-1)}{L^{2d}} |\Gamma|(|\Gamma| - 1) - \frac{2}{L^{2d}} \sum_{\substack{p,q=1 \\ p<q}}^{N} \left(\sum_{x,y \in \Gamma} \operatorname{Re} \left(e^{i(k_p - k_q)\cdot x} e^{i(k_q - k_p)\cdot y} \right) - |\Gamma| \right) \tag{109b}$$

$$= \frac{N}{L^d}|\Gamma| + \frac{N(N-1)}{L^{2d}}|\Gamma|(|\Gamma|-1) + \frac{|\Gamma|N(N-1)}{L^{2d}} - \frac{2}{L^{2d}} \sum_{\substack{p,q=1 \\ p<q}}^{N} \left| \sum_{x\in\Gamma} e^{i(k_p-k_q)\cdot x} \right|^2 \tag{109c}$$

$$\leq N\frac{|\Gamma|}{|\Lambda|} + N(N-1)\frac{|\Gamma|^2}{|\Lambda|^2}. \tag{109d}$$

With this and $\langle \Psi_k, N_\Gamma \Psi_k\rangle = N|\Gamma|/|\Lambda|$ we finally get for the variance of N_Γ in an eigenstate Ψ_k that

$$\langle \Psi_k, N_\Gamma^2 \Psi_k\rangle - (\langle \Psi_k, N_\Gamma \Psi_k\rangle)^2 \leq N\frac{|\Gamma|}{|\Lambda|}\left(1 - \frac{|\Gamma|}{|\Lambda|}\right). \tag{110}$$

4.13 Proof of Proposition 6

As in the proof of Proposition 4 we first assume that there are no k_l, k_m such that $k_l = -k_m$. In this case, we can without loss of generality assume that $k_j \geq 0$ for all j. As we have already computed $\langle \phi, N_\Gamma \phi\rangle$ in Proposition 4, it only remains to compute $\langle \phi, N_\Gamma^2 \phi\rangle$. To this end, we express ϕ again in the basis of the $\Psi_{k'}$ with $k'_j = \pm k_j$, see (94). Then we get

$$\langle \phi, N_\Gamma^2 \phi\rangle = \sum_{k',k''} \alpha_{k'}^* \alpha_{k''} \langle \Psi_{k'}, N_\Gamma^2 \Psi_{k''}\rangle \tag{111a}$$

$$= \sum_{k'} |\alpha_{k'}|^2 \langle \Psi_{k'}, N_\Gamma^2 \Psi_{k'}\rangle + \sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle \Psi_{k'}, N_\Gamma^2 \Psi_{k''}\rangle. \tag{111b}$$

For the first sum we obtain with the help of (109d) that

$$\sum_{k'} |\alpha_{k'}|^2 \langle \Psi_{k'}, N_\Gamma^2 \Psi_{k'}\rangle \leq N\frac{|\Gamma|}{|\Lambda|} + N(N-1)\frac{|\Gamma|^2}{|\Lambda|^2}. \tag{112}$$

For the second sum first note that

$$\sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle \Psi_{k'}, N_\Gamma^2 \Psi_{k''}\rangle$$

$$= \sum_{x,y\in\Gamma} \sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle \tag{113a}$$

$$= \sum_{x\in\Gamma} \sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle \Psi_{k'}, c_x^\dagger c_x \Psi_{k''}\rangle + \sum_{\substack{x,y\in\Gamma \\ x\neq y}} \sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle. \tag{113b}$$

The first sum can be estimated as in the proof of Proposition 4, i.e.,

$$\left| \sum_{x\in\Gamma} \sum_{k'\neq k''} \alpha_{k'}^* \alpha_{k''} \langle \Psi_{k'}, c_x^\dagger c_x \Psi_{k''}\rangle \right| \leq \ln N + 1. \tag{114}$$

For the second sum in (113b) we start by noting that similarly to (104f) we have for $x \neq y$ and $k' \neq k''$ that

$$\begin{aligned}&\langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle \\ &\quad = \frac{1}{L^N} \sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn}(\sigma) \sum_{\substack{l,m=1\\ l\neq m}}^{N} e^{i(k''_{\sigma^{-1}(l)}-k'_l)x} e^{i(k''_{\sigma^{-1}(m)}-k'_m)y} \prod_{\substack{n=1\\ n\neq l,m}}^{N} \left(\sum_{x_n \in \Lambda} e^{i(k''_{\sigma^{-1}(n)}-k'_n)x_n} \right).\end{aligned} \tag{115}$$

From this formula we see that if $k' \neq k''$ then only terms where one or two entries of k' and k'' are different give non-vanishing contributions.

We shall separate the second sum in (113b) into two sums where k' and k'' differ in one or two components, respectively. Suppose first that k' and k'' differ in only one component. Let us evaluate $\langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle$ in the case that $k'_1 = k_1 > 0$ and $k''_1 = -k_1$. If $\sigma = \mathrm{id}$, we get the contribution

$$\frac{N-1}{L^2}\left(e^{-2ik_1x} + e^{-2ik_1y}\right). \tag{116}$$

The only other permutations that yield non-vanishing terms are transpositions of the form τ_{1p} with $p > 1$. Altogether, these transpositions give the contribution

$$-\frac{1}{L^2}\sum_{p=2}^{N} e^{i(-k_1-k'_p)x} e^{i(k'_p-k_1)y} + e^{i(k'_p-k_1)x} e^{i(-k_1-k'_p)y}. \tag{117}$$

One obtains analogous expressions in the case that k' and k'' do not differ in the first, but in another component.

Our goal is to estimate

$$\sum_{\substack{x,y\in\Gamma\\ x\neq y}} \sum_{\substack{k',k''\text{differ}\\ \text{in 1 component}}} \alpha^*_{k'}\alpha_{k''}\langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle \tag{118}$$

First suppose that $0 < k_j < \pi$ for all j. To facilitate the computation, we first let the sum over $x, y \in \Gamma, x \neq y$ run over all $x, y \in \Gamma$ and later estimate the terms where $x = y$. Considering $k'_j > 0$ and $k'_j < 0$ separately we can write

$$\begin{aligned}&\sum_{x,y\in\Gamma} \sum_{\substack{k',k''\text{differ}\\ \text{in 1 component}}} \alpha^*_{k'}\alpha_{k''}\langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle \\ &\qquad = \sum_{x,y\in\Gamma}\sum_{j=1}^{N}\sum_{k':k'_j>0} \chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''} 2\mathrm{Re}\left(\langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle\right)\end{aligned} \tag{119}$$

Using the expressions for $\langle \Psi_{k'}, c_x^\dagger c_x c_y^\dagger c_y \Psi_{k''}\rangle$ derived above, this equals

$$\frac{2}{L^2}\mathrm{Re}\Bigg(\sum_{x,y=1}^{|\Gamma|}\sum_{j=1}^{N}\Bigg[(N-1)\left(e^{-2ik_jx}+e^{-2ik_jy}\right)\sum_{k':k'_j>0}\chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''} \tag{120}$$
$$-\sum_{\substack{p=1\\p\neq j}}^{N}\left(e^{i(-k_j-k_p)x}e^{i(k_p-k_j)y}+e^{i(k_p-k_j)x}e^{i(-k_j-k_p)y}\right)\sum_{k':k'_j,k'_p>0}\chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''}$$
$$-\sum_{\substack{p=1\\p\neq j}}^{N}\left(e^{i(-k_j+k_p)x}e^{i(-k_p-k_j)y}+e^{i(-k_p-k_j)x}e^{i(-k_j+k_p)y}\right)$$
$$\times\sum_{k':k'_j>0,k'_p<0}\chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''}\Bigg]\Bigg)$$

Carrying out the summations over x and y we arrive at

$$\frac{2}{L^2}\mathrm{Re}\Bigg(\sum_{j=1}^{N}\Bigg[2|\Gamma|(N-1)\frac{e^{-2ik_j}-e^{-2ik_j(|\Gamma|+1)}}{1-e^{-2ik_j}}\sum_{k':k'_j>0}\chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''}$$
$$-2\sum_{\substack{p=1\\p\neq j}}^{N}\Bigg(\frac{e^{-i(k_j+k_p)}-e^{-i(k_j+k_p)(|\Gamma|+1)}}{1-e^{-i(k_p+k_j)}}\frac{e^{i(k_p-k_j)}-e^{i(k_p-k_j)(|\Gamma|+1)}}{1-e^{i(k_p-k_j)}}$$
$$\times\sum_{k':k'_j,k'_p>0}\chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''}\Bigg)$$
$$-2\sum_{\substack{p=1\\p\neq j}}^{N}\Bigg(\frac{e^{-i(k_j-k_p)}-e^{-i(k_j-k_p)(|\Gamma|+1)}}{1-e^{-i(k_j-k_p)}}\frac{e^{-i(k_p+k_j)}-e^{-i(k_p+k_j)(|\Gamma|+1)}}{1-e^{-i(k_p+k_j)}}$$
$$\times\sum_{k':k'_j>0,k'_p<0}\chi_{\{k''_j=-k'_j;k''_l=k'_l,l\neq j\}}\alpha^*_{k'}\alpha_{k''}\Bigg)\Bigg]\Bigg) \tag{121}$$

Taking the absolute value, the sums over k' can again be upper bounded by 1. As in the proof of Proposition 4, we compute that

$$\frac{4}{L^2}\sum_{j=1}^{N}\left||\Gamma|(N-1)\left(\frac{e^{-2ik_j}-e^{-2ik_j(|\Gamma|+1)}}{1-e^{-2ik_j}}\right)\right|\leq\frac{4|\Gamma|(N-1)}{L^2}\sum_{j=1}^{N}\frac{2}{(4j/L)}$$
$$=\frac{2|\Gamma|(N-1)}{L}\sum_{j=1}^{N}\frac{1}{j}\leq\frac{2|\Gamma|(N-1)}{L}(\ln N+1). \tag{122}$$

If $k_{j_0} = 0$ or $k_{j_0} = \pi$ for some j_0 or if there are k_l, k_m such that $k_l = -k_m$, the corresponding term is missing in the sum over j and the upper bound of $N-1$ remains valid.

Next we estimate

$$\frac{4}{L^2}\sum_{j=1}^{N}\left|\sum_{\substack{p=1\\p\neq j}}^{N}\frac{e^{-i(k_j+k_p)}-e^{-i(k_j+k_p)(|\Gamma|+1)}}{1-e^{-i(k_p+k_j)}}\frac{e^{i(k_p-k_j)}-e^{i(k_p-k_j)(|\Gamma|+1)}}{1-e^{i(k_p-k_j)}}\right| \tag{123a}$$

$$\leq \frac{16}{L^2}\sum_{j=1}^{N}\sum_{\substack{p=1\\p\neq j}}^{N}\frac{1}{|1-e^{-i(k_p+k_j)}||1-e^{i(k_p-k_j)}|} \tag{123b}$$

$$\leq \frac{16}{L^2}\sum_{j=1}^{N}\left(\sum_{\substack{p=1\\p\neq j}}^{N}\frac{1}{|1-e^{-i(k_p+k_j)}|^2}\right)^{1/2}\left(\sum_{\substack{p=1\\p\neq j}}^{N}\frac{1}{|1-e^{i(k_p-k_j)}|^2}\right)^{1/2} \tag{123c}$$

$$\leq \frac{16}{L^2}\sum_{j=1}^{N}\left(\sum_{p=1}^{N}\frac{L^2}{16p^2}\right) \tag{123d}$$

$$\leq 2N, \tag{123e}$$

where we used in the last step that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6 < 2$. Note that the last term in (121) is of a similar form and can be estimated along the same lines.

To get an estimate for the overall contribution of $k' \neq k''$ which differ in one component, we still have to estimate the terms in (120) with $x = y$. By a similar computation as before and in the proof of Proposition 4 we find that an upper bound for the absolute value of these terms is given by

$$\frac{12(N-1)}{L^2}\sum_{j=1}^{N}\left|\frac{e^{-2ik_j}-e^{-2ik_j(|\Gamma|+1)}}{1-e^{-2ik_j}}\right| \leq \frac{6(N-1)}{L}(\ln N + 1). \tag{124}$$

Again, if there is a k_{j_0} that is equal to 0 or π or if there are k_l, k_m such that $k_l = -k_m$, the only modification that has to be made is to exclude the terms in the sum over j and therefore the estimates remain valid.

Altogether we find for the contribution of the terms in (113b) with $k' \neq k''$, where k' and k'' differ in exactly one component, the upper bound

$$\frac{2|\Gamma|(N-1)}{L}(\ln N + 1) + 4N + \frac{6(N-1)}{L}(\ln N + 1)\,. \tag{125}$$

Next we turn to the terms in (113b) with $k' \neq k''$ which differ in two components. As before, we can assume that $0 < k_j < \pi$ for all j as by the same reasoning as in the previous computations, the upper bounds remain valid if this condition is relaxed.

Consider (115) in the case $k_1' = k_1 > 0, k_2' = k_2 > 0$ and $k_1'' = -k_1, k_2'' = -k_2$. Then the permutation $\sigma = \mathrm{id}$ gives the contribution

$$\frac{1}{L^2}\left(e^{-2ik_1x}e^{-2ik_2y} + e^{-2ik_2x}e^{-2ik_1y}\right) \tag{126}$$

and from the permutation τ_{12} we get

$$-\frac{2}{L^2}e^{-i(k_1+k_2)x}e^{-i(k_1+k_2)y}. \tag{127}$$

If $k_1' = k_1, k_2' = -k_2, k_1'' = -k_1, k_2'' = k_2$, the permutation $\sigma = \mathrm{id}$ gives

$$\frac{1}{L^2}\left(e^{-2ik_1x}e^{2ik_2y} + e^{2ik_2x}e^{-2ik_1y}\right) \tag{128}$$

and τ_{12} yields the term

$$-\frac{2}{L^2}e^{i(k_2-k_1)x}e^{i(k_2-k_1)y}. \tag{129}$$

Summing again first over all $x, y \in \Gamma$ and later estimating the terms with $x = y$ which were added to facilitate the computation we get

$$\begin{aligned}\frac{2}{L^2}\mathrm{Re}\Bigg(\sum_{x,y=1}^{|\Gamma|}\sum_{\substack{l,m=1\\ l\neq m}}^{N}\Bigg[&\left(e^{-2ik_lx}e^{-2ik_my} + e^{-2ik_mx}e^{-2ik_ly} - 2e^{-i(k_l+k_m)x}e^{-i(k_l+k_m)y}\right)\\ &\times\sum_{k':k_l',k_m'>0}\chi_{\{k_l''=-k_l',k_m''=-k_m';k_j''=k_j,j\neq l,m\}}\alpha_{k'}^*\alpha_{k''}\\ +&\left(e^{-2ik_lx}e^{2ik_my} + e^{2ik_mx}e^{-2ik_ly} - 2e^{i(k_m-k_l)x}e^{i(k_m-k_l)y}\right)\\ &\times\sum_{k':k_l'>0,k_m'<0}\chi_{\{k_l''=-k_l',k_m''=-k_m';k_j''=k_j,j\neq l,m\}}\alpha_{k'}^*\alpha_{k''}\Bigg]\Bigg)\end{aligned} \tag{130}$$

Using similar estimates as previously, we can bound the absolute value of this term by

$$2(\ln N + 1)^2 + 4N. \tag{131}$$

Next we note that the terms with $x = y$ in this sum vanish.

