On $\mathbb{K}^*$-surfaces of integral degree and full intrinsic quadric surfaces

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
Katharina Lydia Király
aus Ulm

Tübingen
2024

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation: 27.02.2025
Dekan: Prof. Dr. Thilo Stehle
1. Berichterstatter: Prof. Dr. Jürgen Hausen
2. Berichterstatter: apl. Prof. Dr. Ivo Radloff

CONTENTS

DANKSAGUNG

An dieser Stelle möchte ich den zahlreichen Personen danken, ohne deren Unterstützung diese Arbeit nicht möglich gewesen wäre.

Ganz besonders möchte ich Professor Dr. Jürgen Hausen für die hervorragende Betreuung meiner Promotion danken. Des Weiteren danke ich Professor Dr. Ivo Radloff, dass er sich bereit erklärt hat, der Zweitgutachter dieser Arbeit zu sein. Außerdem möchte ich meiner Koautorin Professor Dr. Milena Wrobel für die Zusammenarbeit am gemeinsamen Paper danken. Darüberhinaus danke ich Dr. Tim Binz für die gemeinsame Arbeit am fortlaufenden Projekt über Ricci-Solitons.

Der Arbeitsgemeinschaft Algebra danke ich für die gute Arbeitsatmosphäre und die Gespräche während der Kaffeepausen, der Spaziergänge, der Gesellschaftsspiele- und Filmeabende.

Anschließend möchte ich mich bei meiner Familie und meiner zweiten Familie Binz für die anhaltende Unterstützung bedanken.

Auch danke ich meinen Freund*innen: Elena Auberer für das gemeinsame Musizieren, Martin Bohnert für die Fahrradtouren und zahlreichen Ausflüge, Nikolai Edeko für das gemeinsame Kuchenbacken, Marie Kocsa für die Partys, Sina Langkamp und Patrick Matt für die Kochabende in der WG, und natürlich für die entspannte und bereichernde Zeit während des Studiums und der Promotion.

Zu guter Letzt möchte ich mich bei meinem Freund Dr. Tim Binz dafür bedanken, dass er mich stets unterstützt, mich liebt, an mich glaubt, mein Zuhause ist, und dafür, dass kein Weg für ihn zu weit ist, um an meiner Seite zu sein.

INTRODUCTION

This dissertation is devoted to the study of quasismooth, rational, projective surfaces with $\mathbb{K}^*$-action, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. Inside this class we exhibit and explore two infinite series. Firstly, the surfaces of Picard number one and integral degree. Secondly, the full intrinsic quadric surfaces, that is, those whose Cox ring is defined by a single quadratic equation. Let us present our results more in detail.

We begin with the quasismooth, rational, projective surfaces of Picard number one and integral degree $\mathcal{K}^2$ that admit a non-trivial, effective action of an algebraic torus $\mathbb{T}$. We provide an explicit description which involves the solutions of the *squared Markov type equation*, that means the triples $u = (u_0, u_1, u_2)$ of positive integers satisfying

$$(u_0 + u_1 + u_2)^2 \ = \ a u_0 u_1 u_2,$$

where a is a positive integer. The solution set $S(a) \subseteq \mathbb{Z}^3_{>0}$ of these equations is non-empty only for $a = 1, 2, 3, 4, 5, 6, 8, 9$ and the solutions $u \in S(a)$ admit an explicit stepwise construction; see Section 2.1 for an elementary treatment.

If the torus $\mathbb{T}$ is of dimension two, then we are concerned with a projective toric surface Z of Picard number one, also called a *fake weighted projective plane*. Any such Z has divisor class group $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ and we can encode Z in terms of its *degree matrix* Q, listing the classes of the three coordinate divisors as columns:

$$Q \ = \ [q_0, q_1, q_2] \ = \ \begin{bmatrix} u_0 & u_1 & u_2 \\ \bar{\eta}_0 & \bar{\eta}_1 & \bar{\eta}_2 \end{bmatrix}, \qquad q_i = (u_i, \bar{\eta}_i) \in \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}.$$

If the degree $a = \mathcal{K}^2$ of Z is an integer, then we obtain $u \in S(\mu a)$ for the first row of Q. That means in particular that, up to reordering, the triple u is of the shape

$$u \ = \ (x_0^2, \xi_1 x_1^2, \xi_2 x_2^2), \qquad x_0, x_1, x_2 \in \mathbb{Z}, \qquad (\xi_1, \xi_2) \ = \ \begin{cases} (1,1), & \mu a = 9, \\ (1,2), & \mu a = 8, \\ (2,3), & \mu a = 6, \\ (1,5), & \mu a = 5. \end{cases}$$

Our first result explicitly describes all fake weighted projective planes of integral degree $\mathcal{K}^2$ by means of 24 infinite series of degree matrices; note that all members of a series with ID (a-μ-η) have the degree $\mathcal{K}^2 = a$ and the order μ of the torsion part of their divisor class group in common.

Theorem 1. *Let Z be a fake weighted projective plane of integral degree. Then $Z \cong Z(Q)$ for a degree matrix Q from the following data table:*

(9-1-0) $\mathcal{K}^2 = 9$ $\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \end{bmatrix}$ $S(9)$	*(8-1-0)* $\mathcal{K}^2 = 8$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \end{bmatrix}$ $S(8)$	*(6-1-0)* $\mathcal{K}^2 = 6$ $\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \end{bmatrix}$ $S(6)$
(5-1-0) $\mathcal{K}^2 = 5$ $\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \end{bmatrix}$ $S(5)$	*(4-2-1)* $\mathcal{K}^2 = 4$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $S(8)$ $\mathbb{Z}/2\mathbb{Z}$	*(3-3-2)* $\mathcal{K}^2 = 3$ $\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $S(9)$ $\mathbb{Z}/3\mathbb{Z}$
(3-2-1) $\mathcal{K}^2 = 3$ $\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $S(6)$ $\mathbb{Z}/2\mathbb{Z}$	*(2-4-1)* $\mathcal{K}^2 = 2$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $S(8)$ $\mathbb{Z}/4\mathbb{Z}$	*(2-4-3)* $\mathcal{K}^2 = 2$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}$ $S(8)$ $\mathbb{Z}/4\mathbb{Z}$
(2-3-1) $\mathcal{K}^2 = 2$ $\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $S(6)$ $\mathbb{Z}/3\mathbb{Z}$	*(2-3-2)* $\mathcal{K}^2 = 2$ $\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $S(6)$ $\mathbb{Z}/3\mathbb{Z}$	*(1-9-2)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $S(9)$ $\mathbb{Z}/9\mathbb{Z}$
(1-9-5) $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}$ $S(9)$ $\mathbb{Z}/9\mathbb{Z}$	*(1-9-8)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{8} \end{bmatrix}$ $S(9)$ $\mathbb{Z}/9\mathbb{Z}$	*(1-8-1)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $S(8)$ $\mathbb{Z}/8\mathbb{Z}$
(1-8-3) $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}$ $S(8)$ $\mathbb{Z}/8\mathbb{Z}$	*(1-8-5)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}$ $S(8)$ $\mathbb{Z}/8\mathbb{Z}$	*(1-8-7)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{7} \end{bmatrix}$ $S(8)$ $\mathbb{Z}/8\mathbb{Z}$
(1-6-1) $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $S(6)$ $\mathbb{Z}/6\mathbb{Z}$	*(1-6-5)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}$ $S(6)$ $\mathbb{Z}/6\mathbb{Z}$	*(1-5-1)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $S(5)$ $\mathbb{Z}/5\mathbb{Z}$
(1-5-2) $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $S(5)$ $\mathbb{Z}/5\mathbb{Z}$	*(1-5-3)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}$ $S(5)$ $\mathbb{Z}/5\mathbb{Z}$	*(1-5-4)* $\mathcal{K}^2 = 1$ $\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{4} \end{bmatrix}$ $S(5)$ $\mathbb{Z}/5\mathbb{Z}$

Moreover, each of the listed degree matrices defines a fake weighted projective plane of the degree given in its data package. Finally, any two distinct degree matrices from the table define non-isomorphic fake weighted projective planes, except they stem from (1-9-$*$) or (1-8-$*$) and are both taken from one of the sets:

$$\left\{ \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{8} \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 & 1 & 4 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}, \begin{bmatrix} 1 & 1 & 4 \\ \bar{0} & \bar{1} & \bar{8} \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{7} \end{bmatrix} \right\}.$$

The proof of Theorem 1 is provided in Section 2.3 and is split according to the possible degrees $\mathcal{K}^2$; see Proposition 2.3.2, 2.3.4, 2.3.5, 2.3.7 and 2.3.10. The arguments essentially use the unique encoding of fake weighted projective planes via *adjusted degree matrices* as provided by Proposition 2.2.12 and, as well, the various divisibility properties of the entries of the solution triples of the squared Markov type equations observed in Section 2.1 and 2.3.

In Section 2.4, we take a closer look at the possible singularities of our fake weighted projective planes; by normality, these are at most the three toric fixed points. Recall that a *(cyclic) T-singularity* is a quotient singularity $\mathbb{K}^2/C(dk^2)$ by the subgroup $C(dk^2) \subseteq \mathbb{K}^*$ of the dk^2-th roots of unity, acting via

$$\zeta \cdot (z_1, z_2) \ = \ (\zeta z_1, \zeta^{dpk-1} z_2),$$

where p and k are coprime. Proposition 2.4.5 to 2.4.9 provide the constellations of non-trivial local Gorenstein indices and possible T-singularities for all fake weighted projective planes of integral degree. As an immediate consequence, one obtains the following.

Theorem 2. *Let Z be a fake weighted projective plane of integral degree. If Z is isomorphic to a member of one of the series*

(2-3-1), (1-8-1), (1-8-5), (1-6-1), (1-5-1), (1-5-2), (1-5-3),

then Z has three singularities and precisely one of them is a T-singularity. Otherwise, the surface Z has at most T-singularities.

The class of the fake weighted projective planes of integral degree contains in particular all fake weighted projective planes having at most T-singularities; see for instance [15, Prop. 2.6]. Thus, the classification results of Section 2.4 complement the classification [15, Thm. 4.1] of T-singular projective toric surfaces of Picard number one in the sense that we add three more series, namely (1-9-2), (1-9-5) and (1-8-3), to Table 2 given there; see also Example 2.6.5 and 2.6.7.