Putting everything together, we finally arrive at

$$\begin{aligned}\langle\phi, N_\Gamma^2\phi\rangle \leq{}& N\frac{|\Gamma|}{|\Lambda|} + N(N-1)\frac{|\Gamma|^2}{|\Lambda|^2} + \ln N + 1 + \frac{2|\Gamma|(N-1)}{L}(\ln N + 1) + 4N\\ &+ \frac{6(N-1)}{L}(\ln N + 1) + 2(\ln N + 1)^2 + 4N\end{aligned} \tag{132a}$$

$$\leq N^2 \frac{|\Gamma|^2}{|\Lambda|^2} + 2N \ln N + 11N + 2(\ln N)^2 + 11 \ln N + 9. \tag{132b}$$

Because of $(a^2 - b^2) = (a-b)(a+b)$ for $a, b \in \mathbb{R}$ it follows from Proposition 4 that

$$\left| \langle \phi, N_\Gamma \phi \rangle^2 - N^2 \frac{|\Gamma|^2}{|\Lambda|^2} \right| \leq (\ln N + 1) \left(\ln N + 1 + 2N \frac{|\Gamma|}{|\Lambda|} \right) \tag{133}$$

and therefore

$$\langle \phi, N_\Gamma \phi \rangle^2 \geq N^2 \frac{|\Gamma|^2}{|\Lambda|^2} - (\ln N + 1) \left(\ln N + 1 + 2N \frac{|\Gamma|}{|\Lambda|} \right). \tag{134}$$

With this we finally obtain

$$\langle \phi, N_\Gamma^2 \phi \rangle - \langle \phi, N_\Gamma \phi \rangle^2 \leq 4N \ln N + 13N + 3(\ln N)^2 + 13 \ln N + 10\,. \tag{135}$$

For $N \geq 46$ it holds that $13N + 3(\ln N)^2 + 13 \ln N + 10 < 4N \ln N$. This finishes the proof.

4.14 Proof of Proposition 3

The proof of a similar result for $d = 1$ given by Tasaki in [33] can be adapted to our situation with only very small modifications. Note that the proof in the present situation in the case that $d = 1$ was already given by Tasaki in an earlier arXiv version of his paper, however, the model was slightly changed in later versions.

Similarly as in the proof of Lemma 4 in [33] we first show that

$$\langle \Psi_k, e^{\lambda N_\Gamma} \Psi_k \rangle \leq \left(\mu e^\lambda + (1-\mu) \right)^N \tag{136}$$

for any $\lambda \in (0, 1]$. To this end note that

$$e^{\lambda N_\Gamma / 2} |\Psi_k\rangle = b_{k_1}^\dagger \dots b_{k_N}^\dagger |\Phi_{\mathrm{vac}}\rangle \tag{137}$$

with

$$b_k^\dagger := \frac{1}{L^{d/2}} \left(e^{\lambda/2} \sum_{x \in \Gamma} e^{ik \cdot x} c_x^\dagger + \sum_{x \in \Lambda \setminus \Gamma} e^{ik \cdot x} c_x^\dagger \right). \tag{138}$$

We then obtain that

$$\langle \Psi_k, e^{\lambda N_\Gamma} \Psi_k \rangle = \langle \Phi_{\mathrm{vac}}, b_{k_N} \dots b_{k_1} b_{k_1}^\dagger \dots b_{k_N}^\dagger \Phi_{\mathrm{vac}} \rangle \leq \prod_{j=1}^N \| b_{k_j} b_{k_j}^\dagger \|. \tag{139}$$

For any j, the operator $b_{k_j} b_{k_j}^\dagger$ is self-adjoint and

$$\left\{b_{k_j}, b_{k_j}^\dagger\right\} = \frac{1}{L^d}\left(e^\lambda \sum_{x,y\in\Gamma} e^{ik_j\cdot(x-y)}\{c_y, c_x^\dagger\} + e^{\lambda/2} \sum_{\substack{x\in\Gamma\\ y\in\Lambda\setminus\Gamma}} e^{ik_j\cdot(x-y)}\{c_y, c_x^\dagger\}\right.$$

$$\left. + e^{\lambda/2} \sum_{\substack{x\in\Lambda\setminus\Gamma\\ y\in\Gamma}} e^{ik_j\cdot(x-y)}\{c_y, c_x^\dagger\} + \sum_{x,y\in\Lambda\setminus\Gamma} e^{ik_j\cdot(x-y)}\{c_y, c_x^\dagger\}\right)$$

$$= \frac{1}{L^d}\left(e^\lambda|\Gamma| + |\Lambda| - |\Gamma|\right) = \mu e^\lambda + (1-\mu). \quad (140)$$

Next note that this implies

$$(b_{k_j} b_{k_j}^\dagger)^2 = \left(\mu e^\lambda + (1-\mu)\right) b_{k_j} b_{k_j}^\dagger - b_{k_j} b_{k_j} b_{k_j}^\dagger b_{k_j}^\dagger = (\mu e^\lambda + (1-\mu)) b_{k_j} b_{k_j}^\dagger. \quad (141)$$

The last step follows from $b_{k_j} b_{k_j} = 0$, which can be seen as follows: From the definition of the b_{k_j} we immediately obtain

$$b_{k_j} b_{k_j} = \frac{1}{L^d}\left(e^\lambda \sum_{x,y\in\Gamma} e^{-ik_j\cdot(x+y)} c_x c_y + e^{\lambda/2} \sum_{\substack{x\in\Gamma\\ y\in\Lambda\setminus\Gamma}} e^{-ik_j\cdot(x+y)} c_x c_y\right.$$

$$\left. + e^{\lambda/2} \sum_{\substack{x\in\Lambda\setminus\Gamma\\ y\in\Gamma}} e^{-ik_j\cdot(x+y)} c_x c_y + \sum_{x,y\in\Lambda\setminus\Gamma} e^{-ik_j\cdot(x+y)} c_x c_y\right). \quad (142)$$

If $x = y$, then $c_x c_y = 0$. In the first and fourth sum, for every term $c_x c_y$ with $x \neq y$ also the term $c_y c_x$ occurs and because of $c_x c_y = -c_y c_x$ and the same prefactors, the terms cancel. Therefore, the first and fourth sum vanishes. By a similar argumentation, the second sum is equal to the (-1) times the third sum, i.e., they cancel, and altogether we obtain that $b_{k_j} b_{k_j} = 0$.

It follows from (141) that the eigenvalues of the self-adjoint operator $b_{k_j} b_{k_j}^\dagger$ are 0 and $\mu e^\lambda + (1-\mu)$: If $\alpha \in \mathbb{R}$ is an eigenvalue of $b_{k_j} b_{k_j}^\dagger$ with eigenfunction ϕ, then $b_{k_j} b_{k_j}^\dagger|\phi\rangle = \alpha|\phi\rangle$ and therefore $(b_{k_j} b_{k_j}^\dagger)^2|\phi\rangle = \alpha b_{k_j} b_{k_j}^\dagger|\phi\rangle$, but at the same time $(b_{k_j} b_{k_j}^\dagger)^2|\phi\rangle = (\mu e^\lambda + (1-\mu)) b_{k_j} b_{k_j}^\dagger|\phi\rangle$, i.e., either $\alpha = \mu e^\lambda + (1-\mu)$ or $b_{k_j} b_{k_j}^\dagger|\phi\rangle = 0$ which implies $\alpha = 0$ as $\phi \neq 0$. Therefore we conclude that $\|b_{k_j} b_{k_j}^\dagger\| = \mu e^\lambda + (1-\mu)$ and hence

$$\langle \Psi_k, e^{\lambda N_\Gamma} \Psi_k\rangle \leq \left(\mu e^\lambda + (1-\mu)\right)^N. \quad (143)$$

The rest of the proof of (29) is exactly the same as in the first arXiv version of [33], but we include it here for the convenience of the reader. First note that the projection onto the non-equilibrium space can be bounded in terms of characteristic functions as

$$\left\|P_{\mathrm{neq},\eta}\Psi_k\right\|^2 \leq \langle \Psi_k, 1_{N_\Gamma/N-\mu\geq\eta}(N_\Gamma)\Psi_k\rangle + \langle \Psi_k, 1_{N_\Gamma/N-\mu\leq-\eta}(N_\Gamma)\Psi_k\rangle. \tag{144}$$

In the first term, the characteristic function is bounded above by $e^{\lambda(N_\Gamma/N-\mu-\eta)N}$ for all $\lambda \geq 0$. Using inequality (143) we obtain

$$\langle \Psi_k, 1_{N_\Gamma/N-\mu\geq\eta}(N_\Gamma)\Psi_k\rangle \leq \left[(\mu e^{\lambda(1-\mu)} + (1-\mu)e^{-\lambda\mu})e^{-\lambda\eta}\right]^N \tag{145}$$

Using the power series representation for the exponential function one finds that for all $0 < \mu < 1, |\lambda| < 1$

$$\begin{aligned}\mu e^{\lambda(1-\mu)} + (1-\mu)e^{-\lambda\mu} &\leq 1 + \frac{\mu(1-\mu)}{2}\lambda^2 + \frac{\mu(1-\mu)}{2}\lambda^2\sum_{l=3}^{\infty}\frac{1}{l!} \\ &\leq 1 + \frac{3}{4}\mu(1-\mu)\lambda^2 \leq e^{\frac{3}{4}\mu(1-\mu)\lambda^2}.\end{aligned} \tag{146}$$

Choosing $\lambda = 2\eta/(3\mu(1-\mu))$, which is smaller than 1 by assumption, results in the bound

$$\langle \Psi_k, 1_{N_\Gamma/N-\mu\geq\eta}(N_\Gamma)\Psi_k\rangle \leq e^{-\frac{\eta^2}{3\mu(1-\mu)}N}. \tag{147}$$

For the second term in (144), note that it equals $\langle \Psi_k, 1_{N_{\Lambda\setminus\Gamma}/N-(1-\mu)\geq\eta}(N_{\Lambda\setminus\Gamma})\Psi_k\rangle$. Now we can apply the bound (147) with Γ replaced by $\Lambda\setminus\Gamma$ and μ by $(1-\mu)$ and conclude that

$$\left\|P_{\mathrm{neq},\eta}\Psi_k\right\|^2 < 2e^{-\frac{\eta^2}{3\mu(1-\mu)}N}. \tag{148}$$

The last statement of Proposition 3, that $D_E \geq 2^{Nd}$ if $N < L/2d$, can be verified as follows. Choose Nd distinct positive integers less than $L/2$, multiply them by $2\pi/L$, and call them, in any order, k_{ia} with $i = 1, \dots, N$ and $a = 1, \dots, d$. Write $k_i = (k_{i1}, \dots, k_{id})$ and $\tilde{k}_i = k_{\pi(i)}$ for the permuted version such that $\tilde{k} = (\tilde{k}_1, \dots, \tilde{k}_N) \in \tilde{\mathcal{K}}^N$. Thus, $\Psi_{\tilde{k}}$ as in (28) is an eigenvector of H_0^{fF}, and the corresponding eigenvalue is $-2\sum_{i=1}^N\sum_{a=1}^d \cos k_{ia}$. Now consider the 2^{Nd} Nd-vectors k' with $k'_{ia} = \pm k_{ia}$; none of them is a permutation of any other, so suitable permutations yield 2^{Nd} distinct elements $\tilde{k}'$ of $\tilde{\mathcal{K}}^N$. The corresponding $\Psi_{\tilde{k}'}$ have the same eigenvalue, so the eigenspace must at least have dimension 2^{Nd}.

5 Conclusions

If a Hamiltonian H satisfies the ETH in the form that every eigenvector is in MATE (which we show in Theorem 2 is the case for the free Fermi gas on a 1d lattice), then every initial pure state ψ_0 will thermalize in the sense of MATE. However, for practical purposes a weaker statement seems just as useful: that in the subspace $\mathcal{H}_\nu$ corresponding to any macro state ν, most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ will thermalize. And for this conclusion, a weaker condition than ETH for *all* eigenbases suffices, as our Theorem 1 shows: that the eigenbasis of H is a purely random eigenbasis of the highly degenerate H_0, of which at least *one* eigenbasis satisfies the ETH (2). For practical purposes again, an arbitrarily small generic perturbation can be added to a Hamiltonian H_0, yielding a non-degenerate H whose eigenbasis is indeed arbitrarily close to a purely random eigenbasis of H_0. We suspect that such an H will in general not itself satisfy the ETH (see Section 3.3), but we are not able to prove this. As a concrete example, these general considerations apply to the free Fermi gas on a d-dimensional lattice with $d > 1$.