We turn to the non-toric $\mathbb{K}^*$-surfaces of Picard number one. Any quasismooth, rational, projective $\mathbb{K}^*$-surface of Picard number one can be realized as a surface in *fake weighted projective space*, that means a toric threefold of Picard number one in the following way. Consider an integral matrix $P \in \mathrm{Mat}(3, 4; \mathbb{Z})$ of the form

$$\begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix}, \qquad \begin{array}{c} 0 \le d_1 < l_1 \le l_2, \quad \gcd(l_i, d_i) = 1, \\ d_0 + \frac{d_1}{l_1} + \frac{d_2}{l_2} \ < \ 0 \ < \ \frac{d_1}{l_1} + \frac{d_2}{l_2} \end{array}.$$

Then there is a unique projective fan having the columns v_1, v_2, v_3, v_4 of P as its primitive ray generators and the associated toric variety Z is a fake weighted projective space; the matrix P is also called a generator matrix of Z. Moreover, inside Z, we define a surface by

$$X \ := \ \overline{\{t \in \mathbb{T}^3;\ 1 + z_1 + z_2 = 0\}} \ \subseteq \ Z,$$

where $\mathbb{T}^3 \subseteq Z$ is the acting torus of Z. It turns out that X is a projective, rational, quasismooth, del Pezzo $\mathbb{K}^*$-surface of Picard number one acted on effectively by $\mathbb{K}^*$ via

$$t \cdot z \ = \ (z_1, z_2, tz_3).$$

Conversely, every quasismooth, rational, projective $\mathbb{K}^*$-surface of Picard number one arises in this way. The toric divisors of Z corresponding to the last two columns v_3, v_4 of P cut out two invariant divisors on X, each containing a $\mathbb{K}^*$-orbit B_1, B_2 with a non-trivial finite isotropy group of order l_1, l_2. Moreover, these $B_i \subseteq X$ are the only orbits with non-trivial finite isotropy groups and the latter ones are the groups $C(l_i) \subseteq \mathbb{K}^*$ of l_i-th roots of unit. As a consequence of more specific considerations, see Corollary 1.5.11, we observe:

Corollary 1. *Let X be a quasismooth, rational, projective $\mathbb{K}^*$-surface of Picard number one. Then $X/C(l_1)$ and $X/C(l_2)$ are fake weighed projective planes.*

The $\mathbb{K}^*$-surfaces of integral degree naturally correspond to certain pairs of fake weighted projective planes from Theorem 1. Every quasismooth, rational, projective $\mathbb{K}^*$-surface X of Picard number one arising from a matrix P as above admits a toric degeneration $Z_1 \leftsquigarrow X \rightsquigarrow Z_2$, where Z_1 and Z_2 are the fake weighted projective planes given by the generator matrices

$$\tilde{P}_1 \ := \ \begin{bmatrix} l_1 & l_1 & -l_2 \\ d_1 & d_1 + l_1 d_0 & d_2 \end{bmatrix}, \qquad \tilde{P}_2 \ := \ \begin{bmatrix} l_2 & l_2 & -l_1 \\ d_2 & d_2 + l_2 d_0 & d_1 \end{bmatrix}$$

It turns out that Z_1 and Z_2 are of the same degree as X; see Construction 2.5.1 and Proposition 2.5.6. We will call two fake weighted projective planes *adjacent* if they arise as such degenerations from a common $\mathbb{K}^*$-surface. Theorem 2.5.13 ensures that every fake weighted projective plane Z_1 of integral degree admits an adjacent partner Z_2. A more concise formulation of adjacency in terms of degree matrices and supporting notions are given in Definition 2.5.12 and 2.5.14. This allows us to assign to any pair (Q_1, Q_2) of adjacent degree matrices a $\mathbb{K}^*$-surface $X(Q_1, Q_2)$ and leads to the following.

Theorem 3. *Let X be a non-toric, quasismooth, rational, projective $\mathbb{K}^*$-surface of Picard number one with $\mathcal{K}_X^2 \in \mathbb{Z}$. Then $X \cong X(Q_1, Q_2)$ with a non-toric, ordered pair of adjacent degree matrices. Moreover, distinct ordered pairs (Q_1, Q_2) of adjacent degree matrices yield non-isomorphic $\mathbb{K}^*$-surfaces.*

We turn to our second series, the *full intrinsic quadrics.* The name refers to the property that the Cox ring of X is defined by a single homogeneous quadratic relation of full rank; see [4]. Intrinsic quadrics exist as well in higher dimensions and form an explicit example class closely beneath the toric varieties which are characterized by having a polynomial ring as Cox ring; see [6, 12, 30] for sample work.

As we will see in Theorem 3.1.3, every full intrinsic quadric surface X is projective, normal, $\mathbb{Q}$-factorial, rational and allows a non-trivial action of the multiplicative group $\mathbb{K}^*$. This allows us to realize X as a surface in a specific toric threefold Z. More precisely, consider integral $3 \times n$ matrices P of the form

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}.$$

Given such P, fix a complete fan Σ in $\mathbb{Z}^3$ having the columns of P as its primitive ray generators, let Z be the associated toric threefold and set

$$X \;:=\; \overline{\{t \in \mathbb{T}^3;\ 1 + z_1 + z_2 = 0\}} \;\subseteq\; Z,$$

where $\mathbb{T}^3 \subseteq Z$ is the standard 3-torus with the coordinates z_1, z_2, z_3. Then X is a full intrinsic quadric surface. The Picard number of X is $\rho(X) = n - 3$ and X comes with the effective $\mathbb{K}^*$-action given on $X \cap \mathbb{T}^3$ by

$$t \cdot z \;=\; (z_1, z_2, tz_3).$$

Our main results are Theorem 3.2.5, 3.3.5 and 3.4.5, provide an *explicit and redundance free presentation of all full intrinsic quadric surfaces* via their defining matrices P in terms of local Gorenstein indices and local class group orders of the possibly singular points: for each of the possible Picard numbers $\rho(X) = 1, 2, 3$, we find four infinite series, each depending on two local Gorenstein indices, ι^+, ι^- and on $\rho(X) - 1$ local class group orders, bounded by ι^+, ι^-.

All full intrinsic quadric surfaces X turn out to be log del Pezzo surfaces; see Proposition 3.5.1. We use our main results to study their geometry. For instance, Theorem 3.5.3, 3.5.23 and 3.5.27 deliver upper and lower bounds on the degree $\mathcal{K}_X^2$, the log canonicity ε_X and the Picard index $\mathfrak{p}_X$, all in terms of the Gorenstein index ι_X; in particular, we obtain

$$\tfrac{2}{\iota_X} \;\le\; \mathcal{K}_X^2 \le \tfrac{9}{2} + \tfrac{9}{2\iota_X}, \quad \tfrac{2}{\iota_X} \;\le\; \varepsilon_X \;\le\; \tfrac{3}{\sqrt{\iota_X}}, \quad \iota_X \;\le\; \mathfrak{p}_X \;\le\; \tfrac{32}{3}\,(2\iota_X - 1)^2\,\iota_X^3 + \tfrac{49}{3}.$$

Another outcome of Theorem 3.2.5, 3.3.5 and 3.4.5 is the following explicit (infinite) list of all full intrinsic quadric complex surfaces admitting a Kähler-Einstein metric; see Theorem 3.5.32 for the precise formulation and more background.

Theorem 4. *The full intrinsic quadric complex surfaces admitting a Kähler-Einstein metric are precisely those constructed from a matrix P of the shape*

$\rho = 1,\ 2 \nmid \iota :$

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota - 1 & -\iota - 1 & 1 & 1 \end{bmatrix},$$

$\rho = 3,\quad 2 \nmid \iota,\quad -\iota + 1 \leq c \leq -2,$
$\max(c, -2\iota - 2c) \leq d \leq -\iota - 1 - c :$

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota & -\iota - c - d & 0 & c & 0 & d \end{bmatrix},$$

$\rho = 1,\ 4 \mid \iota :$

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota}{2} - 1 & -\frac{\iota}{2} - 1 & 1 & 1 \end{bmatrix},$$

$\rho = 3,\quad -2\iota + 1 \leq c \leq -2,$
$\max(c, -4\iota - 2c) \leq d \leq -2\iota - 1 - c :$

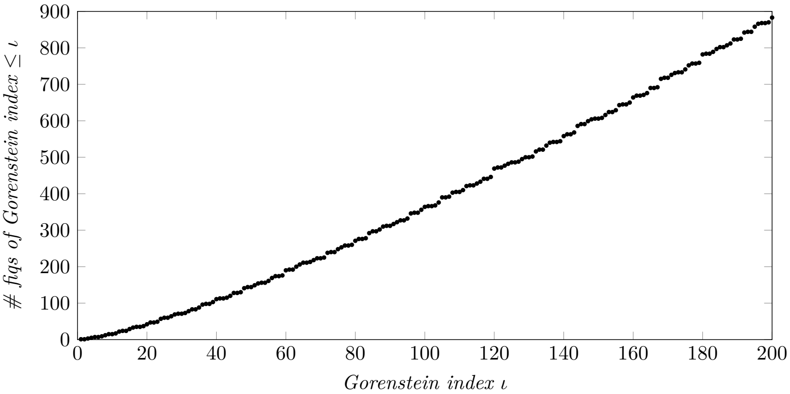
$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota & -2\iota - c - d & 0 & c & 0 & d \end{bmatrix}.$$

Here, $\rho = 1, 3$ *is the Picard number and* $\iota \in \mathbb{Z}_{\geq 1}$ *the Gorenstein index of the resulting full intrinsic quadric complex surface* X *arising from the matrix* P.

Finally, our description yields a filtration of the whole infinite class of full intrinsic quadric surfaces into finite subclasses by bounding the Gorenstein index. This allows, for instance, counting results as the following.

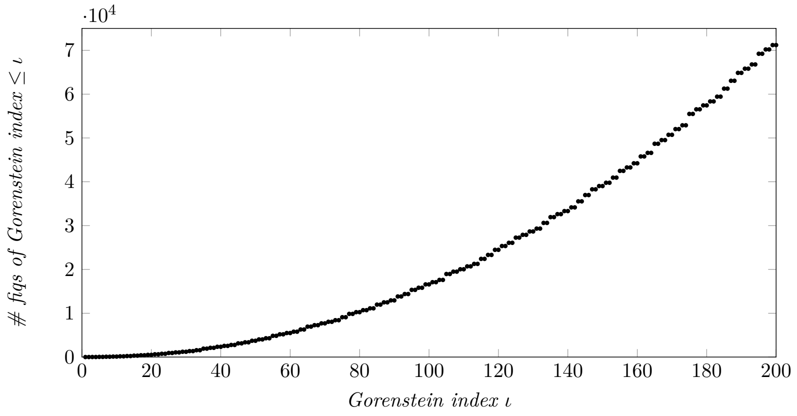
Theorem 5. *Up to isomorphy, there are precisely 15 538 339 full intrinsic quadric surfaces of Gorenstein index at most 200.*

(i) *In Picard number one, we find in total 883 full intrinsic quadric surfaces of Gorenstein index at most 200, filtered as follows:*



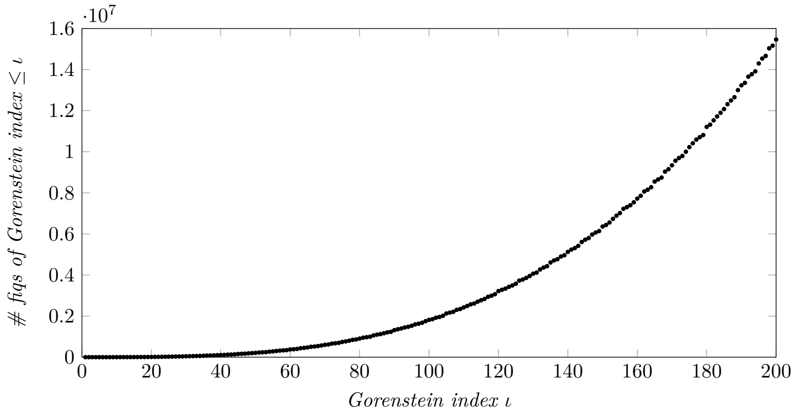
Exactly 150 full intrinsic quadric complex surfaces of Picard number one and Gorenstein index at most 200 admit a Kähler-Einstein metric.

(ii) *In Picard number two, we find in total 71 198 full intrinsic quadric surfaces of Gorenstein index at most 200, filtered as follows:*



In Picard number two, there are no full intrinsic quadric complex surfaces at all admitting a Kähler-Einstein metric.

(iii) *In Picard number three, we find in total 15 466 258 full intrinsic quadric surfaces of Gorenstein index at most 200, filtered as follows:*



Exactly 1 006 633 full intrinsic quadric complex surfaces of Picard number three and Gorenstein index at most 200 admit a Kähler-Einstein metric.

Finally, we consider anticanonical embeddings of Kähler-Einstein full intrinsic quadric surfaces. Our main result settles the case of Picard number one and odd Gorenstein index; observe that assertion (i) of the following theorem reproduces the item number two in the list of [33, Thm. 8].

Theorem 6. *Let X be a full intrinsic quadric surface of Picard number one and of Gorenstein index ι admitting a Kähler-Einstein metric.*

(i) *If $\iota = 1$, then X is anticanonically embedded as a surface into $\mathbb{P}(1,1,1,2)$ with defining equation g of the degree 4 given in homogeneous coordinates as*

$$g \; = \; -U_1U_2U_4 + U_3^4 + 2U_3^2U_4 + U_4^2.$$

(ii) *If $\iota \geq 3$ is odd, then X is anticanonically embedded as a complete intersection into $\mathbb{P}(1,2,2,\iota,\iota)$ with two defining equations, g of degree 4 and h degree 2ι, given in homogeneous coordinates as*

$$\begin{aligned} g &= U_1^4 - U_2U_3, \\ h &= \Big(2U_1^2 + U_2 + U_3\Big) \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} U_2^k U_3^{\iota-1-k} \\ &\quad - (U_2 - U_3)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} U_2^k U_3^{\iota-2-k} - U_4U_5. \end{aligned}$$

CHAPTER

ONE

$\mathbb{K}^*$-SURFACES OF PICARD NUMBER ONE AND BACKGROUND

We gather the necessary (known) background from toric geometry and torus actions of complexity one. Section 1.4 and 1.5 contain also new results, parts of which have been made public in Section 3 of the joint article [27] with Milena Wrobel and Jürgen Hausen.

1.1 Toric geometry

Throughout the whole thesis $\mathbb{K}$ denotes an algebraically closed field of characteristic zero. In this section we briefly gather the necessary background from toric geometry, in particular the Cox ring and Cox's quotient construction for toric varieties. We refer to [3, 10, 11, 13] as introductory texts.

The *standard n-torus* is the direct product $\mathbb{T}^n = (\mathbb{K}^*)^n$. By a homomorphism of tori $\mathbb{T}^n \to \mathbb{T}^m$ we mean a group homomorphism, which is also a morphism of varieties. For instance, the *characters* χ^u on $\mathbb{T}^n$ and the *one-parameter groups* λ_v of $\mathbb{T}^n$ are precisely the homomorphisms

$$\begin{array}{llcllcl} \chi^u\colon \mathbb{T}^n \longrightarrow \mathbb{K}^*, & (t_1,\ldots,t_n) & \mapsto & t_1^{u_1}\cdots t_n^{u_n}, & u=(u_1,\ldots,u_n) & \in & \mathbb{Z}^n, \\ \lambda_v\colon \mathbb{K}^* \longrightarrow \mathbb{T}^n, & t & \mapsto & (t^{v_1},\ldots,t^{v_n}), & v=(v_1,\ldots,v_n) & \in & \mathbb{Z}^n. \end{array}$$

A *toric variety* is an irreducible variety Z together with an algebraic action $\mathbb{T}^n \times Z \to Z$ and a *base point* $z_0 \in Z$ such that the orbit map $\mathbb{T}^n \to Z,\ t \mapsto t\cdot z_0$ is an open embedding. Unless otherwise stated, we assume a toric variety to be *normal.* A *toric morphism* of toric varieties Z and Z' is a pair $(\varphi,\tilde{\varphi})$, where $\varphi : Z \to Z'$ is a morphism with $\varphi(z_0) = z_0'$ and $\tilde{\varphi}\colon \mathbb{T}^n \to \mathbb{T}^m$ is a homomorphism of tori such that $\varphi(t\cdot z) = \tilde{\varphi}(t)\cdot\varphi(z)$ holds for all $t \in \mathbb{T}^n$ and $z \in Z$.

By a *cone* in $\mathbb{Z}^n$ we mean a convex polyhedral cone $\sigma \subseteq \mathbb{Q}^n$ (also $\sigma \subseteq \mathbb{R}^n$). We denote the dual cone of $\sigma \subseteq \mathbb{Q}^n$ by $\sigma^\vee \subseteq \mathbb{Q}^n$. With the additive monoid $\sigma^\vee \cap \mathbb{Z}^n \subseteq \mathbb{Z}^n$ one associates the $\mathbb{K}$-algebra

$$\mathbb{K}[\sigma^\vee \cap \mathbb{Z}^n] \ := \ \bigoplus_{u \in \sigma^\vee \cap \mathbb{Z}^n} \mathbb{K}\chi^u, \qquad \chi^u \cdot \chi^{u'} \ := \ \chi^{u+u'}.$$

A *fan* in $\mathbb{Z}^n$ is a finite collection Σ of pointed, convex, polyhedral cones in $\mathbb{Z}^n$ such that for all $\sigma \in \Sigma$ and all faces $\tau \preccurlyeq \sigma$, we have $\tau \in \Sigma$ and for any two $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'. A map of fans Σ in $\mathbb{Z}^n$ and Σ' in $\mathbb{Z}^m$ is a homomorphism $F\colon \mathbb{Z}^n \to \mathbb{Z}^m$ such that for every $\sigma \in \Sigma$ there is a $\sigma' \in \Sigma'$ with $F(\sigma) \subseteq \sigma'$. We denote by $\Sigma^{\max}$ the set of all maximal cones of Σ.

Note that we have natural identifications $\mathbb{K}[\mathbb{Z}^n] = \mathcal{O}(\mathbb{T}^n)$ and thus $\mathbb{T}^n = \operatorname{Spec} \mathbb{K}[\mathbb{Z}^n]$. For a cone σ in $\mathbb{Z}^n$, the associated affine toric variety is $Z_\sigma := \operatorname{Spec} \mathbb{K}[\sigma^\vee \cap \mathbb{Z}^n]$ with the base point $z_0 \in Z_\sigma$ determined by $\chi^u(z_0) = 1$ for all $u \in \sigma^\vee$. The $\mathbb{T}^n$-action $\mu\colon \mathbb{T}^n \times Z_\sigma \to Z_\sigma$ is given by its comorphism

$$\mu^*\colon \mathbb{K}[\sigma^\vee \cap \mathbb{Z}^n] \ \longrightarrow \ \mathbb{K}[\sigma^\vee \cap \mathbb{Z}^n] \otimes \mathbb{K}[\mathbb{Z}^n], \qquad \chi^u \ \mapsto \ \chi^u \otimes \chi^u.$$

For a fan Σ in $\mathbb{Z}^n$, the associated toric variety Z arises from gluing the affine Z_σ, $\sigma \in \Sigma$, along the open toric subvarieties $Z_{\sigma \cap \sigma'} \subseteq Z_\sigma, Z_{\sigma'}$. This process respects the $\mathbb{T}^n$-action and the base points. The construction is functorial in the sense that every morphism of fans defines a toric morphism between the associated toric varieties.

We call a matrix $P \in \operatorname{Mat}(n, r; \mathbb{Z})$ a *(projective) generator matrix* if its columns are pairwise different, primitive and span $\mathbb{Q}^n$ as a vector space (as a convex cone). For a toric variety Z associated with a fan Σ in $\mathbb{Z}^n$ with primitive ray generators $v_1, \ldots, v_r$ the following statements are equivalent:

(i) $P = [v_1, \ldots, v_r]$ is a generator matrix.
(ii) The fan Σ is non-degenerate.
(iii) We have $\mathcal{O}(Z)^* = \mathbb{K}^*$.
(iv) The variety Z has no torus factor.

Here, a fan Σ in $\mathbb{Z}^n$ (and its associated toric variety Z) is called *non-degenerate* if its cones are not contained in a proper vector subspace of $\mathbb{Q}^n$. If one of the statements hold in the above situation, then we say that Σ or that Z has P as a generator matrix.

Let Z be the toric variety arising from a fan Σ in $\mathbb{Z}^n$. With every cone $\sigma \in \Sigma$, one associates the (well-defined) *limit point*

$$z_\sigma \ := \ \lim_{t \to 0} \lambda_v(t) \cdot z_0 \in X, \ \text{ where } v \in \sigma^\circ \cap \mathbb{Z}^n.$$

The dimension of $\mathbb{T}^n \cdot z_\sigma$ equals $n - \dim(\sigma)$. In particular, z_σ is a fixed point of the $\mathbb{T}^n$-action for every full dimensional cone σ. The rays $\varrho_1, \ldots, \varrho_r$ of Σ, that means the one-dimensional cones, correspond to the $\mathbb{T}^n$ *-invariant prime divisors* via $D_i := \overline{\mathbb{T}^n \cdot z_{\varrho_i}}$.

Recall that the *divisor class group* of a normal, irreducible variety X is the factor group of all Weil divisors modulo all principal divisors:

$$\mathrm{Cl}(X) \ := \ \mathrm{WDiv}(X) \ / \ \mathrm{PDiv}(X).$$

Proposition 1.1.1 (See [3, Sec. 2.1.3], [10, Sec. 5])**.** *Let Z be a toric variety with generator matrix $P \in \mathrm{Mat}(n, r; \mathbb{Z})$. Moreover, consider the projection*

$$Q\colon \mathbb{Z}^r \longrightarrow K, \qquad\qquad K \ := \ \mathbb{Z}^r/\mathrm{im}(P^*),$$

where P^* is the transpose of P and $e_i \in \mathbb{Z}^r$ are the i-th canonical basis vector. Then the divisor class group of Z and the classes of the $\mathbb{T}^n$-invariant divisors D_i are

$$\mathrm{Cl}(Z) \ = \ K, \qquad\qquad [D_i] \ = \ w_i \ := \ Q(e_i).$$

We briefly recall the construction of the Cox ring of a normal irreducible variety X with only constant invertible global functions and finitely generated divisor class group. We refer to [4, 19] and [3, Sec. 1.4, Sec. 2.1.3] for more background. For a Weil divisor D on X, denote by $\mathcal{O}(D)$ the associated sheaf of sections. Then the *Cox sheaf* $\mathcal{R}$ and the *Cox ring* $\mathcal{R}(X)$ of X are defined as

$$\mathcal{R} \ := \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{O}(D), \qquad\qquad \mathcal{R}(X) \ := \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

Proposition 1.1.2 (See [9], also [3, Sec. 2.1.3], [10, Sec. 5])**.** *Let Z be a toric variety arising from a fan Σ with generator matrix $P \in \mathrm{Mat}(n, r; \mathbb{Z})$. Further, let $D_1, \ldots, D_r$ be the $\mathbb{T}^n$-invariant prime divisors of Z. Then the Cox ring of Z and its $\mathrm{Cl}(Z)$-grading are given as*

$$\mathcal{R}(Z) \ = \bigoplus_{[D] \in \mathrm{Cl}(Z)} \mathbb{K}[T_1, \ldots, T_r]_{[D]} \ = \ \mathbb{K}[T_1, \ldots, T_r], \qquad \deg(T_i) \ = \ [D_i] \ \in \ \mathrm{Cl}(Z).$$

Consider again a normal irreducible variety X with only constant invertible global functions and finitely generated divisor class group. If the Cox ring $\mathcal{R}(X)$ is finitely generated then we obtain a diagram

$$\begin{tikzcd}
\mathrm{Spec}_X \mathcal{R} \arrow[r, phantom, "=:"] & \hat{X} \arrow[d, "p"', "/\!\!/ H_X"] \arrow[r, phantom, "\subseteq"] & \bar{X} \arrow[r, phantom, ":="] & \mathrm{Spec}\, \mathcal{R}(X) \\
& X & &
\end{tikzcd}$$

Here $\bar{X}$ is the *total coordinate space* and $H_X := \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$ is the *characteristic quasitorus* of X; recall that a *quasitorus* is an algebraic group isomorphic to a direct product $\mathbb{T}^n \times G$ with G finite abelian. The $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$ defines an action of H_X on $\bar{X}$. The relative spectrum $\hat{X}$ is an invariant open subvariety of the spectrum $\bar{X}$ and the canonical morphism $p\colon \hat{X} \to X$, called the *characteristic space* over X, is a good quotient for this action. We refer to [3, Sec. 1.6] for more background.