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B.2. Long-Time Behavior of Typical Pure States from Thermal Equilibrium Ensembles

Long-Time Behavior of Typical Pure States from Thermal Equilibrium Ensembles

Cornelia Vogel*

Abstract

We consider an isolated macroscopic quantum system in a pure state ψ_t evolving unitarily in a separable Hilbert space $\mathcal{H}$ and take for granted that different macro states ν correspond to mutually orthogonal subspaces $\mathcal{H}_\nu \subset \mathcal{H}$. Let P_ν be the projection to $\mathcal{H}_\nu$. "Normal typicality" is the statement (true for some Hamiltonians) that for all initial states $\psi_0 \in \mathcal{H}$ and most $t \geq 0$, $\|P_\nu \psi_t\|^2$ is close to d_ν/D, where $d_\nu = \dim \mathcal{H}_\nu$ and $D = \dim \mathcal{H} < \infty$. The statement becomes valid for all Hamiltonians if "all $\psi_0 \in \mathcal{H}$" is replaced by "most $\psi_0 \in \mathcal{H}_\mu$" (most w.r.t. the uniform distribution on the sphere $\mathbb{S}(\mathcal{H}_\mu)$) with an arbitrary macro state μ and d_ν/D by a t- and ψ_0-independent quantity $M_{\mu\nu}$ [S. Teufel, R. Tumulka, C. Vogel, J. Stat. Phys., 190, 63 (2023)]. In the present work, we generalize this result from the uniform distribution to a much more general class of measures, so-called GAP measures. For any density matrix ρ on $\mathcal{H}$, GAP(ρ) is the most spread out distribution on $\mathbb{S}(\mathcal{H})$ with density matrix ρ. If ρ is a canonical density matrix, GAP(ρ) is a quantum analog of the canonical ensemble. We show that also for GAP(ρ)-most $\psi_0 \in \mathcal{H}$, the superposition weight $\|P_\nu \psi_t\|^2$ is close to a fixed value $M_{\rho P_\nu}$ for most $t \geq 0$. Moreover, we prove a similar result for certain bounded operators B instead of P_ν and for finite times. The main ingredient of the proof is an improvement of bounds on the GAP(ρ)-variance of $\langle\psi|B|\psi\rangle$ which were first obtained by Reimann [P. Reimann, J. Stat. Phys., 132, 921 (2008)].

Key words: von Neumann's quantum ergodic theorem; equilibration; thermalization; Gaussian adjusted projected (GAP) measure; Scrooge measure; random wave function; quantum statistical mechanics; macroscopic quantum systems; long-time behavior

1 Introduction

We consider an closed macroscopic quantum system in a pure state and investigate its long-time behavior. This approach in order to study equilibration and thermaliza-

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tion has been very fruitful over the last three decades; important discoveries include *dynamical typicality* [2, 26, 1, 32, 31, 33, 42], *canonical typicality* [24, 8, 9, 27, 17, 43], the *eigenstate thermalization hypothesis* (ETH) from a physicist's as well as a mathematician's point of view [6, 38, 5, 4, 7], and, only very recently, the rigorous study of thermalization of a system of free fermions [35, 41, 40, 34].

Another important result was the rediscovery and further elaboration of *normal typicality* [16, 18, 30] which goes back to von Neumann [46]. The goal of the present work is to generalize normal typicality from the uniform distribution on the sphere to so-called *GAP measures*. To any density matrix ρ on $\mathcal{H}$ we can associate a measure $\mathrm{GAP}(\rho)$ which is, roughly speaking, the most spread-out distribution over $\mathbb{S}(\mathcal{H})$ with density matrix ρ. For certain density matrices these measures arise as thermal distributions of quantum states and they can be regarded as a quantum analog of the canonical ensemble from classical statistical mechanics [21, 20].

We start with recalling the notion of normal typicality and first introduce the setting: Following von Neumann [46], we take for granted that the system's Hilbert space $\mathcal{H}$ can be decomposed into mutually orthogonal subspaces $\mathcal{H}_\nu$ ("macro spaces") associated with different macro states ν such that

$$\mathcal{H} = \bigoplus_\nu \mathcal{H}_\nu. \tag{1}$$

Von Neumann thought of the macro spaces $\mathcal{H}_\nu$ as being the joint eigenspaces of a set of mutually commuting self-adjoint and highly degenerate operators $M_1, \dots, M_K$ ("macroscopic observables"). Each macro space is then characterized by a list of eigenvalues, $\nu = (m_1, \dots, m_K)$ of the operators $M_1, \dots, M_K$ and the elements from a macro space $\mathcal{H}_\nu$ look "macroscopically the same" in the sense that every $\psi \in \mathcal{H}_\nu$ is an eigenfunction of all M_j with corresponding eigenvalue m_j. Usually, a coarse-grained version of the system's Hamiltonian H is among the M_j and therefore each $\mathcal{H}_\nu$ is a subset of an *energy shell* $\mathcal{H}_{\mathrm{mc}} = \mathbb{1}_{[E-\Delta E,E]}(H)\mathcal{H}$. Here, $[E - \Delta E, E]$ is a micro-canonical energy interval with ΔE (the resolution of macroscopic energy measurements) being small on the macroscopic scale but sufficiently large on the microscopic scale such that $[E - \Delta E, E]$ contains a very large (but typically finite) number of eigenvalues of H. This implies in particular that even if we allow $\mathcal{H}$ to be infinite-dimensional, the $\mathcal{H}_\nu$ should be of finite dimension. Moreover, in each energy shell there usually is one macro space that contains most dimensions of the shell and which we associate with thermal equilibrium and denote by $\mathcal{H}_{\mathrm{eq}}$.

Let P_ν be the orthogonal projection to $\mathcal{H}_\nu$ and let $\mathbb{S}(\mathcal{H}) = \{\psi \in \mathcal{H} : \|\psi\| = 1\}$ denote the sphere in $\mathcal{H}$. If $D := \dim \mathcal{H} < \infty$, for most $\phi \in \mathbb{S}(\mathcal{H})$ (where "most" refers to the uniform distribution on $\mathbb{S}(\mathcal{H})$) we have that

$$\|P_\nu \phi\|^2 \approx \frac{d_\nu}{D} \quad \forall \nu, \tag{2}$$

where $d_\nu := \dim \mathcal{H}_\nu$ provided that d_ν and D are sufficiently large [16]. If the eigenbasis of H is chosen purely randomly (i.e., according to the Haar measure) among all orthonormal bases (and under some further technical not very restrictive assumptions), every initial wave function $\psi_0 \in \mathbb{S}(\mathcal{H})$ evolves such that for most times $t \geq 0$,

$$\|P_\nu \psi_t\|^2 \approx \frac{d_\nu}{D}, \tag{3}$$

where $\psi_t = \exp(-iHt)\psi_0$ (we set $\hbar = 1$), provided that d_ν and D are sufficiently large [16, 18, 30, 46]. This phenomenon is known as "normal typicality". In particular, if the macro state ν represents thermal equilibrium, we write $d_\nu = d_{\mathrm{eq}}$ and have that for every $\psi_0 \in \mathbb{S}(\mathcal{H})$,

$$\|P_{\mathrm{eq}} \psi_t\|^2 \approx \frac{d_{\mathrm{eq}}}{D} \approx 1 \tag{4}$$

for most $t \geq 0$, i.e., every state spends most of the time in thermal equilibrium. Therefore we have thermalization in the sense that also non-equilibrium initial states sooner or later reach the thermal equilibrium macro space and stay there for most of the time. This sense of thermal equilibrium is often called "macroscopic thermal equilibrium" (MATE), see, e.g., [13, 14, 15, 39, 25, 19, 23]. Note that here it is important that the statement holds true for *every* initial wave function; if it was only true for "most" ψ_0, the statement would not tell us much about thermalization because "most" states in $\mathbb{S}(\mathcal{H})$ are in thermal equilibrium anyway.

Von Neumann's assumptions on the Hamiltonian are not physically realistic because the energy eigenbasis of H is unrelated to the decomposition of $\mathcal{H}$ into the macro spaces and therefore the system goes very rapidly from any (possibly very non-equilibrium) macro space almost immediately to the thermal equilibrium macro space $\mathcal{H}_{\mathrm{eq}}$ [10, 11, 12]. However, if the system starts in a very non-equilibrium state, it should pass through larger and larger (and therefore less far away from thermal equilibrium) macro spaces until it finally reaches $\mathcal{H}_{\mathrm{eq}}$. In [42], the notion of generalized normal typicality was therefore generalized in the following way: for a general Hamiltonian H, for most $\psi_0 \in \mathbb{S}(\mathcal{H}_\mu)$ (where "most" again refers to the uniform distribution over $\mathbb{S}(\mathcal{H}_\mu)$) with a possibly non-equilibrium macro state μ and most times $t \geq 0$,

$$\|P_\nu \psi_t\|^2 \approx M_{\mu\nu} \quad \forall \nu \tag{5}$$

for suitable values $M_{\mu\nu}$ provided that the eigenvalues and eigenvalue gaps of H are not too highly degenerate. We remark that the $M_{\mu\nu}$ depend also an μ but we still expect that in relevant cases, $M_{\mu\nu} \approx d_\nu/D$. While (2) holds in the sense that the absolute as well as the relative errors are small, (5) was in [44] only proved in the sense that the absolute error is small. In order to show that the relative errors are small as well, we need a lower bound on the quantities $M_{\mu\nu}$. In [44] such a lower bound was obtained in the case that the Hamiltonian is of the form $H = H_0 + V$ where H_0 is a deterministic Hermitian matrix and V is a Hermitian random Gaussian perturbation by making use of results from random matrix theory. We remark that (5) can also be shown for arbitrary bounded operators B, i.e., with $\|P_\nu \psi_t\|^2$ replaced by $\langle \psi_t | B | \psi_t \rangle$ and $M_{\mu\nu}$ replaced by a suitable quantity $M_{\mu B}$. Moreover, a finite time result is available, however, the times required for a small error are extremely large, see, e.g., [42] for a detailed discussion.

In the present work we generalize (5) in the sense that we replace the notion of "most" which there refers to the uniform distribution over the sphere $\mathbb{S}(\mathcal{H}_\mu)$ of some macro space $\mathcal{H}_\mu$ with some other measure on $\mathbb{S}(\mathcal{H})$, namely with *GAP measures*. The acronym GAP stands for *Gaussian adjusted projected* measure [22, 21] and it refers to one possible way of how these measures can be constructed.[1] As mentioned above, to any density matrix ρ on $\mathcal{H}$ we can associate a measure GAP(ρ) on $\mathbb{S}(\mathcal{H})$, see Section 2 for a mathematically precise definition and construction of GAP(ρ). We remark already here that GAP(ρ) can, in contrast to the uniform distribution, also be defined on separable Hilbert spaces, i.e., the Hilbert space does not have to be of finite dimension. If $\rho = \rho_{\mathrm{can}}$, i.e., if ρ is of the form

$$\rho = \rho_{\mathrm{can}} = \frac{1}{Z} e^{-\beta H}, \tag{6}$$

where β is the inverse temperature and Z a normalization constant, GAP(ρ) arises as the thermal equilibrium distribution of the wave function, see [21, 20] for details.

Our main result in the present paper is that also for most $\psi_0 \in \mathbb{S}(\mathcal{H})$, where "most" now refers to GAP(ρ), for most $t \geq 0$,

$$\langle \psi_t | B | \psi_t \rangle \approx M_{\rho B} \tag{7}$$

for suitable quantities $M_{\rho B}$ provided that $\|\rho\|$ is small, $\|B\|$ is not too large and the eigenvalues and eigenvalue gaps of H are not too highly degenerate. Here, B is any operator if $\mathcal{H}$ is finite-dimensional. If $\dim \mathcal{H} = \infty$, we have to restrict the class of operators B, roughly speaking, to operators which act non-trivially only on finitely many eigenspaces of H and whose image is contained in the span of only finitely many eigenspaces. As discussed above, the projections P_ν to the macro spaces should fulfill this assumption as the macro spaces are usually contained in an energy shell which typically is finite-dimensional. Note that here we restrict our considerations to the absolute errors; however, if $B = P_\nu$ or $\rho = P_\nu / d_\nu$ for some macro state ν, the bounds from [44] apply. Our result shows that for GAP(ρ)-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ the curve $t \mapsto \langle \psi_t | B | \psi_t \rangle$ is nearly constant in the long run $t \to \infty$. We remark that for

[1] GAP measures we first introduced by Jozsa, Robb and Wootters [22] in 1994 in an information theoretical context. More precisely, they showed that GAP(ρ) minimizes the so-called "accessible information" under the constraint that its density matrix is given by ρ. They named it *Scrooge measure*, referring to Ebenezer Scrooge, the protagonist of Charles Dickens' novella *A Christmas Carol* (1843) who is a very stingy character. The authors chose this name as the GAP measure is "particularly stingy with its information".

GAP(ρ)-most $\psi_0 \in \mathbb{S}(\mathcal{H})$, the curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ is nearly deterministic ("dynamical typicality"); this has been shown in [43]. The generalization of (generalized) normal typicality to GAP(ρ) reveals that it is not a phenomenon specific for the uniform distribution but it is also valid for a broader class of natural and physically relevant distributions which can also be defined in infinite-dimensional Hilbert spaces. As some GAP measures are a quantum analog of the canonical ensemble whereas the uniform distribution on an energy shell $\mathcal{H}_{\mathrm{mc}}$ is an analog micro-canonical ensemble, our results can be regarded as expressing a version of equivalence of ensembles.

The remainder of this paper is organized as follows: In Section 2, we give some background and present the mathematical setup. In Section 3, we formulate and discuss our main result. In Section 4, we provide the proofs. Finally, in Section 5, we conclude.

2 Background and Mathematical Setup

In the following, we make precise some notions that appeared in the introduction, give a mathematically rigorous definition of the GAP measures and introduce the system's Hamiltonian and its relevant quantities that we will need to state our main result in Section 3. Throughout this paper we assume that $\mathcal{H}$ is a separable Hilbert space, i.e., that it has either a finite or countably infinite orthonormal basis.

Measures of "most". Let $\mathbb{P}$ be a probability measure on $\mathbb{S}(\mathcal{H})$ and let $\varepsilon > 0$. We say that, w.r.t. $\mathbb{P}$, a statement $s(\psi)$ is true for $(1-\varepsilon)$-most $\psi \in \mathbb{S}(\mathcal{H})$ if

$$\mathbb{P}\left\{\psi \in \mathbb{S}(\mathcal{H}) : s(\psi) \text{ holds}\right\} \geq 1-\varepsilon. \tag{8}$$

Let $T, \delta > 0$. Analogously we say that a statement $S(t)$ is true for $(1-\delta)$-most $t \in [0,T]$ if

$$\frac{1}{T}\,\lambda\left\{t \in [0,T] : S(t) \text{ holds}\right\} \geq 1-\delta, \tag{9}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$. Moreover, we say that a statement $S(t)$ is true for $(1-\delta)$-most $t \in [0,\infty)$ (or also $t \geq 0$) if

$$\liminf_{T\to\infty} \frac{1}{T}\,\lambda\left\{t \in [0,T] : S(t) \text{ holds}\right\} \geq 1-\delta. \tag{10}$$

Time averages. Let $T > 0$ and let $f : \mathbb{R}_+ \to \mathbb{C}$. We define the finite time average $\langle\,\cdot\,\rangle_T$ of f over the interval $[0,T]$ by

$$\langle f(t)\rangle_T := \frac{1}{T}\int_0^T f(t)\,dt. \tag{11}$$

Moreover, the infinite time average of f is given by

$$\overline{f(t)} := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt \tag{12}$$

whenever this limit exists.

Norms. Throughout this paper we need two norms, the operator norm and the trace norm. Let M be an operator on $\mathcal{H}$. Its *operator norm* is given by

$$\|M\| := \sup_{\|\psi\|=1} \|M\psi\|. \tag{13}$$

If M is self-adjoint, $\|M\|$ is equal to the largest absolute eigenvalue of M. In general, $\|M\|$ is equal to the square root of the largest eigenvalue of M^*M where M^* is the adjoint operator of M.

The *trace norm* of M is defined as

$$\|M\|_{\mathrm{tr}} := \mathrm{tr}\,|M| = \mathrm{tr}\,\sqrt{M^*M}. \tag{14}$$

If M is self-adjoint, $\|M\|_{\mathrm{tr}}$ is equal to the sum of the absolute eigenvalues of M.

Density matrix. Let $\mathbb{P}$ be a probability measure on $\mathbb{S}(\mathcal{H})$. From the measure $\mathbb{P}$ we can obtain a density matrix $\rho_{\mathbb{P}}$ on $\mathcal{H}$ by

$$\rho_{\mathbb{P}} := \int_{\mathbb{S}(\mathcal{H})} \mathbb{P}(d\psi)\, |\psi\rangle\langle\psi| \tag{15}$$

and we say that the measure $\mathbb{P}$ has density matrix $\rho_{\mathbb{P}}$. Note that this density matrix always exists, see, e.g., Lemma 1 in [45]. If the mean of $\mathbb{P}$ is equal to zero, $\rho_{\mathbb{P}}$ is the covariance matrix of $\mathbb{P}$.

GAP measure. In this paragraph, we give a mathematically rigorous definition of the measure $\mathrm{GAP}(\rho)$ on $\mathbb{S}(\mathcal{H})$ for a density matrix ρ on $\mathcal{H}$. There are several equivalent definitions of $\mathrm{GAP}(\rho)$ in the case that $\mathcal{H}$ is finite-dimensional, see [20]. In the following, we present the one which gave the GAP measures their name as this construction can be done in a similar way if $\mathcal{H}$ is infinite-dimensional. For simplicity, we restrict ourselves to the finite-dimensional case and refer to [45] for the case that $\dim \mathcal{H} = \infty$.

Let ρ be a density matrix on a finite-dimensional Hilbert space $\mathcal{H}$. Then we can write it as

$$\rho = \sum_n p_n |n\rangle\langle n|, \tag{16}$$

where the $|n\rangle$ form an orthonormal eigenbasis of ρ with corresponding eigenvalues p_n.

As the name *Gaussian adjusted projected measure* suggests, we start from a Gaussian measure $G(\rho)$ on $\mathcal{H}$; it is constructed as follows: Let (Z_n) be a sequence of independent complex-valued centered Gaussian random variables[2] with variances

$$\mathbb{E}|Z_n|^2 = p_n. \tag{17}$$

The Gaussian measure $G(\rho)$ is then defined as the distribution of the random vector

$$\Psi^G := \sum_n Z_n|n\rangle. \tag{18}$$

While $G(\rho)$ is centered and has density matrix ρ, it is not a distribution on the sphere $\mathbb{S}(\mathcal{H})$: It follows immediately from the definition of Ψ^G together with the fact that the p_n sum up to 1 that $\mathbb{E}\|\Psi^G\|^2 = 1$ but in general $\|\Psi^G\| \neq 1$.

In the next step, we have to adjust the measure $G(\rho)$; this adjustment is necessary because if we would project $G(\rho)$ directly to the sphere $\mathbb{S}(\mathcal{H})$, the resulting measure would not have the property that it has density matrix ρ. We define the *adjusted Gaussian measure* $\mathrm{GA}(\rho)$ on $\mathcal{H}$ by

$$\mathrm{GA}(\rho)(d\psi) := \|\psi\|^2 G(\rho)(d\psi), \tag{19}$$

i.e., we multiply the density of $G(\rho)$ by the factor $\|\psi\|^2$. It follows from $\mathbb{E}\|\Psi^G\|^2 = 1$ that also $\mathrm{GA}(\rho)$ is a probability measure on $\mathcal{H}$.

In the last step of the construction, we project the measure $\mathrm{GA}(\rho)$ on $\mathcal{H}$ to the sphere $\mathbb{S}(\mathcal{H})$. To this end, let Ψ^{GA} be a $\mathrm{GA}(\rho)$-distributed random vector. Then we define $\mathrm{GAP}(\rho)$ to be the distribution of the random vector

$$\Psi^{\mathrm{GAP}} := \frac{\Psi^{\mathrm{GA}}}{\|\Psi^{\mathrm{GA}}\|}. \tag{20}$$

Obviously, this measure is a distribution on $\mathbb{S}(\mathcal{H})$ and a short computation shows that $\mathrm{GAP}(\rho)$ indeed has density matrix ρ:

$$\begin{aligned}\rho_{\mathrm{GAP}(\rho)} &= \int_{\mathbb{S}(\mathcal{H})} \mathrm{GAP}(\rho)(d\psi)|\psi\rangle\langle\psi| = \int_{\mathcal{H}} \mathrm{GA}(\rho)(d\psi)\frac{1}{\|\psi\|^2}|\psi\rangle\langle\psi| \\ &= \int_{\mathcal{H}} G(\rho)(d\psi)|\psi\rangle\langle\psi| = \rho. \end{aligned}\tag{21}$$

[2]Recall that a centered complex-valued Gaussian random variable Z with variance $\sigma^2 > 0$ has independent real and imaginary part $\mathrm{Re}\, Z$ and $\mathrm{Im}\, Z$, $\mathbb{E}\mathrm{Re}\, Z = \mathbb{E}\mathrm{Im}\, Z = 0$ and $\mathbb{E}(\mathrm{Re} Z)^2 = \mathbb{E}(\mathrm{Im}\, Z)^2 = \sigma^2/2$.

The Hamiltonian. In this paper we consider Hamiltonians H with spectral decomposition

$$H = \sum_{e\in\mathcal{E}} e\,\Pi_e, \tag{22}$$

where $\mathcal{E}$ denotes the set of distinct eigenvalues of H and Π_e is the projection onto the eigenspace of H with corresponding eigenvalue e. We define $d_E := |\mathcal{E}|$, where $|\{\cdot\}|$ denotes the number of elements of the set $\{\cdot\}$, and $D_E := \max_{e\in\mathcal{E}} \operatorname{tr}(\Pi_e)$, i.e., d_E is the number of distinct eigenvalues of H and D_E is the maximum degeneracy of an eigenvalue of H. Moreover, we define the maximal gap degeneracy D_G by

$$D_G := \max_{E\in\mathbb{R}} |\{(e,e') \in \mathcal{E}\times\mathcal{E} : e \neq e' \text{ and } e-e' = E\}|\,. \tag{23}$$

Let $\kappa > 0$. The maximal number of gaps in an energy interval of length κ is given by

$$G(\kappa) := \max_{E\in\mathbb{R}} |\{(e,e') \in \mathcal{E}\times\mathcal{E} : e \neq e' \text{ and } e-e' \in [E, E+\kappa)\}| \tag{24}$$

and obviously $D_G = \lim_{\kappa\to 0^+} G(\kappa)$.

If $\mathcal{H}$ is infinite-dimensional, these quantities are not necessarily finite. In particular, d_E and D_E cannot both be finite. Let B be an operator on $\mathcal{H}$. We will see in the proofs that only those eigenvalues e, e' with $\Pi_e B \Pi_{e'} \neq 0$ contribute to the quantities we want to compute and we will require that there are only finitely many such e, e' which will imply that the sums over e, e' are effectively finite. We therefore define for an operator B the set of "contributing" eigenvalues by

$$\mathcal{E}_B := \{e \in \mathcal{E} : \exists e' \in \mathcal{E} \text{ such that } \Pi_e B \Pi_{e'} \neq 0 \text{ or } \Pi_{e'} B \Pi_e \neq 0\}. \tag{25}$$

Moreover, we also define the quantities d_E, D_E, D_G and $G(\kappa)$ relative to B:

$$d_{E,B} := |\mathcal{E}_B|, \tag{26}$$

$$D_{E,B} := \max_{e\in\mathcal{E}_B} \operatorname{tr}(\Pi_e), \tag{27}$$

$$D_{G,B} := \max_{E\in\mathbb{R}} |\{(e,e') \in \mathcal{E}_B\times\mathcal{E}_B : e \neq e' \text{ and } e-e' = E\}|\,, \tag{28}$$

$$G_B(\kappa) := \max_{E\in\mathbb{R}} |\{(e,e') \in \mathcal{E}_B\times\mathcal{E}_B : e \neq e' \text{ and } e-e' \in [E, E+\kappa)\}|\,. \tag{29}$$

3 Main Result

After having introduced the relevant quantities and notations in the previous section, we are now ready to state our main result.

Theorem 1 (Normal Typicality for GAP(ρ))**.** *Let B be an operator with $d_{E,B} < \infty$ and let ρ be a density matrix with $\|\rho\| < 1/4$ on a separable Hilbert space $\mathcal{H}$. Let $\dim \mathcal{H} \geq 4$, let $\varepsilon, \delta, \kappa, T > 0$ and define*

$$M_{\rho B} := \sum_{e \in \mathcal{E}} \operatorname{tr}(\rho \Pi_e B \Pi_e). \tag{30}$$

Then w.r.t. GAP(ρ), *$(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for $(1-\delta)$-most $t \in [0, T]$*

$$\Big| \langle \psi_t | B | \psi_t \rangle - M_{\rho B} \Big| \leq \left(\frac{188}{\varepsilon \delta} \|B\|^2 \|\rho\| D_{E,B} G_B(\kappa) \left(1 + \frac{8 \log_2 d_{E,B}}{\kappa T} \right) \right)^{1/2} \tag{31}$$

Moreover, w.r.t. GAP(ρ), *$(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for $(1-\delta)$-most $t \in [0, \infty)$,*

$$\Big| \langle \psi_t | B | \psi_t \rangle - M_{\rho B} \Big| \leq \left(\frac{188}{\varepsilon \delta} \|B\|^2 \|\rho\| D_{E,B} D_{G,B} \right)^{1/2}. \tag{32}$$

Thus, as soon as $\|\rho\| \ll D_{E,B} D_{G,B} \|B\|^2$, i.e., as soon as the largest eigenvalue of ρ is small, no eigenvalue and no gap of H is hugely degenerate and $\|B\|$ is not too large, for most initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$, where "most" refers to GAP(ρ), the expectation value $\langle \psi_t | B | \psi_t \rangle$ of B in the state ψ_t is close to the fixed value $M_{\rho B}$ for most times $t \geq 0$. Put another way, the curve $t \mapsto \langle \psi_t | B | \psi_t \rangle$ is nearly constant in the long run $t \to \infty$. The finite-time result shows that as soon as $\|\rho\| \ll D_{E,B} G_B(\kappa) \|B\|^2$ and T is large enough, for most (w.r.t. GAP(ρ)) initial states $\psi_0 \in \mathbb{S}(\mathcal{H})$ the expectation $\langle \psi_t | B | \psi_t \rangle$ is close to $M_{\rho B}$ for most times $t \in [0, T]$. Unfortunately, the times T required to make the upper bound in (31) small are usually extremely large: for example for a system of N particles we need that $T \gg \exp(N)$, see, e.g., the discussion in [42] after Theorem 4.

We remark that for $\rho = P_\mu / d_\mu$ for some macro state μ, GAP(ρ) becomes the uniform distribution over the sphere $\mathbb{S}(\mathcal{H}_\mu)$, we have $\|\rho\| = 1/d_\mu$ and in this case we basically recover the results from [42]. Note that in Theorem 1 for any B only the eigenvalues in $\mathcal{E}_B$ and gaps in $\mathcal{E}_B \times \mathcal{E}_B$ contribute to the quantities $D_{E,B}, d_{E,B}, D_{G,B}$ and $G_B(\kappa)$. However, if $\mathcal{H}$ is finite-dimensional, these quantities can obviously be replaced by D_E, d_E, D_G and $G(\kappa)$. The technical assumption on B that $|\mathcal{E}_B| < \infty$ is needed to ensure that the sums over e appearing in the proofs are finite and it therefore is for example unproblematic to interchange the sums over e with the time average.

Motivated by the phenomenon of normal typicality [16, 18, 30, 46], in [42] the case that $B = P_\nu$ for some macro state ν was of particular interest. As discussed in the introduction, the P_ν usually project to macro spaces $\mathcal{H}_\nu$ which are contained in a finite-dimensional energy shell. Thus the P_ν fulfill the assumption of Theorem 1 that $|\mathcal{E}_{P_\nu}| < \infty$. We therefore see that not only with respect to the uniform distribution over some subspace $\mathcal{H}_\mu$ but also with respect to GAP(ρ), for most initial wave functions ψ_0 the superposition weights $\|P_\nu\psi_t\|^2$ are, in the long run, close to some fixed value. In the case of normal typicality, this value was given by d_ν/D, where $D = \dim\mathcal{H}$. Here, if $D < \infty$, we also expect that $M_{\rho P_\nu}$ is usually close to d_ν/D; we can argue that this should be the case for example if H is non-degenerate and satisfies the eigenstate thermalization hypothesis in the sense that for every eigenstate $|m\rangle$ of H we have that $\langle m|P_\nu|m\rangle \approx \langle P_\nu\rangle_{\mathrm{mic}} = \mathrm{tr}(P_\nu)/D = d_\nu/D$ where $\langle\,\cdot\,\rangle_{\mathrm{mic}}$ denotes the micro-canonical expectation. Then we find

$$M_{\rho P_\nu} = \sum_m \langle m|\rho|m\rangle\langle m|P_\nu|m\rangle \approx \frac{d_\nu}{D}\sum_m \langle m|\rho|m\rangle = \frac{d_\nu}{D}. \tag{33}$$

If $\dim\mathcal{H} = \infty$ (and there are infinitely many macro states ν), the $\|P_\nu\psi_t\|^2$ form a null sequence for every ψ_0 and t.