Proposition 1.1.3 (See [9], also [3, Sec. 2.1.3], [10, Sec. 5])**.** *Let Z be a toric variety arising from a fan Σ with generator matrix $P = (p_{ij}) \in \mathrm{Mat}(n, r; \mathbb{Z})$. Then P defines a homomorphism*

$$p\colon \mathbb{T}^r \longrightarrow \mathbb{T}^n, \qquad t \mapsto (t_1^{p_{11}} \cdots t_r^{p_{1r}}, \ldots, t_1^{p_{n1}} \cdots t_r^{p_{nr}}).$$

With the fan $\hat{\Sigma} := \{\delta_0 \preceq \mathbb{Q}^r_{\geq 0};\ P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}$, its associated toric variety $\hat{Z}$ and the toric morphism $p\colon \hat{Z} \to Z$ defined by the map $P\colon \mathbb{Z}^r \to \mathbb{Z}^n$ of the fans $\hat{\Sigma}$ and Σ, we arrive at the diagram

$$\begin{tikzcd} \hat{Z} \arrow[r, phantom, "\subseteq"] \arrow[d, "p"', "/\!\!/ H_Z"] & \bar{Z} \arrow[r, phantom, "="] & \mathbb{K}^n \\ Z & & \end{tikzcd}$$

where $\bar{Z}$ is the total coordinate space of Z, we have $H_Z = \ker(p) \subseteq \mathbb{T}^r$ for the characteristic quasitorus of Z and $p\colon \hat{Z} \to Z$ is the characteristic space over Z.

Remark 1.1.4 (See [9], also [3, Prop. 2.1.3.4], [10, Sec. 5])**.** Let Z be a toric variety with quotient presentation $p\colon \hat{Z} \to Z$ as in Proposition 1.1.3. Then every p-fiber contains a unique closed H_Z-orbit. The presentation in *Cox coordinates* of a point $x \in Z$ is

$$x \;=\; [z_1, \ldots, z_r], \qquad \text{where } z = (z_1, \ldots, z_r) \in p^{-1}(x) \text{ with } H_Z \cdot z \subseteq \hat{Z} \text{ closed.}$$

Thus, $[z]$ and $[z']$ represent the same point $x \in Z$ if and only if z and z' lie in the same closed H_Z-orbit of $\hat{Z}$. For instance, the limit points $z_\sigma \in Z$, where $\sigma \in \Sigma$, are given in Cox coordinates as

$$z_\sigma \;=\; [\varepsilon_1, \ldots, \varepsilon_r], \qquad \varepsilon_i \;=\; \begin{cases} 0, & P(e_i) \in \sigma, \\ 1, & P(e_i) \notin \sigma. \end{cases}$$

Let X again be a normal irreducible variety. The *local class group* $\mathrm{Cl}(X, x)$ of a point x of X is the group of Weil divisors modulo the subgroup of divisors that are principal on some open neighbourhood of x. We denote by $\mathrm{cl}(x)$ the order of $\mathrm{Cl}(X, x)$. A *Cartier divisor* is a Weil divisor that is locally principal. Moreover, the *Picard group* of X is the group of Cartier divisors modulo the subgroup of principal divisors:

$$\mathrm{Pic}(X) \;:=\; \mathrm{CDiv}(X) \;/\; \mathrm{PDiv}(X).$$

Proposition 1.1.5 (See [3, Prop. 2.4.2.3, Cor. 2.4.2.4])**.** *Let Z be a toric variety arising from a fan Σ with generator matrix $P \in \mathrm{Mat}(n, r; \mathbb{Z})$. Further, set*

$$K \;:=\; \mathbb{Z}^r / \mathrm{im}(P^*), \qquad K_\sigma \;:=\; \langle Q(e_i);\ P(e_i) \notin \sigma \rangle$$

and denote by $Q\colon \mathbb{Z}^r \to K$ the canonical projection. Then the local class groups of the limit points $z_\sigma \in Z$ and the Picard group of Z are given by

$$\mathrm{Cl}(Z, z_\sigma) \;=\; K / K_\sigma, \qquad \mathrm{Pic}(Z) \;=\; \bigcap_{\sigma \in \Sigma} K_\sigma \;\subseteq\; \mathrm{Cl}(Z).$$

We say that a cone σ in $\mathbb{Z}^n$ is *simplicial* if it is generated by a linearly independent family $v_1, \dots, v_r \in \mathbb{Z}^n$. Moreover, a cone σ in $\mathbb{Z}^n$ is called *regular* if it is generated by a family $v_1, \dots, v_r \in \mathbb{Z}^n$ that can be completed to a basis of $\mathbb{Z}^n$.

Recall that a point $x \in X$ of a normal, irreducible variety X is $\mathbb{Q}$-*factorial* if every Weil divisor admits a non-zero multiple that is principal near x. Moreover, X is called $\mathbb{Q}$-*factorial* if each of its points is $\mathbb{Q}$-factorial.

Proposition 1.1.6 (See [10, Thm. 3.1.19])**.** *Let Z be a toric variety arising from a non-degenerate fan Σ, having $P \in \operatorname{Mat}(n, r; \mathbb{Z})$ as generator matrix and consider the limit points $z_\sigma \in Z$, where $\sigma \in \Sigma$. Then the following holds.*

(i) *The point $z_\sigma \in Z$ is $\mathbb{Q}$-factorial if and only if σ is simplicial.*
(ii) *The point $z_\sigma \in Z$ is smooth if and only if σ is regular.*

Construction 1.1.7 (Fake weighted projective spaces)**.** Consider a projective generator matrix $P = [v_0, \dots, v_r] \in \operatorname{Mat}(r, r+1; \mathbb{Z})$. Further, let Σ be the fan in $\mathbb{Z}^r$ with maximal cones $\sigma_i := \operatorname{cone}(v_j;\, j \neq i)$. Then we call the associated toric variety $Z = Z(P)$ a *fake weighted projective space.*

Remark 1.1.8 (See also [27, Rem. 2.3])**.** Consider $P = [v_0, \dots, v_r]$ as in Construction 1.1.7 and the associated fake weighted projective space Z. The *fake weight vector* associated with P is

$$w \;=\; w(P) \;=\; (w_0, \dots, w_r) \in \mathbb{Z}^{r+1}_{>0}, \qquad w_i \;:=\; |\det(v_j;\, j = 0, \dots, r,\, j \neq i)|.$$

For the divisor class group and the local class groups of the toric fixed points $z(i) \coloneqq z_{\sigma_i}$, having i-th homogeneous coordinate one and all others zero, we obtain

$$\operatorname{Cl}(Z) \;=\; \mathbb{Z}^r/\operatorname{im}(P^*) \cong \mathbb{Z} \oplus \Gamma, \qquad |\Gamma| \;=\; \gcd(w_0, \dots, w_r), \qquad \operatorname{cl}(z(i)) \;=\; w_i;$$

see [3, Prop. 2.1.4.1]. Moreover, with the homomorphism $p\colon \mathbb{T}^{r+1} \to \mathbb{T}^r$ from Proposition 1.1.3, the characteristic quasitorus of Z is

$$H_Z \;=\; \ker(p) \;\cong\; \mathbb{K}^* \oplus G,$$

where the splitting $H_Z = \mathbb{K}^* \oplus G$ on the right hand side is induced by the splitting $\operatorname{Cl}(Z) \cong \mathbb{Z} \oplus \Gamma$ of character groups. Moreover, the Cox coordinates for the toric fixed points in $Z(P)$ are as given in Remark 1.1.4, i.e.

$$z(0) \;\coloneqq\; [1, 0, \dots, 0], \quad z(1) \;\coloneqq\; [0, 1, 0, \dots, 0], \quad \dots, \quad z(n) \;\coloneqq\; [0, \dots, 0, 1].$$

Remark 1.1.9. Consider projective generator matrices P, P' and the associated fake weighted projective spaces Z, Z'. Then one has $Z \cong Z'$ if and only if $P' = S \cdot P \cdot T$ holds with a unimodular matrix S and a permutation matrix T.

Remark 1.1.10 (See also [27, Rem. 2.4])**.** Let Z arise from Construction 1.1.7 and $w = w(P)$ be as in Remark 1.1.8. Then $\operatorname{Cl}(Z)$ is torsion free if and only if $w \in \mathbb{Z}^{r+1}$ is primitive. In the latter case, Z equals the *weighted projective space* $\mathbb{P}(w_0, \dots, w_r)$.

Remark 1.1.11. According to Proposition 1.1.5 and Proposition 1.1.6 the fake weighted projective spaces are precisely the $\mathbb{Q}$-factorial projective toric varieties of Picard number one.

Definition 1.1.12. Consider a finitely generated abelian group $K = \mathbb{Z}^m \oplus \Gamma$, whith Γ finite. A *degree matrix* in K is a matrix

$$Q \;=\; \begin{bmatrix} \omega_1 & \dots & \omega_r \\ \eta_1 & \dots & \eta_r \end{bmatrix}, \qquad \omega_i \in \mathbb{Z}^m, \quad \eta_i \in \Gamma$$

such that any $r-1$ of the pairs $(\omega_i, \eta_i) \in \mathbb{Z}^m \oplus \Gamma$ generate $K = \mathbb{Z}^m \oplus \Gamma$ as an abelian group.