We now give a brief outline of the proof of Theorem 1. An important ingredient for the proof is an upper bound for the variance of $\langle\psi|A|\psi\rangle$ where A is a bounded operator on $\mathcal{H}$ and $\psi \sim \mathrm{GAP}(\rho)$. Reimann [29] proved such an upper bound under the assumption that A is self-adjoint and $\mathcal{H}$ is finite-dimensional. In [43] the proof was generalized to arbitrary bounded operators A on a separable Hilbert space $\mathcal{H}$. Unfortunately, the bounds obtained there are not sufficiently sharp for our purpose as it would lead in Theorem 1 to an upper bound of the order

$$\mathrm{tr}\,\rho^2 \sum_{e\neq e'} \|\Pi_e B \Pi_{e'}\|^2 \leq \|\rho\| \sum_{e,e'} \mathrm{tr}(\Pi_{e'} B^* \Pi_e B \Pi_{e'}) = \|\rho\|\,\mathrm{tr}(B^*B) \tag{34}$$

and for example in the special case that $\rho = P_\mu/d_\mu$ and $B = P_\nu$ this would yield an upper bound of the order d_ν/d_μ which is not necessarily small (here we ignored the possible degeneracy of the eigenvalues and gaps of H to simplify the discussion). Therefore, using similar techniques as in [29, 43], we prove in Lemma 1 a slightly improved upper bound for the variance that allows us to obtain a bound of the order $\|B\|\,\mathrm{tr}\,\rho^2$ in Theorem 1. After having shown this improved upper bound for the variance of $\langle\psi|A|\psi\rangle$, we proceed to compute upper bounds for the expectation w.r.t. $\mathrm{GAP}(\rho)$ of the infinite time variance of $\langle\psi_t|B|\psi_t\rangle$, a related quantity where the outer infinite time average is replaced by a finite one and the variance w.r.t. $\mathrm{GAP}(\rho)$ of the infinite time average $\overline{\langle\psi_t|B|\psi_t\rangle}$ in Proposition 1 making use of the bound for the variance of $\langle\psi|A|\psi\rangle$. In these computations, many traces and products and sums over traces appear and we use similar methods as in [42] to bound them. Finally, with the help of these bounds and Markov's and Chebyshev's inequality we are able to prove Theorem 1.

Remark 1 (Applying Lévy's Lemma)**.** Another strategy of proof of Theorem 1 is to make use of Lévy's Lemma for GAP measures, a concentration-of-measure-type result for Lipschitz continuous functions on the sphere. We quote the result from [43]:

Theorem 2 (Lévy's Lemma for GAP measures [43])**.** *Let $\mathcal{H}$ be a separable Hilbert space, let $f:\mathbb{S}(\mathcal{H})\to\mathbb{C}$ be a Lipschitz continuous function with Lipschitz constant η, let ρ be a density matrix on $\mathcal{H}$ and let $\varepsilon\geq 0$. Then,*

$$\mathrm{GAP}(\rho)\Big\{\psi\in\mathbb{S}(\mathcal{H}):\big|f(\psi)-\mathbb{E}_\rho(f)\big|>\varepsilon\Big\}\leq 12\exp\left(-\frac{C\varepsilon^2}{2\eta^2\|\rho\|}\right), \tag{35}$$

where $\frac{1}{288\pi^2}$.

In fact, one can show that $f(\psi_0):=\langle|\langle\psi_t|B|\psi_t\rangle-M_{\rho B}|^2\rangle_T$ is Lipschitz continuous with Lipschitz constant bounded by $8\|B\|^2$ (and the same holds true for the infinite time average as well). However, as it is also explained in [42], in such a situation Lévy's Lemma does in general not lead to better results than applying Markov's and Chebychev's inequality. Markov's inequality gives

$$f(\psi_0)\leq\frac{\mathbb{E}_\rho f}{\varepsilon} \tag{36}$$

for $(1-\varepsilon)$-most $\psi_0\in\mathbb{S}(\mathcal{H})$ while it follows from Theorem 2 that

$$f(\psi_0)\leq\mathbb{E}_\rho f+\sqrt{\frac{2\eta^2\|\rho\|\log(12/\varepsilon)}{C}}. \tag{37}$$

The expectation $\mathbb{E}_\rho f$ is of the order $\|\rho\|$ while the square root in (37) is of the order $\|\rho\|^{1/2}$, so while the ε-dependence in the result from Lévy's Lemma is better, the dependence on $\|\rho\|$ is better for Markov's inequality. Often, $\|\rho\|$ is extremely small, for example for $\rho=\mathbb{1}_\mathcal{H}/D$ where D is the dimension of the (finite-dimensional) Hilbert space and $\mathbb{1}_\mathcal{H}$ the identity on $\mathcal{H}$, we have that $\|\rho\|=1/D$. In such cases, the better ε-dependence does not compensate the worse $\|\rho\|$-dependence unless ε is extremely small and it is of little interest to consider, e.g., ε smaller than 10^{-200} [3].

Another possibility to obtain bounds on $|\langle\psi_t|B|\psi_t\rangle-M_{\rho B}|$ with a better ε-dependence is to first apply Lévy's Lemma for GAP measures to the function $f(\psi)=\langle\psi_t|B|\psi_t\rangle$ for fixed $t\geq 0$. Note that by Lemma 5 in [28], f is Lipschitz continuous with Lipschitz constant bounded by $2\|B\|$. Then it follows from Theorem 2 that for every $t\geq 0$, w.r.t. $\mathrm{GAP}(\rho)$, $(1-\varepsilon)$-most $\psi_0\in\mathbb{S}(\mathcal{H})$ are such that

$$\left|\langle\psi_t|B|\psi_t\rangle-\mathrm{tr}\left(e^{iHt}Be^{-iHt}\rho\right)\right|\leq\left(\frac{8\|B\|^2\|\rho\|\log(12/\varepsilon)}{C}\right)^{1/2}. \tag{38}$$

Using similar estimates as in the proof of Proposition 1 to bound $|\,\mathrm{tr}(e^{iHt}Be^{-iHt})-M_{\rho B}|$ for $(1-\delta)$-most $t\in[0,T]$ one can show that $(1-\delta)$-most $t\in[0,T]$ are such that for $(1-\varepsilon)$-most $\psi_0\in\mathbb{S}(\mathcal{H})$,

$$\left|\langle\psi_t|B|\psi_t\rangle-M_{\rho B}\right|\leq 4\left(\frac{\|B\|^2\|\rho\|D_{E,B}G_B(\kappa)\log(12/\varepsilon)}{\delta C}\left(1+\frac{8\log_2 d_{E,B}}{\kappa T}\right)\right)^{1/2}. \tag{39}$$

Note that similar estimates can be obtained for the infinite time interval.

As we are interested in a statement of the form that most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for most times the error between $\langle\psi_t|B|\psi_t\rangle$ and $M_{\rho B}$ is small, we have to interchange "most ψ_0" and "most t". When considering a finite time interval $[0,T]$ (however, not for the infinite time interval $[0,\infty)$[3]) this is possible by Footnote 7 in [16] at the expense of making the parameter ε and δ worse. More precisely, if a statement is true for $(1-\delta)$-most $t \in [0,T]$ for $(1-\varepsilon)$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$, then it is also true for $(1-\varepsilon')$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ for $(1-\delta')$-most $t \in [0,T]$ where $\varepsilon' \geq \frac{\varepsilon+\delta-\varepsilon\delta}{\delta'}$.

We do not apply Footnote 7 in [16] directly to (39) but rather go back to (38) and apply it to this statement. We obtain that for every $\varepsilon, \delta > 0$, w.r.t. GAP(ρ), $(1-\varepsilon/\delta)$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for $(1-\delta/2)$-most $t \in [0,T]$,

$$\left|\langle\psi_t|B|\psi_t\rangle - \operatorname{tr}\left(e^{iHt}Be^{-iHt}\rho\right)\right| \leq \left(\frac{8\|B\|^2\|\rho\|\log(12/\varepsilon)}{C}\right)^{1/2}. \tag{40}$$

Altogether we find that, w.r.t. GAP(ρ), $(1-\varepsilon/\delta)$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for $(1-\delta)$-most $t \in [0,T]$,

$$\left|\langle\psi_t|B|\psi_t\rangle - M_{\rho B}\right| \leq 5\left(\frac{\|B\|^2\|\rho\|D_{E,B}G_B(\kappa)\log(12/\varepsilon)}{\delta C}\left(1+\frac{8\log_2 d_{E,B}}{\kappa T}\right)\right)^{1/2}. \tag{41}$$

[3] While a statement that is true for most $\psi_0 \in \mathbb{S}(\mathcal{H})$ for most $t \in [0,\infty)$ is also true for most $t \in [0,\infty)$ for most $\psi_0 \in \mathbb{S}(\mathcal{H})$ (which can be proved similarly as Footnote 7 in [16] and by making use of Fatou's Lemma), the converse does not hold. In general, the quantifier "most $t \in [0,\infty)$" and a "most"-quantifier referring to a probability measure on a probability space and cannot be interchanged. As a simple example consider the interval $[0,1]$ equipped with the Lebesgue measure and let $N \in \mathbb{N}$ be large. We define $I_1 := [0,1/N)^c, I_2 := [1/N, 2/N)^c, \ldots, I_N := [(N-1)/N, 1]^c$, where the complement is taken in $[0,1]$. We define the set

$$S := \bigcup_{m=0}^{\infty}\bigcup_{n=1}^{N}\left(I_n \times \left[2^{mN+n-1}, 2^{mN+n}\right)\right) \subset [0,1]\times[0,\infty).$$

Clearly, for every $t > 1$ we have that $(\psi,t) \in S$ for $(1-1/N)$-most $\psi \in [0,1]$. Therefore, the statement "$(\psi,t) \in S$" is true for most $t \in [0,\infty)$ for most $\psi \in [0,1]$.

Let $\psi \in [0,1]$. Without loss of generality we assume that $\psi \in [0,1/N)$ which implies that $\psi \notin I_1$ and $\psi \in I_k$ for all $k \geq 2$. We define a sequence of times $(T_l)_{l\geq 0}$ by $T_l := 2^{lN+1}$. Then, by construction, $(\psi,t) \notin S$ for all $t \in [2^{lN}, 2^{lN+1})$ and thus $(\psi,t) \notin S$ for at least half of the times $t \in [0,T_l]$. This implies

$$\limsup_{T\to\infty}\frac{1}{T}\lambda\left\{t \in [0,T] : (\psi,t) \notin S\right\} \geq \lim_{l\to\infty}\frac{1}{T_l}\lambda\left\{t \in [0,T_l] : (\psi,t) \notin S\right\} \geq 1/2.$$

As $\psi \in [0,1]$ was arbitrary, we conclude that the statement "$(\psi,t) \in S$" is false for every $\psi \in [0,1]$ for a substantial amount of times $t \in [0,\infty)$.

Effectively, the $1/\varepsilon$ in Theorem 1 is replaced by $\log(12/(\varepsilon\delta))$ and in relevant cases (i.e., $\varepsilon \ll \delta$) this bound is slightly better than the one from Theorem 1 if δ is sufficiently small. $\diamond$

4 Proofs

This section is devoted to the proof of our main result. As already outlined in Section 3, we first prove an improved upper bound for the variance $\mathrm{Var}_\rho\langle\psi|A|\psi\rangle$ with respect to $\mathrm{GAP}(\rho)$ in Section 4.1. With the help of this bound, we prove in Section 4.2 an upper bound for the expected time variance of $\langle\psi_t|B|\psi_t\rangle$, i.e., for

$$\mathbb{E}_\rho\left(\left\langle\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2\right\rangle_T\right) \tag{42}$$

and also for the infinite time average instead of $\langle\cdot\rangle_T$. Moreover, an application of Lemma 1 also yields an upper bound for $\mathrm{Var}_\rho\,\overline{\langle\psi_t|B|\psi_t\rangle}$. After having proved these upper bounds in Section 4.2, the proof of Theorem 1 which we give in Section 4.3 basically reduces to a careful application of Markov's and Chebyshev's inequality.

4.1 An Improved Upper Bound for $\mathrm{Var}_\rho\langle\psi|A|\psi\rangle$

Lemma 1. *Let ρ be a density matrix on a separable Hilbert space $\mathcal{H}$ with eigenvalues $p_n > 0$ such that $p_{\max} = \|\rho\| < 1/4$ and let $\dim\mathcal{H} \geq 4$. For $\mathrm{GAP}(\rho)$-distributed ψ and any bounded operator $A : \mathcal{H} \to \mathcal{H}$,*

$$\mathbb{E}_\rho\langle\psi|A|\psi\rangle = \mathrm{tr}(A\rho), \tag{43}$$

$$\begin{aligned}\mathrm{Var}_\rho\langle\psi|A|\psi\rangle \leq \frac{1}{1-p_{\max}}\Bigg(&\mathrm{tr}(A\rho A^*\rho) + \frac{\mathrm{tr}(A\rho^2A^*\rho)+\mathrm{tr}(A\rho A^*\rho^2)}{1-2p_{\max}}\\ &+\frac{2}{(1-2p_{\max})(1-3p_{\max})}\Bigg[\mathrm{tr}(A\rho^3A^*\rho)+\mathrm{tr}(A\rho^2A^*\rho^2)+\mathrm{tr}(A\rho A^*\rho^3)\\ &+\sum_{m,n}\Big(|\,\mathrm{tr}(A\rho^3P_m)\,\mathrm{tr}(A^*\rho P_n)|+|\,\mathrm{tr}(A\rho^2P_m)\,\mathrm{tr}(A^*\rho^2P_n)|\\ &+|\,\mathrm{tr}(A\rho P_m)\,\mathrm{tr}(A^*\rho^3P_n)|\Big)\Bigg]\Bigg),\end{aligned} \tag{44}$$

where $\mathbb{E}_\rho$ and Var_ρ denote the expectation and variance with respect to $\mathrm{GAP}(\rho)$, $P_n = |n\rangle\langle n|$ and $\{|n\rangle\}$ is an orthonormal eigenbasis of ρ.