Remark 1.1.13. Consider a generator matrix $P \in \operatorname{Mat}(n, r; \mathbb{Z})$, the factor group $K := \mathbb{Z}^r / \operatorname{im}(P^*)$, the projection $Q\colon \mathbb{Z}^r \to K$ and let $K = \mathbb{Z}^m \oplus \Gamma$ be a splitting with Γ finite. Then each $w_i = Q(e_i)$ defines a pair (ω_i, η_i) in $\mathbb{Z}^m \oplus \Gamma$ and we can regard Q as a degree matrix in K

$$Q \;=\; \begin{bmatrix} \omega_1 & \dots & \omega_r \\ \eta_1 & \dots & \eta_r \end{bmatrix}.$$

Example 1.1.14. We discuss the three dimensional fake weighted projective space given by the following 3×4 generator matrix

$$P \;=\; [v_1, v_2, v_3, v_4] \;=\; \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{bmatrix}.$$

We associate with P the complete fan Σ in $\mathbb{Z}^3$ having the four maximal cones

$$\sigma_i \;:=\; \operatorname{cone}(v_j;\, j \neq i), \quad i \;=\; 1, 2, 3, 4.$$

Then the corresponding toric variety Z has P as its generator matrix. By Proposition 1.1.1 the divisor class group and the degree matrix $Q\colon \mathbb{Z}^4 \longrightarrow \operatorname{Cl}(Z)$ are given by

$$\operatorname{Cl}(Z) \;=\; \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \qquad Q \;=\; [w_1, w_2, w_3, w_4] \;=\; \begin{bmatrix} 1 & 3 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}.$$

Proposition 1.1.2 delivers the Cox ring

$$\mathcal{R}(Z) \;=\; \mathbb{K}[T_1, T_2, T_3, T_4], \qquad \deg(T_i) \;=\; w_i.$$

Further, the homomorphism of tori defined by the matrix P is given by

$$p\colon \mathbb{T}^4 \;\longrightarrow\; \mathbb{T}^3, \qquad (t_1, t_2, t_3, t_4) \;\mapsto\; \left(\tfrac{t_3^2}{t_1 t_2}, \tfrac{t_4^4}{t_1 t_2}, \tfrac{t_3 t_4}{t_2}\right).$$

The total coordinate space is $\bar{Z} = \mathbb{K}^4$, the map p extends to a morphism $p\colon \hat{Z} = \mathbb{K}^4 \backslash \{0\} \to Z$ and the characteristic quasitorus is given by $H_Z = \mathbb{K}^* \oplus \{\pm 1\}$.

1.2 Torus actions of complexity one

We recall the necessary background on torus actions of complexity one. The *complexity* of an effective algebraic torus action $\mathbb{T}^m \times X \to X$ on a variety X is the difference $\dim(X) - m$. We refer to [20, 28] for the treatment of normal rational complete varieties with a torus action of complexity one; see also [3, Sec. 3.4]. Here, we follow the approach from [29, Constr. 1.6, Type 2] which also covers semiprojective varieties. We adapt the construction to our setting where the Cox ring admits only one trinomial relation and has no free variables.

Construction 1.2.1 (See [29, Constr. 1.6, Type 2, Thm. 1.7]). Fix $n_0, n_1, n_2 \in \mathbb{Z}_{\geq 1}$ and $0 < s < n-2$, where $n := n_0 + n_1 + n_2$. Consider a generator matrix of the form

$$P \;=\; \begin{bmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \\ d_0 & d_1 & d_2 \end{bmatrix}, \qquad \begin{array}{rcl} l_i & = & (l_{i1}, \ldots, l_{in_i}),\ l_{ij} \in \mathbb{Z}_{\geq 1}, \\ \\ d_i & = & (d_{i1}, \ldots, d_{in_i}),\ d_{ij} \in \mathbb{Z}^s, \end{array}$$

We denote by $v_{ij} \in \mathbb{Z}^{2+s}$ the columns of P. Further, let Z be a toric variety with generator matrix P. We define

$$X \;:=\; \overline{V_{\mathbb{T}^{2+s}}(h)} \;\subseteq\; Z, \qquad h \;:=\; 1 + S_1 + S_2 \;\in\; \mathcal{O}\left(\mathbb{T}^{2+s}\right),$$

where S_1, S_2 are the first two coordinate functions on $\mathbb{T}^{2+s}$. Since h doesn't depend on the last s coordinates, the variety X admits a $\mathbb{T}^s$-action on $X \cap \mathbb{T}^{2+s}$ given by

$$t \cdot x \;=\; (1, 1, t_1, \ldots, t_s) \cdot x.$$

Moreover, the variety X is rational, normal, the $\mathbb{T}^s$-action is effective and is of complexity one. If X is projective, then the v_{ij} must generate $\mathbb{Q}^{2+s}$ as a cone.

Remark 1.2.2 (See [22, Prop. 7.8]). Let $X \subseteq Z$ be as in Construction 1.2.1. Assume that Z arises from a fan Σ in $\mathbb{Z}^{2+s}$. Then, we have $\mathbb{T}^{2+s} \cdot z_\sigma \cap X \neq \emptyset$ for a cone $\sigma \in \Sigma$ if and only if one of the following holds:

(i) We have $v_{0j_0}, v_{1j_1}, v_{2j_2} \in \sigma$ for some $1 \leq j_i \leq n_i$.
(ii) We have $\sigma \subseteq \operatorname{cone}(v_{i1}, \ldots, v_{in_i})$ for exactly one $i \in \{0, 1, 2\}$.

The union over all affine toric charts $Z_\sigma \subseteq Z$, where $\sigma \in \Sigma$ is a maximal cone such that $\mathbb{T}^{2+s} \cdot z_\sigma \cap X \neq \emptyset$, is called the *minimal toric ambient variety* of X; it is the minimal open toric subvariety of Z such that X as a closed subvariety.

Definition 1.2.3. Let P and h be as in Construction 1.2.1 and $p\colon \mathbb{T}^n \to \mathbb{T}^{2+s}$ the homomorphism of tori given by P as in Proposition 1.1.3. The *P-homogenization* of h is the polynomial

$$g \;:=\; T_0^{l_0} p^* h \;=\; T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \qquad \text{where } T_i^{l_i} \;:=\; T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Proposition 1.2.4 (See [3, Constr. 3.2.5.3, Prop. 3.2.5.4])**.** *Let $X \subseteq Z$ and P be as in Construction 1.2.1 and let $K = \mathbb{Z}^n/\mathrm{im}(P^*)$. Then there are natural identifications*

$$\mathrm{Cl}(X) \ = \ K \ = \ \mathrm{Cl}(Z).$$

Each invariant prime divisor D_{ij}^Z in Z defines a prime divisor $D_{ij}^X := X \cap D_{ij}^Z$ in X. With the projection $Q\colon \mathbb{Z}^n \to K$ and $w_{ij} = Q(e_{ij})$, the class of D_{ij}^X is given by

$$[D_{ij}^X] \ = \ w_{ij} \ = \ [D_{ij}^Z].$$

Moreover, with the polynomial ring $\mathbb{K}[T_{ij}] := \mathbb{K}[T_{ij};\, i = 0,1,2,\, j = 1,\ldots,n_i]$ and the P-homogenization $g := T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$ of h, the Cox ring of X is given by

$$\mathcal{R}(X) \ \cong \ \mathbb{K}[T_{ij}]/\langle g\rangle, \quad \deg(T_{ij}) \ = \ [D_{ij}^X].$$

In particular, $\mathcal{R}(X) \cong \mathcal{R}(Z)/\langle g\rangle$. Moreover with $\bar{X} := V(g)$ in $\bar{Z} := \mathbb{K}^n$ we obtain a commutative diagram involving the total coordinate spaces and the characteristic spaces

$$\begin{tikzcd}
V(g) \arrow[r, phantom, "="] & \bar{X} \arrow[r, phantom, "\subseteq"] & \bar{Z} \arrow[r, phantom, "="] & \mathbb{K}^n \\
& \hat{X} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] \arrow[d, "/\!\!/H_X"', "p"] & \hat{Z} \arrow[u, phantom, "\cup" rotate=90] \arrow[d, "p"', "/\!\!/H_Z"] & \\
& X \arrow[r] & Z &
\end{tikzcd}$$

where the characteristic quasitorus H_X and the the characteristic quasitorus H_Z both are equal to $\mathrm{Spec}\,\mathbb{K}[K]$.

We call X *quasismooth* if X is $\mathbb{Q}$-factorial and if $\hat{X}$ is smooth.

Example 1.2.5. Let $n_0 = n_1 = n_2 = 2$, $n = 6$ and let $s = 1$. We discuss the complexity one variety given by the following 3×6 generator matrix

$$P \ = \ \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -3 & 0 & -4 & 0 & -1 \end{bmatrix}.$$

We associate with P the fan Σ in $\mathbb{Z}^3$ having the five maximal cones

$$\sigma_{135} \ := \ \mathrm{cone}\,(v_{01}, v_{11}, v_{21})\,, \qquad \sigma_{246} \ := \ \mathrm{cone}\,(v_{02}, v_{12}, v_{22})\,,$$

$$\tau_{12} \ := \ \mathrm{cone}\,(v_{01}, v_{02})\,, \qquad \tau_{34} \ := \ \mathrm{cone}\,(v_{11}, v_{12})\,, \qquad \tau_{56} \ := \ \mathrm{cone}\,(v_{21}, v_{22})\,.$$

Then the corresponding toric variety Z has P as its generator matrix and we obtain a rational, normal, projective variety X given by

$$X \ := \ \overline{V_{\mathbb{T}^3}(1 + S_1 + S_2)}.$$

The five maximal cones $\sigma \in \Sigma$ from above fulfill $\mathbb{T}^{2+s} \cdot z_\sigma \cap X \neq \emptyset$. Hence Z is the minimal toric ambient variety of X. According to Proposition 1.2.4, we obtain the divisor class group as

$$\mathrm{Cl}(X) \;=\; \mathrm{Cl}(Z) \;=\; \mathbb{Z}^6/\mathrm{im}(P^*) \;=\; \mathbb{Z}^3$$

and the projection $Q\colon \mathbb{Z}^6 \to \mathbb{Z}^3$ yields a degree matrix

$$Q \;=\; [w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{22}] \;=\; \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 4 & -2 \end{bmatrix}.$$

The homomorphism of tori defined by the matrix P is given by

$$p\colon \mathbb{T}^6 \;\longrightarrow\; \mathbb{T}^3, \qquad (t_{01},\, t_{02},\, t_{11},\, t_{12},\, t_{21}, t_{22}) \;\mapsto\; \left(\tfrac{t_{11}t_{12}}{t_{01}t_{02}},\; \tfrac{t_{21}t_{22}}{t_{01}t_{02}}\; \tfrac{t_{01}}{t_{02}^3 t_{12}^4 t_{22}}\right).$$

Then the P-homogenization of $1 + S_1 + S_2$ is $g \coloneqq T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}$. Furthermore, the Cox ring has a representation

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{22}]/\langle g \rangle, \qquad \deg(T_{ij}) \;=\; w_{ij}.$$

Lastly, with $H \coloneqq \ker(p) = \mathbb{T}^3$, we are provided with the commutative diagram

$$\begin{tikzcd}
V(T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}) \arrow[r, phantom, "="] & \bar{X} \arrow[r, phantom, "\subseteq"] & \bar{Z} \arrow[r, phantom, "="] & \mathbb{K}^6 \\
\bar{X} \setminus \{0\} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, phantom, "="] & \hat{X} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] \arrow[d, "/\!\!/H"', "p"] & \hat{Z} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, phantom, "="] \arrow[d, "p"', "/\!\!/H"] & \mathbb{K}^6 \setminus \{0\}. \arrow[u, phantom, sloped, "\subseteq"] \\
 & X \arrow[r] & Z &
\end{tikzcd}$$

Remark 1.2.6. Due to Proposition 1.2.4, we have Cox coordinates on any $X \subseteq Z$ arising from Construction 1.2.1.

Proposition 1.2.7 (See [3, Prop. 2.4.2.3, Cor. 2.4.2.4])**.** *Let $X \subseteq Z$, where Z arises from a fan Σ in $\mathbb{Z}^{2+s}$, be as in Construction 1.2.1. Consider $x \in X \cap \mathbb{T}^{2+s} \cdot z_\sigma$, where $\sigma \in \Sigma$.*

(i) *The local class group of x is given by $\mathrm{Cl}(X, x) = \mathrm{Cl}(Z, z_\sigma)$.*
(ii) *The point $x \in X$ is $\mathbb{Q}$-factorial if and only if $z_\sigma \in Z$ is $\mathbb{Q}$-factorial.*

Moreover, if Z is the minimal toric ambient variety, then the Picard group of X is given as $\mathrm{Pic}(X) = \mathrm{Pic}(Z)$.

Construction 1.2.8 (See [3, Sec. 4.4], [22, Prop. 3.16])**.** Let $X \subseteq Z$ be as in Construction 1.2.1 and assume that Z arises from a fan Σ in $\mathbb{Z}^{2+s}$. We denote by Σ_0 the fan consisting only of the rays of Σ, that is

$$\Sigma_0 \;\coloneqq\; \{\varrho_{ij};\, i = 0, 1, 2,\, j = 1, \ldots, n_i\} \cup \{\{0\}\}.$$

and denote by Z_0 the associated toric variety. Further, we set $X_0 := Z_0 \cap X$. Then $Z_0 \subseteq Z$ is invariant under the torus $\mathbb{T}^{2+s}$ and $X_0 \subseteq X$ is invariant under $\mathbb{T}^s$. Denote by $\Sigma_{\mathbb{P}_2}$ the fan of the projective space $\mathbb{P}_2$. We define a map F of fans Σ_0 in $\mathbb{Z}^{2+s}$ and $\Sigma_{\mathbb{P}_2}$ in $\mathbb{Z}^2$ given by

$$F\colon \mathbb{Z}^{2+s} \longrightarrow \mathbb{Z}^2, \qquad z \mapsto (z_1, z_2).$$

This map defines a toric morphism between the associated toric varieties $\pi_Z\colon Z_0 \to \mathbb{P}_2$ given on the tori by $t \mapsto (t_1, t_2)$. Moreover, we obtain a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "/\!\!/\mathbb{T}^s"', "\pi_X"] & Z_0 \arrow[d, "\pi_Z"', "/\!\!/\mathbb{T}^s"] \\
V_{\mathbb{P}_2}(U_0 + U_1 + U_2) \cong \mathbb{P}_1 \arrow[r] & \mathbb{P}_2,
\end{tikzcd}$$

where $\pi_X := \pi_Z|_X$ and π_Z are categorical quotients with respect to the actions of $\mathbb{T}^s$ on X and Z respectively. For $c_0 = [1, 0], c_1 = [0, 1]$ and $c_2 = [-1, -1]$ we obtain

$$\overline{\pi_X^{-1}(c_i)} \;=\; \bigcup_{j=1}^{n_i} D_{ij}^X \subseteq X, \qquad \overline{\pi_Z^{-1}(C_i)} \;=\; \bigcup_{j=1}^{n_i} D_{ij}^Z \subseteq Z$$

with the toric $\mathbb{T}^2$-invariant divisors $C_0, C_1, C_2 \subseteq \mathbb{P}_2$. The points $c_0, c_1, c_2 \in \mathbb{P}_1$ are the *critical values* of the quotient map π_X. The map π_X is the *maximal orbit quotient* of X from [22].

Theorem 1.2.9 (See [20, 28])**.** *Let X be a normal, semiprojective, rational variety with a torus action of complexity one and with $\mathcal{O}(X)^* = \mathbb{K}^*$. Further, let $\pi_X\colon X_0 \dashrightarrow \mathbb{P}_1$ be its maximal orbit quotient. If π_X admits exactly three critical values then X is equivariantly isomorphic to a variety arising from Construction 1.2.1.*

Remark 1.2.10 (See [3, Prop. 3.3.3.2, Prop. 3.4.4.1])**.** Let $X \subseteq Z$ be as in Construction 1.2.1 and consider its Cox ring as provided by Proposition 1.2.4. Then for $k = 0, 1, 2$ the anticanonical divisor class of X is given as

$$w_X \;=\; \sum_{i,j} \deg(T_{ij}) - \sum_{j=1}^{n_k} l_{kj} \deg(T_{kj}) \;\in\; \mathrm{Cl}(X).$$

Proposition 1.2.11 (See [18, Prop. 2.17])**.** *Let $X \subseteq Z$ be as in Construction 1.2.1 and let Z be projective. Moreover, consider ample divisors D^Z on Z and D^X on X given by*

$$D^Z \;=\; \sum_{i,j} \alpha_{ij} D_{ij}^Z, \qquad D^X \;=\; \sum_{i,j} \alpha_{ij} D_{ij}^X.$$

Then the associated affine cones $\tilde{X} \subseteq \tilde{Z}$ over $X \subseteq Z$ arise via Construction 1.2.1 from a generator matrix $\tilde{P}$ with $s' := s + 1$, i.e. the $(2 + s') \times n$ stack matrix

$$\tilde{P} \;:=\; \begin{bmatrix} P \\ \alpha \end{bmatrix}, \qquad \alpha \;=\; (\alpha_{ij}) \;\in\; \mathbb{Z}^n.$$

Moreover, the fan $\tilde{\Sigma}$ is given as the fan of faces of the cone $\tilde{\sigma} \subseteq \mathbb{Q}^{2+s'}$ generated by the columns of $\tilde{P}$. Using the projection $F\colon \mathbb{Z}^{2+s'} \to \mathbb{Z}^{2+s}$ onto the first $2+s$ coordinates, we obtain the representation

$$\Sigma \;=\; \{F(\tilde{\sigma}_0);\ \tilde{\sigma}_0 \prec \tilde{\sigma}\}$$

and the cone projection $\tilde{Z} \setminus \{\tilde{z}\} \to Z$ restricts to the cone projection $\tilde{X} \setminus \{\tilde{x}\} \to X$, where $\tilde{x} = \tilde{z}$ is the common apex of the affine cones $\tilde{X} \subseteq \tilde{Z}$.

1.3 Toric degenerations

We gather the necessary concepts and tools on toric degenerations of semiprojective rational varieties with a torus action of complexity one from [18]. The semiprojective case includes in particular affine varieties.

Construction 1.3.1 (See [18, Constr. 4.1, Rem. 4.2])**.** Consider a fan Σ with generator matrix $P \in \mathrm{Mat}(2+s, n; \mathbb{Z})$ as in Construction 1.2.1. Given integers $\kappa = 0, 1, 2$ and $\ell \geq 1$, we associate the generator matrices P_κ with P by adding a row and a column to P as follows

$$P_0 = \begin{bmatrix} -l_0 & -\ell & l_1 & 0 \\ -l_0 & -\ell & 0 & l_2 \\ d_0 & 0 & d_1 & d_2 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad P_1 \;=\; \begin{bmatrix} -l_0 & l_1 & \ell & 0 \\ -l_0 & 0 & 0 & l_2 \\ d_0 & d_1 & 0 & d_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -l_0 & l_1 & 0 & 0 \\ -l_0 & 0 & l_2 & \ell \\ d_0 & d_1 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We denote the new column by $v_{\kappa n_\kappa + 1} \in \mathbb{Z}^{3+s}$. We assign to each $P_\kappa \in \mathrm{Mat}(3+s, n+1; \mathbb{Z})$ a fan Σ_κ in $\mathbb{Z}^{3+s}$. It is determined by its maximal cones

$$(\sigma \times 0) + \mathbb{Q}_{\geq 0} \cdot v_{\kappa n_\kappa + 1}, \qquad \sigma \in \Sigma^{\max}.$$

We denote by Z the toric variety arising from Σ, by Z_κ the toric variety arising from Σ_κ and by $\mathcal{X}_\kappa \subseteq Z_\kappa$ the variety of complexity one arising from the generator matrix P_κ as in Construction 1.2.1. Furthermore, let F_κ be the linear isomorphisms given by

$$F_0\colon \mathbb{Z}^{3+s} \;\longrightarrow\; \mathbb{Z}^{3+s}, \quad e_i \;\mapsto\; \begin{cases} e_i & \text{, if } i = 1, \ldots, 2+s, \\ (\ell, \ell, 0, 1) & \text{, else,} \end{cases}$$

$$F_1\colon \mathbb{Z}^{3+s} \;\longrightarrow\; \mathbb{Z}^{3+s}, \quad e_i \;\mapsto\; \begin{cases} e_i & \text{, if } i = 1, \ldots, 2+s, \\ (-\ell, 0, 0, 1) & \text{, else,} \end{cases}$$

$$F_2\colon \mathbb{Z}^{3+s} \;\longrightarrow\; \mathbb{Z}^{3+s}, \quad e_i \;\mapsto\; \begin{cases} e_i & \text{, if } i = 1, \ldots, 2+s, \\ (0, -\ell, 0, 1) & \text{, else.} \end{cases}$$

Then we have a commutative diagram, where both downwards arrows represent the projection onto the $(3+s)$-th coordinate:

$$\begin{tikzcd}
\mathbb{Z}^{3+s} \arrow[rr, "F_\kappa", "\cong"'] \arrow[dr] & & \mathbb{Z}^{3+s} \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}$$

The map F_κ is an isomorphism of fans from Σ_κ in $\mathbb{Z}^{3+s}$ to the fan product of Σ in $\mathbb{Z}^{2+s}$ and the fan of faces of $\mathbb{Q}_{\geq 0}$ in $\mathbb{Z}$. This yields a commutative diagram of the associated toric morphisms

$$\begin{tikzcd}
Z_\kappa \arrow[rr, "\varphi_\kappa", "\cong"'] \arrow[dr, "\Psi_\kappa"'] & & Z \times \mathbb{K} \arrow[dl, "\mathrm{pr}_\mathbb{K}"] \\
& \mathbb{K} &
\end{tikzcd}$$

where Ψ_κ is given in Cox coordinates by $[z_{ij}] \mapsto z_{\kappa n_\kappa+1}$. Restricting $\Psi_\kappa \colon Z_\kappa \to \mathbb{K}$ gives a family of morphisms $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$. Further, for $\zeta \in \mathbb{K}$ we denote the fibers by $\mathcal{X}_{\kappa,\zeta} := \psi_\kappa^{-1}(\zeta)$.

Remark 1.3.2 (See [18, Rem. 4.3])**.** Let $X \subseteq Z$ be as in Construction 1.2.1 and consider the variety $\mathcal{X}_\kappa \subseteq Z_\kappa$ given by Construction 1.3.1. Then we have

$$\mathrm{Cl}(\mathcal{X}_\kappa) \;=\; \mathbb{Z}^{n+1}/\mathrm{im}(P_\kappa^*) \;=\; \mathrm{Cl}(X), \qquad \mathcal{R}(\mathcal{X}_\kappa) \;=\; \mathbb{K}[T_{ij}]/\langle g_\kappa \rangle,$$

where the new variable $T_{\kappa n_\kappa+1}$ is of $\mathrm{Cl}(\mathcal{X}_\kappa)$-degree zero and all other variables T_{ij} have the same $\mathrm{Cl}(\mathcal{X}_\kappa)$-degree in $\mathcal{R}(\mathcal{X}_\kappa)$ as they have in $\mathcal{R}(X)$. Moreover, the defining relation g_κ arises from g by replacing $T_\kappa^{l_\kappa}$ with $T_\kappa^{l_\kappa} T_{\kappa n_\kappa+1}^{\ell}$.