Proof. The formula for the expectation has already been proved in [43, Proposition 1]. For the proof of (44) we adapt the proof for the upper bound on the variance from

[43] which in turn follows closely the proof of Reimann [29] who only considered the case that A is self-adjoint and $\mathcal{H}$ is finite-dimensional. For convenience of the reader we give most of the details here.

We first assume that $D := \dim \mathcal{H} < \infty$. Recall that the variance of a complex-valued random variable Z is defined as

$$\mathrm{Var}_\rho(Z) = \mathbb{E}_\rho(|Z - \mathbb{E}_\rho Z|^2) = \mathbb{E}_\rho|Z|^2 - |\mathbb{E}_\rho Z|^2. \tag{45}$$

Because of the invariance of Var_ρ under the addition of constants, i.e., $\mathrm{Var}_\rho(Z + a) = \mathrm{Var}_\rho(Z)$ for all constants $a \in \mathbb{C}$, we can assume without loss of generality that $\mathbb{E}_\rho\langle\psi|A|\psi\rangle = \mathrm{tr}(A\rho) = 0$. As a consequence, we only have to compute $\mathbb{E}_\rho(\langle\psi|A|\psi\rangle\langle\psi|A|\psi\rangle^*) = \mathbb{E}_\rho(\langle\psi|A|\psi\rangle\langle\psi|A^*|\psi\rangle)$.

Let $\{|n\rangle : n = 1, \ldots, D\}$ be an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of ρ and let $\psi \in \mathbb{S}(\mathcal{H})$. Then,

$$\langle\psi|A|\psi\rangle\langle\psi|A^*|\psi\rangle = \sum_{m,n,m',n'} \langle\psi|m\rangle\langle m|A|n\rangle\langle n|\psi\rangle\langle\psi|n'\rangle\langle n'|A^*|m'\rangle\langle m'|\psi\rangle \tag{46a}$$

$$= \sum_{m,n,m',n'} c_m^* c_n c_{n'}^* c_{m'} A_{mn} A_{m'n'}^*, \tag{46b}$$

where $c_m = \langle m|\psi\rangle$ and $A_{mn} = \langle m|A|n\rangle$.

Reimann [29] computed the fourth moments $\mathbb{E}_\rho(c_m^* c_n c_{n'}^* c_{m'})$ and obtained that they vanish except in the two cases (i) $m = n$ and $m' = n'$, (ii) $m = m'$ and $n = n'$, and that

$$\mathbb{E}_\rho\left(|c_m|^2|c_n|^2\right) = p_m p_n (1 + \delta_{mn}) K_{mn}, \tag{47}$$

where

$$K_{mn} = \int_0^\infty (1 + xp_m)^{-1}(1 + xp_n)^{-1} \prod_{l=1}^{D} (1 + xp_l)^{-1}\, dx. \tag{48}$$

We compute

$$\begin{aligned}\mathbb{E}_\rho\left(\langle\psi|A|\psi\rangle\langle\psi|A^*|\psi\rangle\right) &= \sum_{m,n} p_n p_m (1 + \delta_{mn}) K_{mn} A_{mm} A_{nn}^* \\ &\quad + \sum_{m,n} p_n p_m (1 + \delta_{mn}) K_{mn} |A_{mn}|^2 - 2\sum_n p_n^2 K_{nn} |A_{nn}|^2 \qquad (49a)\\ &= \sum_{m,n} \left[A_{mm} A_{nn}^* + |A_{mn}|^2\right] p_n p_m K_{mn} \qquad (49b)\end{aligned}$$

and as in [29] we write K_{mn} as

$$K_{mn} = K^{(0)} - (p_m + p_n) K^{(1)} + 2\kappa_{mn} \left(p_m^2 + p_m p_n + p_n^2\right) K^{(2)}, \tag{50}$$

where

$$K^{(k)} = \frac{1}{k!}\int_0^\infty x^k \prod_{l=1}^{D}(1+xp_l)^{-1}dx, \quad k = 0,1,2, \tag{51}$$

and $\kappa_{mn} \in [0,1]$. This follows from a Taylor expansion of $g_{mn}(x) := (1+xp_m)^{-1}(1+xp_n)^{-1}$ up to second order.

With (50) we obtain for the first term in (49b) that

$$\begin{aligned}
&\sum_{m,n} A_{mm}A^*_{nn}p_np_m\left(K^{(0)} - (p_m+p_n)K^{(1)} + 2\kappa_{mn}(p_m^2+p_mp_n+p_n^2)K^{(2)}\right) \\
&\qquad = K^{(0)}\left(\sum_m A_{mm}p_m\right)\left(\sum_n A^*_{nn}p_n\right) - K^{(1)}\left[\sum_m A_{mm}p_m^2\sum_n A^*_{nn}p_n\right. \\
&\qquad\quad \left.+\sum_m A_{mm}p_m\sum_n A^*_{nn}p_n^2\right] + 2K^{(2)}\sum_{m,n}A_{mm}A^*_{nn}\kappa_{mn}\left(p_m^3p_n + p_m^2p_n^2 + p_mp_n^3\right) \qquad (52a) \\
&\qquad = 2K^{(2)}\sum_{m,n}A_{mm}A^*_{nn}\kappa_{mn}\left(p_m^3p_n+p_m^2p_n^2+p_mp_n^3\right), \qquad (52b)
\end{aligned}$$

where we used that

$$\begin{aligned}
\sum_m A_{mm}p_m &= \sum_m \langle m|A|m\rangle\langle m|\rho|m\rangle = \sum_{m,m'}\langle m|A|m'\rangle\langle m'|\rho|m\rangle \\
&= \sum_m \langle m|A\rho|m\rangle = \mathrm{tr}(A\rho) = 0,
\end{aligned} \tag{53}$$

and similarly $\sum_n A^*_{nn}p_n = 0$. Therefore an upper bound for the absolute value of the first term in (49b) is given by

$$\begin{aligned}
&2K^{(2)}\sum_{m,n}|A_{mm}A^*_{nn}|(p_m^3p_n+p_m^2p_n^2+p_mp_n^3) \\
&\quad = 2K^{(2)}\sum_{m,n}\big(|\,\mathrm{tr}(A\rho^3P_m)\,\mathrm{tr}(A^*\rho P_n)| + |\,\mathrm{tr}(A\rho^2P_m)\,\mathrm{tr}(A^*\rho^2P_n)| \\
&\qquad + |\,\mathrm{tr}(A\rho P_m)\,\mathrm{tr}(A^*\rho^3P_n)|\big),
\end{aligned} \tag{54}$$

where $P_n = |n\rangle\langle n|$. Here we used for the first term that

$$A_{mm}A^*_{nn}p_m^3p_n = \langle m|A\rho^3|m\rangle\langle n|A^*\rho|n\rangle = \mathrm{tr}(A\rho^3P_m)\,\mathrm{tr}(A^*\rho P_n) \tag{55}$$

and similarly for the other terms.

For the second term in (49b) we find

$$
\begin{aligned}
&K^{(0)} \sum_{m,n} |A_{mn}|^2 p_n p_m - K^{(1)} \sum_{m,n} |A_{mn}|^2 \left(p_n^2 p_m + p_n p_m^2\right) \\
&\quad + 2K^{(2)} \sum_{m,n} |A_{mn}|^2 \kappa_{mn} (p_n p_m^3 + p_m^2 p_n^2 + p_n^3 p_m) \qquad (56a) \\
&\quad = K^{(0)} \operatorname{tr}(A\rho A^* \rho) - K^{(1)} \left(\operatorname{tr}(A\rho^2 A^* \rho) + \operatorname{tr}(A\rho A^* \rho^2)\right) \\
&\qquad + 2K^{(2)} \sum_{m,n} |A_{mn}|^2 \kappa_{mn} (p_n p_m^3 + p_m^2 p_n^2 + p_n^3 p_m), \qquad (56b)
\end{aligned}
$$

where we used that

$$
\sum_{m,n} |A_{nm}|^2 p_n p_m = \sum_{m,n} A_{mn} A^*_{mn} p_n p_m = \sum_{m,n} \langle m|A\rho|n\rangle \langle n|A^*\rho|m\rangle = \operatorname{tr}(A\rho A^*\rho) \qquad (57)
$$

and similarly $\sum_{m,n} A_{mn} A^*_{mn} p_n^2 p_m = \operatorname{tr}(A\rho^2 A^* \rho)$ and $\sum_{m,n} A_{mn} A^*_{mn} p_n p_m^2 = \operatorname{tr}(A\rho A^* \rho^2)$.

An upper bound for (56b) is given by

$$
\begin{aligned}
&K^{(0)} \operatorname{tr}(A\rho A^* \rho) + K^{(1)} (\operatorname{tr}(A\rho^2 A^* \rho) + \operatorname{tr}(A\rho A^* \rho^2)) \\
&\qquad + 2K^{(2)} \sum_{m,n} |A_{mn}|^2 (p_n p_m^3 + p_m^2 p_n^2 + p_n^3 p_m) \qquad (58a) \\
&\quad = K^{(0)} \operatorname{tr}(A\rho A^* \rho) + K^{(1)} (\operatorname{tr}(A\rho^2 A^* \rho) + \operatorname{tr}(A\rho A^* \rho^2)) \\
&\qquad + 2K^{(2)} \left(\operatorname{tr}(A\rho^3 A^* \rho) + \operatorname{tr}(A\rho^2 A^* \rho^2) + \operatorname{tr}(A\rho A^* \rho^3)\right). \qquad (58b)
\end{aligned}
$$

Thus we arrive at

$$
\begin{aligned}
&\operatorname{Var}_\rho \langle\psi|A|\psi\rangle \\
&\quad \leq K^{(0)} \operatorname{tr}(A\rho A^* \rho) + K^{(1)} \left(\operatorname{tr}(A\rho^2 A^* \rho) + \operatorname{tr}(A\rho A^* \rho^2)\right) \\
&\quad + 2K^{(2)} \Big(\operatorname{tr}(A\rho^3 A^* \rho) + \operatorname{tr}(A\rho^2 A^* \rho^2) + \operatorname{tr}(A\rho A^* \rho^3) \\
&\quad + \sum_{m,n} \left(|\operatorname{tr}(A\rho^3 P_m) \operatorname{tr}(A^* \rho P_n)| + |\operatorname{tr}(A\rho^2 P_m) \operatorname{tr}(A^* \rho^2 P_n)| \right. \\
&\quad \left. + |\operatorname{tr}(A\rho P_m) \operatorname{tr}(A^* \rho^3 P_n)| \right) \Big). \qquad (59)
\end{aligned}
$$

In [29] it was shown that

$$
K^{(k)} \leq \prod_{j=1}^{k+1} \frac{1}{1 - j p_{\max}}, \quad k = 0, 1, 2. \qquad (60)
$$

With the help of this inequality we finally obtain

$$\begin{aligned}\mathrm{Var}_\rho\langle\psi|A|\psi\rangle \leq \frac{1}{1-p_{\max}}\Bigg(&\mathrm{tr}(A\rho A^*\rho) + \frac{\mathrm{tr}(A\rho^2A^*\rho)+\mathrm{tr}(A\rho A^*\rho^2)}{1-2p_{\max}}\\ &+\frac{2}{(1-2p_{\max})(1-3p_{\max})}\Bigg[\mathrm{tr}(A\rho^3A^*\rho)+\mathrm{tr}(A\rho^2A^*\rho^2)+\mathrm{tr}(A\rho A^*\rho^3)\\ &+\sum_{m,n}\big(|\,\mathrm{tr}(A\rho^3P_m)\,\mathrm{tr}(A^*\rho P_n)|+|\,\mathrm{tr}(A\rho^2P_m)\,\mathrm{tr}(A^*\rho^2P_n)|\\ &+|\,\mathrm{tr}(A\rho P_m)\,\mathrm{tr}(A^*\rho^3P_n)|\big)\Bigg]\Bigg). \end{aligned} \tag{61}$$

This concludes the proof in the finite-dimensional case.

Now suppose that $\dim\mathcal{H}=\infty$. We proceed similarly as in the proof of Proposition 1 in [43] and for convenience of the reader, we give in the following the full argument. Let $\{|k\rangle : k\in\mathbb{N}\}$ be an orthonormal eigenbasis of ρ with corresponding positive eigenvalues $p_{\max}=\|\rho\|=p_1\geq p_2\geq\ldots$. We approximate ρ in trace norm by finite-rank density matrices ρ_k, $k\in\mathbb{N}$, defined by

$$\rho_k := \sum_{m=1}^{k-1} p_m|m\rangle\langle m| + \left(\sum_{m=k}^{\infty} p_m\right)|k\rangle\langle k|. \tag{62}$$

Because of $\|\rho_k-\rho\|_{\mathrm{tr}}\to 0$ as $k\to\infty$, it follows from Theorem 3 in [45] that $\mathrm{GAP}(\rho_k)\Rightarrow\mathrm{GAP}(\rho)$, i.e., the measures $\mathrm{GAP}(\rho_k)$ converge weakly to the measure $\mathrm{GAP}(\rho)$. Note that for k large enough, $\sum_{m\geq k}p_m<p_1=p_{\max}$ and therefore $p_{\max,k}=p_{\max}$ where $p_{\max,k}$ denotes the largest eigenvalue of ρ_k. We define the functions $f_k, f:\mathbb{S}(\mathcal{H})\to\mathbb{R}$ by

$$f_k(\psi) = |\langle\psi|A|\psi\rangle - \mathrm{tr}(A\rho_k)|^2\,, \tag{63}$$

$$f(\psi) = |\langle\psi|A|\psi\rangle - \mathrm{tr}(A\rho)|^2\,. \tag{64}$$

It follows from

$$|\mathrm{tr}(A\rho_k)-\mathrm{tr}(A\rho)| = |\mathrm{tr}(A(\rho_k-\rho))| \leq \|A\|\,\mathrm{tr}(|\rho_k-\rho|) = \|A\|\|\rho_k-\rho\|_{\mathrm{tr}}\to 0 \tag{65}$$

that $f_k\to f$ uniformly in ψ and this implies $\mathrm{GAP}(\rho_k)(f_k)-\mathrm{GAP}(\rho_k)(f)\to 0$ where we introduced the notation

$$\mathrm{GAP}(\rho)(f) := \int_{\mathbb{S}(\mathcal{H})} f(\psi)\, d\mathrm{GAP}(\rho)(\psi). \tag{66}$$