Example 1.3.3. Let $X \subseteq Z$ arise from the generator matrix P as in Example 1.2.5. The corresponding generator matrices P_κ are given by

$$P_0 \;=\; \begin{bmatrix} -1 & -1 & -\ell & 1 & 1 & 0 & 0 \\ -1 & -1 & -\ell & 0 & 0 & 1 & 1 \\ 1 & -3 & 0 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_1 \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & \ell & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & -3 & 0 & -4 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_2 \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & \ell \\ 1 & -3 & 0 & -4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Further, we assign a fan Σ_κ to each $\kappa = 0, 1, 2$, which is determined by the following five maximal cones

$$\begin{array}{lll} \mathrm{cone}\left(v_{01}, v_{11}, v_{21}, v_{\kappa n_\kappa+1}\right), & \mathrm{cone}\left(v_{02}, v_{12}, v_{22}, v_{\kappa n_\kappa+1}\right), & \\ \mathrm{cone}\left(v_{01}, v_{02}, v_{\kappa n_\kappa+1}\right), & \mathrm{cone}\left(v_{11}, v_{12}, v_{\kappa n_\kappa+1}\right), & \mathrm{cone}\left(v_{21}, v_{22}, v_{\kappa n_\kappa+1}\right). \end{array}$$

The respective Cox rings are given as

$$\begin{aligned} \mathcal{R}(\mathcal{X}_0) &= \mathbb{K}[T_{01}, T_{02}, T_{03}, T_{11}, T_{12}, T_{21}, T_{22}]/\langle T_{01}T_{02}T_{03}^{\ell} + T_{11}T_{12} + T_{21}T_{22} \rangle, \\ \mathcal{R}(\mathcal{X}_1) &= \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}]/\langle T_{01}T_{02} + T_{11}T_{12}T_{13}^{\ell} + T_{21}T_{22} \rangle, \\ \mathcal{R}(\mathcal{X}_2) &= \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{22}, T_{23}]/\langle T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}T_{23}^{\ell} \rangle. \end{aligned}$$

The fiber $\mathcal{X}_{\kappa,0}$ is a possibly non-normal toric variety. We recall how to construct the associated fan and characterize, when the fiber is normal. We adapt [18, Constr. 4.5] to our setting; namely to the two cases of generator matrix $P \in \mathrm{Mat}(2+s, n; \mathbb{Z})$ with $s = 1$ and $s = 2$.

Construction 1.3.4 (See [18, Constr. 4.5])**.** Consider a fan Σ with generator matrix $P \in \mathrm{Mat}(3, n; \mathbb{Z})$ as in Construction 1.2.1. Here $e_1, e_2, e_3 \in \mathbb{Q}^3$ are the canonical basis vectors and we set $e_0 := -e_1 - e_2$. We define leaves

$$\tau_0 \;:=\; \mathrm{cone}(e_0) + \mathrm{lin}(e_3), \qquad \tau_1 \;:=\; \mathrm{cone}(e_1) + \mathrm{lin}(e_3), \qquad \tau_2 \;:=\; \mathrm{cone}(e_2) + \mathrm{lin}(e_3).$$

Then we define the *tropical variety* to be

$$\operatorname{trop}(X) \;=\; \tau_0 \;\cup\; \tau_1 \;\cup\; \tau_2.$$

For $\kappa = 0, 1, 2$, the *antitropical coordinates* of a vector $v \in \mathbb{Z}^3 \cap \operatorname{lin}(\tau_\kappa)$ are $\eta_\kappa^{-1}(v) \in \mathbb{Z}^2$, where

$$\eta_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \eta_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Moreover, for $\kappa = 0, 1, 2$, the fans Δ_κ in $\mathbb{Z}^3$ and $\Delta_\kappa^{\mathrm{at}}$ in $\mathbb{Z}^2$ are given by

$$\Delta_\kappa \;=\; \{\sigma \cap \operatorname{lin}(\tau_\kappa)\,;\; \sigma \in \Sigma\}, \qquad \Delta_\kappa^{\mathrm{at}} \;=\; \{\eta_\kappa^{-1}(\sigma \cap \operatorname{lin}(\tau_\kappa)\,;\; \sigma \in \Sigma\}.$$

Example 1.3.5. We continue the example from Example 1.3.3. First, we intersect the maximal cones with the respective linear subspace and exemplarily treat the case $\kappa = 0$.

$$\begin{aligned}
\sigma_{135} \;\cap\; \operatorname{lin}(e_0, e_3) &\;=\; \operatorname{cone}\left(v_{01}, (1,1,0)\right), \\
\sigma_{246} \;\cap\; \operatorname{lin}(e_0, e_3) &\;=\; \operatorname{cone}\left(v_{02}, (1,1,-5)\right), \\
\tau_{12} \;\cap\; \operatorname{lin}(e_0, e_3) &\;=\; \operatorname{cone}\left(v_{01}, v_{02}\right), \\
\tau_{34} \;\cap\; \operatorname{lin}(e_0, e_3) &\;=\; \{0\}, \\
\tau_{56} \;\cap\; \operatorname{lin}(e_0, e_3) &\;=\; \{0\}.
\end{aligned}$$

Then the cones are given in antitropical coordinates by

$$\begin{aligned}
\mu_0^{-1}\left(\sigma_{135} \cap \operatorname{lin}(e_0, e_3)\right) &\;=\; \operatorname{cone}\left((1,-1), (0,1)\right), \\
\mu_0^{-1}\left(\sigma_{246} \cap \operatorname{lin}(e_0, e_3)\right) &\;=\; \operatorname{cone}\left((-3,-1), (-5,1)\right), \\
\mu_0^{-1}\left(\tau_{12} \cap \operatorname{lin}(e_0, e_3)\right) &\;=\; \operatorname{cone}\left((1,-1), (-3,-1)\right).
\end{aligned}$$

Similar computations for the other two cases give us the generator matrices $P_{\kappa,0}$ of the fans $\Delta_\kappa^{\mathrm{at}}$:

$$P_{0,0} = \begin{bmatrix} 1 & -3 & 0 & -5 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad P_{1,0} = \begin{bmatrix} 0 & -4 & 1 & -4 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad P_{2,0} = \begin{bmatrix} 0 & -1 & 1 & -7 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Construction 1.3.6 (See [18, Constr. 4.5])**.** Consider a fan Σ such that its generator matrix $P \in \operatorname{Mat}(4, n; \mathbb{Z})$ is as in Construction 1.2.1. Here $e_1, e_2, e_3, e_4 \in \mathbb{Q}^4$ are the canonical basis vectors and we set $e_0 := -e_1 - e_2$. We define leaves

$$\tau_0 \;:=\; \operatorname{cone}(e_0) + \operatorname{lin}(e_3, e_4), \quad \tau_1 \;:=\; \operatorname{cone}(e_1) + \operatorname{lin}(e_3, e_4), \quad \tau_2 \;:=\; \operatorname{cone}(e_2) + \operatorname{lin}(e_3, e_4).$$

Then we define the *tropical variety* to be

$$\operatorname{trop}(X) \;=\; \tau_0 \;\cup\; \tau_1 \;\cup\; \tau_2.$$

For $\kappa = 0, 1, 2$, the *antitropical coordinates* of $v \in \mathbb{Z}^4 \cap \operatorname{lin}(\tau_\kappa)$ are given by $\eta_\kappa^{-1}(v) \in \mathbb{Z}^4$, where

$$\eta_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \eta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Furthermore, the fans Δ_κ in $\mathbb{Z}^4$ and $\Delta_\kappa^{\mathrm{at}}$ in $\mathbb{Z}^3$ are given by

$$\Delta_\kappa \;=\; \{\sigma \cap \operatorname{lin}(\tau_\kappa)\,;\ \sigma \in \Sigma\}, \qquad \Delta_\kappa^{\mathrm{at}} \;=\; \{\eta_\kappa^{-1}(\sigma \cap \operatorname{lin}(\tau_\kappa)\,;\ \sigma \in \Sigma\}.$$

Proposition 1.3.7 (See [18, Prop. 4.6])**.** *Let $X \subseteq Z$ and $\mathcal{X}_\kappa \subseteq Z_\kappa$ be as provided by Construction 1.3.1. Then the following statements hold.*

(i) *The variety $\mathcal{X}_\kappa$ is irreducible and normal and $\psi_\kappa\colon \mathcal{X}_\kappa \to \mathbb{K}$ is a flat family. Moreover, ψ_κ is proper (projective) if Z is complete (projective).*

(ii) *For $\zeta \neq 0$, we have $\mathcal{X}_{\kappa,\zeta} \cong X$ and the fiber $\mathcal{X}_{\kappa,0}$ is an irreducible toric variety.*

(iii) *As a toric variety, the fiber $\mathcal{X}_{\kappa,0}$ is isomorphic to the closure of $\mathbb{T}_\kappa \subseteq \mathbb{T}^{2+s} \subseteq Z$ given by*

$$\mathbb{T}_\kappa \;=\; \begin{cases} t_1 = t_2, & \kappa = 0, \\ t_2 = 1, & \kappa = 1, \\ t_1 = 1, & \kappa = 2. \end{cases}$$

Moreover, $\mathcal{X}_{\kappa,0}$ has the fan Δ_κ in $\mathbb{Z}^{2+s}$ as its convergency fan and the associated toric variety as its normalization.

(iv) *For $J = (j_i;\ i = 0, 1, 2,\ i \neq \kappa)$, set $\sigma(J) := \operatorname{cone}(v_{ij_i};\ i = 0, 1, 2,\ i \neq \kappa)$. Then the fiber $\mathcal{X}_{\kappa,0}$ is normal if and only if for every cone $\sigma(J) \in \Sigma$, we have $l_{ij_i} > 1$ for at most one $i = 0, 1, 2$ distinct from κ.*

(v) *The fiber $\mathcal{X}_{\kappa,0}$ is affine (semiprojective, projective, complete) if Z is affine (semiprojective, projective, complete).*

Definition 1.3.8. We call a toric degeneration ψ_κ resp. κ *special* if the corresponding fiber $\mathcal{X}_{\kappa,0}$ is a normal variety.

Lemma 1.3.9 (See [18, Lem. 4.7])**.** *Let $X \subseteq Z$ and $\mathcal{X}_\kappa \subseteq Z_\kappa$ be as provided by Construction 1.3.1. Then we have $\mathcal{X}_{\kappa,0} = \psi_\kappa^{-1}(0) \subseteq \Psi_\kappa^{-1}(0) = Z$. Moreover, with suitable $b_i \in \mathbb{K}^*$ the vanishing ideal $\mathcal{I}_{\kappa,0}$ of $\mathcal{X}_{\kappa,0}$ is the K-prime binomial ideal*

$$\mathcal{I}_{0,0} \;=\; \langle T_1^{l_1} + b_2 T_2^{l_2} \rangle, \qquad \mathcal{I}_{1,0} \;=\; \langle T_0^{l_0} + b_2 T_2^{l_2} \rangle, \qquad \mathcal{I}_{2,0} \;=\; \langle T_0^{l_0} + b_1 T_1^{l_1} \rangle.$$

Example 1.3.10. We continue the example from Example 1.3.5. Then the fibers $\mathcal{X}_{\kappa,0}$ are given in Cox coordinates as follows:

$$\mathcal{X}_{0,0} = V(T_{11}T_{12} + T_{21}T_{22}), \quad \mathcal{X}_{1,0} = V(T_{01}T_{02} + T_{11}T_{12}), \quad \mathcal{X}_{2,0} = V(T_{01}T_{02} + T_{21}T_{22}).$$

Moreover, we have $l_{ij} = 1$ for all $i, j = 0, 1, 2$. Thus, the fibers $\mathcal{X}_{\kappa,0}$ are all normal.