As the measures $\mathrm{GAP}(\rho_k)$ converge weakly to $\mathrm{GAP}(\rho)$ and f is continuous, we have that $\mathrm{GAP}(\rho_k)(f)\to\mathrm{GAP}(\rho)(f)$ and therefore $\mathrm{Var}_{\rho_k}\langle\psi|A|\psi\rangle = \mathrm{GAP}(\rho_k)(f_k)\to$

$\mathrm{GAP}(\rho)(f) = \mathrm{Var}_\rho\langle\psi|A|\psi\rangle$. By restricting to the subspace spanned by $\{|n\rangle : n = 1,\dots,k\}$ we obtain from the finite-dimensional case that

$$\begin{aligned}\mathrm{Var}_{\rho_k}\langle\psi|A|\psi\rangle \leq{}& \frac{1}{1-p_{\max,k}}\Bigg(\mathrm{tr}(A\rho_k A^*\rho_k) + \frac{\mathrm{tr}(A\rho_k^2A^*\rho_k)+\mathrm{tr}(A\rho_kA^*\rho_k^2)}{1-2p_{\max,k}}\\ &+\frac{2}{(1-2p_{\max,k})(1-3p_{\max,k})}\Big[\mathrm{tr}(A\rho_k^3A^*\rho_k)+\mathrm{tr}(A\rho_k^2A^*\rho_k^2)+\mathrm{tr}(A\rho_kA^*\rho_k^3)\\ &+\sum_{m,n}\big(|\,\mathrm{tr}(A\rho_k^3P_m)\,\mathrm{tr}(A^*\rho_kP_n)|+|\,\mathrm{tr}(A\rho_k^2P_m)\,\mathrm{tr}(A^*\rho_k^2P_n)|\\ &+|\,\mathrm{tr}(A\rho_kP_m)\,\mathrm{tr}(A^*\rho_k^3P_n)|\big)\Big]\Bigg)\,, \end{aligned} \tag{67}$$

where we chose k large enough such that $p_{\max,k} < 1/4$. We have already discussed that $p_{\max,k} = p_{\max}$ for k large enough. Moreover, we find

$$\|\rho_k^2-\rho^2\|_{\mathrm{tr}} = \|(\rho_k+\rho)(\rho_k-\rho)\|_{\mathrm{tr}} \leq \|\rho_k+\rho\|\|\rho_k-\rho\|_{\mathrm{tr}} \leq 2\|\rho_k-\rho\|_{\mathrm{tr}} \to 0, \tag{68}$$

$$\|\rho_k^3-\rho^3\|_{\mathrm{tr}} = \big\|(\rho_k^2-\rho^2)(\rho_k+\rho)+\rho_k(\rho-\rho_k)\rho\big\|_{\mathrm{tr}} \leq 2\|\rho^2-\rho_k^2\|_{\mathrm{tr}}+\|\rho_k-\rho\|_{\mathrm{tr}} \to 0 \tag{69}$$

and therefore all traces in the third line in (67) converge to the corresponding trace with ρ instead of ρ_k; this can be seen as in (65). The traces in the first and second line also converge to the same expression with ρ_k replaced by ρ; this follows from

$$|\mathrm{tr}(A\rho_kA^*\rho_k)-\mathrm{tr}(A\rho A^*\rho)| \leq |\mathrm{tr}(A\rho_kA^*(\rho_k-\rho))|+|\mathrm{tr}(A(\rho-\rho_k)A^*\rho)| \tag{70a}$$

$$\leq 2\|A\|^2\|\rho_k-\rho\|_{\mathrm{tr}} \to 0 \tag{70b}$$

and similarly for the other terms. Thus taking the limit $k\to\infty$ in (67) shows that the upper bound for the variance remains true in the infinite-dimensional case. □

4.2 Upper Bounds for Some Variances and Averaged Deviations

Before we state and prove these upper bounds, we first note that for any operator B on $\mathcal{H}$ such that $\Pi_{e'}B\Pi_e \neq 0$ only for finitely many e, e' and any $\psi_0 \in \mathbb{S}(\mathcal{H})$ the limit

$$M_{\psi_0B} = \overline{\langle\psi_t|B|\psi_t\rangle} := \lim_{T\to\infty}\frac{1}{T}\int_0^T\langle\psi_t|B|\psi_t\rangle\,dt \tag{71}$$

exists and can be computed as follows:

$$M_{\psi_0B} = \overline{\left\langle\psi_0\left|e^{iHt}\sum_{e\in\mathcal{E}}\Pi_eB\sum_{e'\in\mathcal{E}}\Pi_{e'}e^{-iHt}\right|\psi_0\right\rangle} \tag{72a}$$

$$= \sum_{e,e' \in \mathcal{E}} \overline{e^{i(e-e')t}} \langle \psi_0 | \Pi_e B \Pi_{e'} | \psi_0 \rangle \tag{72b}$$

$$= \sum_{e \in \mathcal{E}} \langle \psi_0 \left| \Pi_e B \Pi_e \right| \psi_0 \rangle \,. \tag{72c}$$

An application of (43) immediately shows that

$$\mathbb{E}_\rho M_{\psi_0 B} = \sum_{e \in \mathcal{E}} \mathrm{tr}(\rho \Pi_e B \Pi_e) = M_{\rho B}. \tag{73}$$

Note that interchanging the sums over $\mathcal{E}$ with the time average and expectation with respect to GAP(ρ) are unproblematic as due to our assumption on B the sums are effectively sums over only finitely many e, e'.

Proposition 1. *Let ρ be a density matrix on a separable Hilbert space $\mathcal{H}$ with eigenvalues $p_n > 0$ and $\|\rho\| < 1/4$, let $\dim \mathcal{H} \geq 4$ and let B be an operator on $\mathcal{H}$ such that $d_{E,B} < \infty$. Then for every $\kappa, T > 0$,*

$$\mathbb{E}_\rho \left(\left\langle \left| \langle \psi_t | B | \psi_t \rangle - \overline{\langle \psi_t | B | \psi_t \rangle} \right|^2 \right\rangle_T \right) \leq 24 \|B\|^2 \|\rho\| D_{E,B} G_B(\kappa) \left(1 + \frac{8 \log_2 d_{E,B}}{\kappa T} \right), \tag{74}$$

$$\mathrm{Var}_\rho \, \overline{\langle \psi_t | B | \psi_t \rangle} \leq 23 \|B\|^2 \|\rho\|. \tag{75}$$

Moreover,

$$\mathbb{E}_\rho \left(\overline{\left| \langle \psi_t | B | \psi_t \rangle - \overline{\langle \psi_t | B | \psi_t \rangle} \right|^2} \right) \leq 24 \|B\|^2 \|\rho\| D_{E,B} D_{G,B}. \tag{76}$$

Proof. We first assume that $\mathcal{H}$ is finite-dimensional. The proof starts as the one of Proposition 1 in [42] but, for convenience of the reader, we repeat the steps here so that it is later, when we discuss the infinite-dimensional case, easier to see which steps remain true and where the subtleties are. We find that

$$\left\langle \left| \langle \psi_t | B | \psi_t \rangle - \overline{\langle \psi_t | B | \psi_t \rangle} \right|^2 \right\rangle_T$$

$$= \left\langle \left| \sum_{e,e' \in \mathcal{E}} e^{i(e-e')t} \langle \psi_0 | \Pi_e B \Pi_{e'} | \psi_0 \rangle - \sum_{e \in \mathcal{E}} \langle \psi_0 | \Pi_e B \Pi_e | \psi_0 \rangle \right|^2 \right\rangle_T \tag{77a}$$

$$= \left\langle \left| \sum_{e \neq e'} e^{i(e-e')t} \langle \psi_0 | \Pi_e B \Pi_{e'} | \psi_0 \rangle \right|^2 \right\rangle_T \tag{77b}$$

$$= \sum_{\substack{e \neq e' \\ e'' \neq e'''}} \left\langle e^{i(e-e'-e''+e''')t} \right\rangle_T \langle \psi_0 | \Pi_e B \Pi_{e'} | \psi_0 \rangle \langle \psi_0 | \Pi_{e'''} B^* \Pi_{e''} | \psi_0 \rangle \tag{77c}$$

$$=: \sum_{\alpha,\beta} v_\alpha^* R_{\alpha\beta} v_\beta, \tag{77d}$$

where we defined for $\alpha = (e, e'), \beta = (e'', e''') \in \mathcal{G} := \{(\bar{e}, \bar{e}') \in \mathcal{E} \times \mathcal{E}, \bar{e} \neq \bar{e}'\}$ the vector $v_\alpha = \langle\psi_0|\Pi_{e'} B^* \Pi_e|\psi_0\rangle$ and the Hermitian matrix

$$R_{\alpha\beta} = \left\langle e^{i(G_\alpha - G_\beta)t}\right\rangle_T \tag{78}$$

with $G_\alpha = e - e'$. Written in this way, we immediately see that

$$\left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \leq \|R\| \sum_\alpha |v_\alpha|^2 \leq \|R\| \sum_{e,e'} |\langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle|^2 . \tag{79}$$

It was proved by Short and Farrelly [36] that the operator norm of R can be bounded as

$$\|R\| \leq G(\kappa) \left(1 + \frac{8 \log_2 d_E}{\kappa T}\right). \tag{80}$$

We further compute

$$\mathbb{E}_\rho \left(\sum_{e,e'} |\langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle|^2 \right) = \sum_{e,e'} \mathrm{Var}_\rho \langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle + |\mathbb{E}_\rho \langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle|^2 \tag{81a}$$

$$= \sum_{e,e'} \mathrm{Var}_\rho \langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle + |\mathrm{tr}(\rho \Pi_e B \Pi_{e'})|^2 , \tag{81b}$$

where we applied Lemma 1 in the second step. It follows again from Lemma 1 that

$$\begin{aligned}
&\sum_{e,e'} \mathrm{Var}_\rho \langle\psi_0|\Pi_e B \Pi_{e'}|\psi_0\rangle \\
&\leq \frac{1}{1-p_{\max}} \sum_{e,e'} \Big(\mathrm{tr}(\Pi_e B \Pi_{e'} \rho \Pi_{e'} B^* \Pi_e \rho) \\
&\quad + \frac{\mathrm{tr}(\Pi_e B \Pi_{e'} \rho^2 \Pi_{e'} B^* \Pi_e \rho) + \mathrm{tr}(\Pi_e B \Pi_{e'} \rho \Pi_{e'} B^* \Pi_e \rho^2)}{1 - 2p_{\max}} \\
&\quad + \frac{2}{(1-2p_{\max})(1-3p_{\max})} \big[\mathrm{tr}(\Pi_e B \Pi_{e'} \rho^3 \Pi_{e'} B^* \Pi_e \rho) + \mathrm{tr}(\Pi_e B \Pi_{e'} \rho^2 \Pi_{e'} B^* \Pi_e \rho^2) \\
&\quad + \mathrm{tr}(\Pi_e B \Pi_{e'} \rho \Pi_{e'} B^* \Pi_e \rho^3) + \sum_{m,n} \big(|\mathrm{tr}(\Pi_e B \Pi_{e'} \rho^3 P_m) \, \mathrm{tr}(\Pi_{e'} B^* \Pi_e \rho P_n)| \\
&\quad + |\mathrm{tr}(\Pi_e B \Pi_{e'} \rho^2 P_m) \, \mathrm{tr}(\Pi_{e'} B^* \Pi_e \rho^2 P_n)| + |\mathrm{tr}(\Pi_e B \Pi_{e'} \rho P_m) \, \mathrm{tr}(\Pi_{e'} B^* \Pi_e \rho^3 P_n)| \big) \big] \Big).
\end{aligned} \tag{82}$$

The main tools for bounding the traces are the Cauchy-Schwarz inequality for the trace which states that $|\operatorname{tr}(A^*B)| \leq \sqrt{\operatorname{tr}(A^*A)\operatorname{tr}(B^*B)}$ for any operators A and B and the inequality $|\operatorname{tr}(AB)| \leq \|A\| \operatorname{tr}(|B|)$; for the latter see, e.g., Theorem 3.7.6 in [37]. We compute

$$\begin{aligned}
\sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} \rho \Pi_{e'} B^* \Pi_e \rho) &= \sum_e \operatorname{tr}\left(B \left(\sum_{e'} \Pi_{e'} \rho \Pi_{e'} \right) B^* \Pi_e \rho \Pi_e \right) && \text{(83a)}\\
&\leq \sum_e \|B\|^2 \left\| \sum_{e'} \Pi_{e'} \rho \Pi_{e'} \right\| \operatorname{tr}(\Pi_e \rho \Pi_e) && \text{(83b)}\\
&\leq \|B\|^2 \|\rho\|, && \text{(83c)}
\end{aligned}$$

where we used that $\| \sum_{e'} \Pi_{e'} \rho \Pi_{e'} \| \leq \|\rho\|$ which follows immediately from

$$\left\| \sum_{e'} \Pi_{e'} \rho \Pi_{e'} \psi \right\|^2 = \sum_{e'} \|\Pi_{e'} \rho \Pi_{e'} \psi\|^2 \leq \|\rho\|^2 \sum_{e'} \|\Pi_{e'} \psi\|^2 = \|\rho\|^2 \|\psi\|^2 \tag{84}$$

for all $\psi \in \mathcal{H}$. In the same way we get

$$\sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} \rho^2 \Pi_{e'} B^* \Pi_e \rho) \leq \|B\|^2 \|\rho\|^2, \tag{85}$$

$$\sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} \rho \Pi_{e'} B^* \Pi_e \rho^2) \leq \|B\|^2 \|\rho\|^2, \tag{86}$$

$$\sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} \rho^3 \Pi_{e'} B^* \Pi_e \rho) \leq \|B\|^2 \|\rho\|^3, \tag{87}$$

$$\sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} \rho^2 \Pi_{e'} B^* \Pi_e \rho^2) \leq \|B\|^2 \|\rho\|^2 \operatorname{tr} \rho^2 \leq \|B\|^2 \|\rho\|^3, \tag{88}$$

$$\sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} \rho \Pi_{e'} B^* \Pi_e \rho^3) \leq \|B\|^2 \|\rho\|^3. \tag{89}$$

Next we turn to the products of traces; we find that

$$\begin{aligned}
&\sum_{e,e'} \sum_{m,n} |\operatorname{tr}(\Pi_e B \Pi_{e'} \rho^3 P_m) \operatorname{tr}(\Pi_{e'} B^* \Pi_e \rho P_n)| \\
&\quad = \sum_{m,n} \sum_{e,e'} |\operatorname{tr}(\Pi_e B \Pi_{e'} \Pi_{e'} P_m \Pi_e) \operatorname{tr}(\Pi_{e'} B^* \Pi_e \Pi_e P_n \Pi_{e'})| p_m^3 p_n && \text{(90a)}\\
&\quad \leq \sum_{m,n} p_m^3 p_n \sum_{e,e'} \left(\operatorname{tr}(\Pi_e B \Pi_{e'} B^*) \operatorname{tr}(\Pi_e P_m \Pi_{e'} P_m) \operatorname{tr}(\Pi_{e'} B^* \Pi_e B) \operatorname{tr}(\Pi_{e'} P_n \Pi_e P_n) \right)^{1/2} && \text{(90b)}\\
&\quad = \sum_{m,n} p_m^3 p_n \sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} B^*) \left(\langle m|\Pi_e|m\rangle \langle m|\Pi_{e'}|m\rangle \langle n|\Pi_e|n\rangle \langle n|\Pi_{e'}|n\rangle \right)^{1/2} && \text{(90c)}
\end{aligned}$$

$$\le \|B\|^2 \sum_{m,n} p_m^3 p_n \sum_{e,e'} \operatorname{tr}(\Pi_e) \left(\langle m|\Pi_e|m\rangle\langle m|\Pi_{e'}|m\rangle\langle n|\Pi_e|n\rangle\langle n|\Pi_{e'}|n\rangle\right)^{1/2} \tag{90d}$$

$$\le \|B\|^2 D_E \sum_{m,n} p_m^3 p_n \left(\sum_e \left(\langle m|\Pi_e|m\rangle\langle n|\Pi_e|n\rangle\right)^{1/2}\right)^2 \tag{90e}$$

$$\le \|B\|^2 D_E \sum_{m,n} p_m^3 p_n \left(\sum_e \langle m|\Pi_e|m\rangle\right)\left(\sum_e \langle n|\Pi_e|n\rangle\right) \tag{90f}$$

$$= \|B\|^2 D_E \sum_{m,n} p_m^3 p_n \tag{90g}$$

$$\le \|B\|^2 D_E \|\rho\|^2. \tag{90h}$$

In the same way we obtain

$$\sum_{e,e'}\sum_{m,n} |\operatorname{tr}(\Pi_e B \Pi_{e'} \rho^2 P_m) \operatorname{tr}(\Pi_{e'} B^* \Pi_e \rho^2 P_n)| \le \|B\|^2 D_E \|\rho\|^2, \tag{91}$$

$$\sum_{e,e'}\sum_{m,n} |\operatorname{tr}(\Pi_e B \Pi_{e'} \rho P_m) \operatorname{tr}(\Pi_{e'} B^* \Pi_e \rho^3 P_n)| \le \|B\|^2 D_E \|\rho\|^2. \tag{92}$$

Finally we estimate

$$\sum_{e,e'} |\operatorname{tr}(\rho \Pi_e B \Pi_{e'})|^2 \le \sum_{e,e'} \operatorname{tr}(\Pi_e B \Pi_{e'} B^*) \operatorname{tr}(\Pi_e \rho \Pi_{e'} \rho) \tag{93a}$$

$$\le \|B\|^2 D_E \sum_{e,e'} \operatorname{tr}(\Pi_e \rho \Pi_{e'} \rho) \tag{93b}$$

$$= \|B\|^2 D_E \operatorname{tr} \rho^2 \tag{93c}$$

$$\le \|B\|^2 D_E \|\rho\|. \tag{93d}$$

Putting everything together we arrive at

$$\begin{aligned} &\mathbb{E}_\rho \left(\left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \right) \\ &\quad \le \|B\|^2 \|\rho\| D_E G(\kappa) \left(1 + \frac{8\log_2 d_E}{\kappa T}\right) \\ &\qquad \cdot \left(1 + \frac{1}{1-\|\rho\|}\left[1 + \frac{2\|\rho\|}{1-2\|\rho\|} + \frac{6(\|\rho\| + \|\rho\|^2)}{(1-2\|\rho\|)(1-3\|\rho\|)}\right]\right) \end{aligned} \tag{94a}$$

$$\le 24 \|B\|^2 \|\rho\| D_E G(\kappa) \left(1 + \frac{8\log_2 d_E}{\kappa T}\right), \tag{94b}$$

where we used the assumption that $\|\rho\| < 1/4$ to bound the bracket $(1 + \frac{1}{1-\|\rho\|}[...])$ in (94a) by[4] 24.

[4] Note that the bracket is a monotone increasing function in $\|\rho\|$ and therefore bounded by its value for $\|\rho\| = 1/4$ which is $71/3 < 24$.

For the variance of $\overline{\langle\psi_t|B|\psi_t\rangle}$ we see from (72c) that we can simply apply Lemma 1 with $A = \sum_e \Pi_e B \Pi_e$ and estimate the occurring traces. We compute

$$\operatorname{tr}\left(\left(\sum_e \Pi_e B \Pi_e\right)\rho\left(\sum_{e'} \Pi_{e'} B^* \Pi_{e'}\right)\rho\right) \leq \left\|\left(\sum_e \Pi_e B \Pi_e\right)\rho\left(\sum_{e'} \Pi_{e'} B^* \Pi_{e'}\right)\right\| \operatorname{tr}\rho \tag{95a}$$

$$\leq \|B\|^2 \|\rho\|, \tag{95b}$$

where we used that $\|\sum_e \Pi_e B \Pi_e\| \leq \|B\|$. Similarly we find that

$$\operatorname{tr}(A\rho^2 A^*\rho), \operatorname{tr}(A\rho A^*\rho^2) \leq \|B\|^2\|\rho\|^2, \tag{96}$$

$$\operatorname{tr}(A\rho^3 A^*\rho), \operatorname{tr}(A\rho^2 A^*\rho^2), \operatorname{tr}(A\rho A^*\rho^3) \leq \|B\|^2\|\rho\|^3, \tag{97}$$

Moreover, we get

$$\sum_{m,n}\left|\operatorname{tr}\left(\sum_e \Pi_e B \Pi_e \rho^3 P_m\right)\operatorname{tr}\left(\sum_{e'} \Pi_{e'} B^* \Pi_{e'} \rho P_n\right)\right| \leq \sum_{m,n} p_m^3 p_n \|B\|^2 \operatorname{tr}(P_m)\operatorname{tr}(P_n) \tag{98a}$$

$$= \|B\|^2 \operatorname{tr}\rho^3 \tag{98b}$$

$$\leq \|B\|^2\|\rho\|^2 \tag{98c}$$

and similarly

$$\sum_{m,n} |\operatorname{tr}(A\rho^2 P_m)\operatorname{tr}(A^*\rho^2 P_n)|, \sum_{m,n} |\operatorname{tr}(A\rho P_m)\operatorname{tr}(A^*\rho^3 P_n)| \leq \|B\|^2\|\rho\|^2. \tag{99}$$

Thus we finally arrive at

$$\operatorname{Var}_\rho \overline{\langle\psi_t|B|\psi_t\rangle} \leq \frac{\|B\|^2\|\rho\|}{1-\|\rho\|}\left(1 + \frac{2\|\rho\|}{1-2\|\rho\|} + \frac{6(\|\rho\| + \|\rho\|^2)}{(1-2\|\rho\|)(1-3\|\rho\|)}\right) \tag{100a}$$

$$\leq 23\|B\|^2\|\rho\|. \tag{100b}$$

Now suppose that $\mathcal{H}$ is infinite-dimensional. Due to our assumption on B, all sums over $e \in \mathcal{E}$ are effectively sums over $e \in \mathcal{E}_B$ and therefore finite. In the finite-dimensional setting, the matrix R defined by (78) is a $d_E(d_E-1) \times d_E(d_E-1)$ matrix and this is where the d_E in the upper bound for $\|R\|$ comes from, see the proof in [36]. In the infinite-dimensional setting, the matrix R can be viewed as a $d_{E,B}(d_{E,B}-1) \times d_{E,B}(d_{E,B}-1)$ matrix and therefore, in the bound for $\|R\|$, d_E has to be replaced by $d_{E,B}$; by the same argument, d_E can also be replaced by $d_{E,B}$ in the finite-dimensional setting. Moreover, we can also change $G(\kappa)$ to $G_B(\kappa)$ and D_E to $D_{E,B}$ as the only "contributing" eigenvalues are the ones in $\mathcal{E}_B$. Besides this change all other steps of the proof remain valid and this proves the bounds also in the infinite-dimensional case.

For the statements about the infinite time average, we want to take the limit $T \to \infty$ in

$$\mathbb{E}_\rho \left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \leq 24\|B\|^2\|\rho\| D_{E,B} G_B(\kappa) \left(1 + \frac{8\log_2 d_{E,B}}{\kappa T}\right). \quad (101)$$

To this end, we first choose κ so small that $G_B(\kappa) = D_{G,B}$. Then we take the limit $T \to \infty$ and use dominated convergence to interchange the limit and $\mathbb{E}_\rho$. This gives (76) and finishes the proof. □

4.3 Proof of Theorem 1

Without loss of generality we assume that all eigenvalues of ρ are positive (otherwise we restrict our considerations to the subspace of $\mathcal{H}$ spanned by the eigenvectors of ρ corresponding to its positive eigenvalues). It follows from Markov's inequality together with Proposition 1 that

$$\mathrm{GAP}(\rho)\left\{ \left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \right.$$
$$\left. \geq \frac{48}{\varepsilon}\|B\|^2\|\rho\| D_{E,B} G_B(\kappa) \left(1 + \frac{8\log_2 d_{E,B}}{\kappa T}\right) \right\} \leq \frac{\varepsilon}{2} \quad (102)$$

and

$$\mathrm{GAP}(\rho) \left\{ \overline{\left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2} \geq \frac{48}{\varepsilon}\|B\|^2\|\rho\| D_{E,B} D_{G,B} \right\} \leq \frac{\varepsilon}{2}. \quad (103)$$

Concerning the finite-time average, this shows that, w.r.t. $\mathrm{GAP}(\rho)$, $(1 - \frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that

$$\left\langle \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right\rangle_T \leq \frac{48}{\varepsilon}\|B\|^2\|\rho\| D_{E,B} G_B(\kappa) \left(1 + \frac{8\log_2 d_{E,B}}{\kappa T}\right). \quad (104)$$

Similarly, we have that, w.r.t. $\mathrm{GAP}(\rho)$, $(1 - \frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that

$$\overline{\left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2} \leq \frac{48}{\varepsilon}\|B\|^2\|\rho\| D_{E,B} D_{G,B}. \quad (105)$$

Let λ denote the Lebesgue measure on $\mathbb{R}$. An application of Markov's inequality and Proposition 1 shows that

$$\frac{1}{T}\lambda\left\{ t \in [0,T] : \left| \langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle} \right|^2 \right.$$

$$\geq \frac{48}{\varepsilon\delta}\|B\|^2\|\rho\| D_{E,B} G_B(\kappa)\left(1+\frac{8\log_2 d_{E,B}}{\kappa T}\right)\Bigg\} \leq \delta. \tag{106}$$

Thus, w.r.t. GAP(ρ), $(1-\frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for $(1-\delta)$-most $t \in [0,T]$,

$$\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right| \leq \left(\frac{48}{\varepsilon\delta}\|B\|^2\|\rho\| D_{E,B} G_B(\kappa)\left(1+\frac{8\log_2 d_{E,B}}{\kappa T}\right)\right)^{1/2}. \tag{107}$$

In the same way, we obtain

$$\liminf_{T\to\infty}\frac{1}{T}\lambda\left\{t\in[0,T]:\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right|^2 \geq \frac{48}{\varepsilon\delta}\|B\|^2\|\rho\| D_{E,B} D_{G,B}\right\} \leq \delta \tag{108}$$

and conclude that, w.r.t. GAP(ρ), $(1-\frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that for $(1-\delta)$-most $t \in [0,\infty)$,

$$\left|\langle\psi_t|B|\psi_t\rangle - \overline{\langle\psi_t|B|\psi_t\rangle}\right| \leq \left(\frac{48}{\varepsilon\delta}\|B\|^2\|\rho\| D_{E,B} D_{G,B}\right)^{1/2}. \tag{109}$$

It follows from Chebyshev's inequality together with Proposition 1 that

$$\mathrm{GAP}(\rho)\left\{\left|\overline{\langle\psi_t|B|\psi_t\rangle} - M_{\rho B}\right| \geq \left(\frac{46\|B\|^2\|\rho\|}{\varepsilon}\right)^{1/2}\right\} \leq \frac{\varepsilon}{2} \tag{110}$$

and thus, w.r.t. GAP(ρ), $(1-\frac{\varepsilon}{2})$-most $\psi_0 \in \mathbb{S}(\mathcal{H})$ are such that

$$\left|\overline{\langle\psi_t|B|\psi_t\rangle} - M_{\rho B}\right| \leq \left(\frac{46\|B\|^2\|\rho\|}{\varepsilon}\right)^{1/2}. \tag{111}$$

Now the claims follow from the triangle inequality.

5 Summary and Conclusions

Our result concerns the long-time behavior of GAP(ρ)-typical pure states ψ_0 from the sphere of a separable Hilbert space where ρ is a density matrix on $\mathcal{H}$ and ψ_0 evolves unitarily according to $\psi_t = \exp(-iHt)\psi_0$. We have seen that for any operator B on $\mathcal{H}$ that satisfies $|\mathcal{E}_B| < \infty$, for GAP(ρ)-most $\psi_0 \in \mathbb{S}(\mathcal{H})$, the curve $t \mapsto \langle\psi_t|B|\psi_t\rangle$ is nearly constant in the long run $t \to \infty$ provided that $\|\rho\|$ is small, $\|B\|$ is not too large and no eigenvalue or eigenvalue gap of the Hamiltonian H is too highly degenerate. In particular, we have argued that our result can be applied to the case that $B = P_\nu$ for some macro state ν. We have provided explicit error bounds that reveal the dependence of the error on these quantities. Our result shows that the concept of normal typicality can be generalized from the uniform distribution on the sphere of the Hilbert space to a much broader class of distributions which can also be defined on infinite-dimensional Hilbert spaces and which for certain density matrices arise naturally as the distribution of wave functions in thermal equilibrium.

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