The following construction provides us with the necessary data for checking the existence of Kähler-Einstein metrics and Sasaki-Einstein metrics later on.

Construction 1.3.11 (See [18, Constr. 5.5])**.** Let $X \subseteq Z$ arise from a generator matrix $P \in \mathrm{Mat}(2+s, n; \mathbb{Z})$ as in Construction 1.2.1 and let Z be projective. Fix α_{ij} such that we obtain an anticanonical class for X

$$w_X \ = \ \sum \alpha_{ij} w_{ij} \ \in \ K \ = \ \mathrm{Cl}(X) \ = \ \mathrm{Cl}(Z).$$

Then the generator matrix $\tilde{P} \in \mathrm{Mat}(3+s, n; \mathbb{Z})$ for an anticanonical cone $\tilde{X}$ over X is given as in Proposition 1.2.11 by

$$\tilde{P} \ = \ \begin{bmatrix} P \\ \alpha \end{bmatrix}, \qquad \alpha \ = \ (\alpha_{ij})$$

and the associated fan $\tilde{\Sigma}$ contains all faces of the cone $\tilde{\sigma} \subseteq \mathbb{R}^{3+s}$ over the columns of $\tilde{P}$. For $\kappa = 0, 1, 2$ set

$$\tilde{\tau}_\kappa \ := \ \eta_\kappa^{-1}(\tilde{\sigma}) \ \subseteq \ \mathbb{R}^{2+s}, \qquad \qquad \tilde{\omega}_\kappa \ := \ (\tilde{\tau}_\kappa)^\vee \ \subseteq \ \mathbb{R}^{2+s},$$

where $\eta_\kappa \colon \mathbb{Z}^{2+s} \to \mathbb{Z}^{3+s}$ are the antitropical coordinates. Set $\imath \colon \mathbb{R}^{1+s} \to \mathbb{R}^{2+s}$, $u \mapsto (u_1, \ldots, u_s, 1, u_{1+s})$ and denote by $\mathrm{conv}(\tilde{\omega}_\kappa)$ the convex hull of the primitive generators of $\tilde{\omega}_\kappa$. Then, we obtain for each κ a polytope

$$\mathcal{C}_\kappa \ := \ \imath^{-1}\left(\mathrm{conv}\left(\tilde{\omega}_\kappa\right)\right) \ \subseteq \ \mathbb{R}^{1+s}.$$

If κ is special, then there is a unique interior lattice point $u_\kappa \in \mathcal{C}_\kappa$. The *moment polytope* is given by

$$\mathcal{B}_\kappa \ := \ \begin{cases} \mathcal{C}_\kappa - u_\kappa, & \kappa \text{ special}, \\ \mathcal{C}_\kappa, & \kappa \text{ not special}. \end{cases}$$

For the case of special κ the following proposition provides the relationship between the fan $\Delta_\kappa^{\mathrm{at}}$ and the cone $\tilde{\tau}_\kappa$.

Proposition 1.3.12 (See [18, Prop. 5.6 (v)])**.** *Let $\tilde{X} \subseteq \tilde{Z}$ and $\tilde{\tau}_\kappa$ be as in Construction 1.3.11 and let κ be special. Furthermore, let $\tilde{v}_1, \ldots, \tilde{v}_k$ be the primitive generators of $\tilde{\tau}_\kappa$ and let $v_i' \in \mathbb{Z}^{2+s}$ arise from $\tilde{v}_i \in \mathbb{Z}^{2+s}$ by replacing the $1+s$-th coordinate with 1. Then*

$$v_i' \ = \ G_\kappa \cdot \tilde{v}_i, \qquad i = 1, \ldots, k,$$

with a unique unimodular matrix $G_\kappa \in \mathrm{GL}(2+s, \mathbb{Z})$. Denote by τ_κ' the cone that has the primitive generators $v_1', \ldots, v_k'$. Let $\mathrm{pr} \colon \mathbb{Z}^{2+s} \to \mathbb{Z}^{1+s}$ denote the projection erasing the $1+s$-th coordinate. Fix an integer $s = 1, 2$. Then

$$v_i \ := \ \mathrm{pr}(\tilde{v}_i), \qquad i = 1, \ldots, k,$$

are the primitive generators of the fan $\Delta_\kappa^{\mathrm{at}}$ from Construction 1.3.4 and Construction 1.3.6. In particular, the generator matrix $P_{\kappa,0}$ of $\mathcal{X}_{\kappa,0}$ is given by

$$P_{\kappa,0} = [v_1, \ldots, v_k]$$

and the convex hull of the column vectors yields the Fano polytope of $\mathcal{X}_{\kappa,0}$. Furthermore, the moment polytope is given by the dual polytope

$$\mathcal{B}_\kappa \ = \ \operatorname{conv}(v_1, \ldots, v_k)^*.$$

Example 1.3.13. We continue the example from Example 1.3.10. The generator matrix for an anticanonical cone $\tilde{X}$ is given by

$$\tilde{P} \ = \ \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -3 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

For the cones $\tilde{\tau}_\kappa$, we obtain

$$\begin{aligned} \tilde{\tau}_0 \ &= \ \operatorname{cone}\left((0,2,1),(1,0,-1),(-3,0,-1),(-5,2,1)\right), \\ \tilde{\tau}_1 \ &= \ \operatorname{cone}\left((1,1,1),(-4,1,1),(0,1,-1),(-4,1,-1)\right), \\ \tilde{\tau}_2 \ &= \ \operatorname{cone}\left((1,1,1),(-7,1,1),(0,1,-1),(-1,1,-1)\right). \end{aligned}$$

The generator matrices $P_{\kappa,0}$ are given by Example 1.3.5. Since the fibers $\mathcal{X}_{\kappa,0}$ are normal for all $k = 0,1,2$, we can apply Proposition 1.3.12. We obtain the Fano polytopes of $\mathcal{X}_{\kappa,0}$ as the convex hull of the column vectors of $P_{\kappa,0}$ and by dualizing the Fano polytopes, the moment polytopes are given as follows

$$\begin{aligned} \mathcal{B}_0 \ &= \ \operatorname{conv}\Big((0,1),(0,-1),(-2,1),\Big(\tfrac{1}{4},\tfrac{1}{4}\Big)\Big), \\ \mathcal{B}_1 \ &= \ \operatorname{conv}\Big((0,1),(0,-1),(-2,1),\Big(\tfrac{1}{4},0\Big)\Big), \\ \mathcal{B}_2 \ &= \ \operatorname{conv}\Big((0,1),(0,-1),(-2,1),\Big(\tfrac{1}{4},\tfrac{3}{4}\Big)\Big). \end{aligned}$$

1.4 Rational projective $\mathbb{K}^*$-surfaces of Picard number one

A $\mathbb{K}^*$-*surface* is an irreducible, normal surface X together with an effective algebraic action $\mathbb{K}^* \times X \to X$ of the multiplicative group $\mathbb{K}^*$. We begin with general observations about $\mathbb{K}^*$-actions, such as those found in the work of Orlik and Wagreich [36]; see also [3, Sect. 5.4].

Whenever $\mathbb{K}^*$ acts on a normal projective variety X, each point $x \in X$ gives rise to a morphism $\mu_x \colon \mathbb{P}_1 \to X$, extending the orbit map $\mathbb{K}^* \to X$, $s \mapsto s \cdot x$. For every $x \in X$ we obtain two corresponding points by $x^0 := \mu_x(0)$ and $x^\infty := \mu_x(\infty)$ which are fixed points of the $\mathbb{K}^*$-action. The properties normality and projectivity ensure that x^0, x^∞ are necessarily distinct. Moreover, there is precisely one *source* $F^+ \subseteq X$ and precisely one *sink* $F^- \subseteq X$. These are irreducible components of the fixed point set such that

$$X^+ \ := \ \{x \in X;\ x^0 \in F^+\}, \qquad\qquad X^- \ := \ \{x \in X;\ x^\infty \in F^-\}$$

are open subsets of X. Now assume that X is a surface. Then there are exactly three types of fixed points. We call $x \in X$

- *elliptic*, if x lies in the closure of infinitely many non-trivial $\mathbb{K}^*$-orbits,
- *hyperbolic*, if x lies in the closure of precisely two non-trivial $\mathbb{K}^*$-orbits,
- *parabolic*, if x lies in the closure of precisely one non-trivial $\mathbb{K}^*$-orbit.

Elliptic and hyperbolic fixed points are isolated and the parabolic fixed points belong to a curve in X. The source and sink each consist of either a single elliptic fixed point or they form a smooth irreducible curve of parabolic fixed points. In the first case we denote by x^+ resp. x^- the unique points forming F^+ resp. F^-. Moreover, every fixed point outside the source or the sink is hyperbolic.

Now we adapt the machinery presented in Section 1.2 to the case of quasismooth $\mathbb{K}^*$-surfaces of Picard number one. The considerations of this section are taken up in the common article [27, Sec. 3].

Construction 1.4.1. Fix $d_i, l_i \in \mathbb{Z}$ and consider a generator matrix $P \in \mathrm{Mat}(3,4;\mathbb{Z})$ of the following shape:

$$P \ := \ [v_1, v_2, v_3, v_4] \ := \ \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix}, \qquad \begin{array}{c} {\scriptstyle 1 \le d_1 \le l_1 \le l_2, \quad \gcd(l_i, d_i) = 1,} \\ {\scriptstyle d_0 + \frac{d_1}{l_1} + \frac{d_2}{l_2} \ < \ 0 \ < \ \frac{d_1}{l_1} + \frac{d_2}{l_2}.} \end{array}$$

Let Z be the fake weighted projective space with generator matrix P and let $X \subseteq Z$ arise from Construction 1.2.1. Then X inherits the $\mathbb{K}^*$-action from Z given on $\mathbb{T}^3 \subseteq Z$ by

$$t \cdot s \ = \ \lambda(t)s \ = \ (s_1, s_2, ts_3),$$

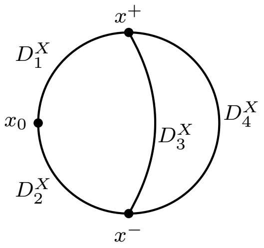
with the one-parameter subgroup $\lambda\colon \mathbb{K}^* \to \mathbb{T}^3, t \mapsto (1,1,t)$. Define $\Sigma' \subseteq \Sigma$ to be the subfan with the maximal cones

$$\sigma^+ := \mathrm{cone}(v_1, v_3, v_4), \qquad \sigma^- := \mathrm{cone}(v_2, v_3, v_4), \qquad \tau_0 := \mathrm{cone}(v_1, v_2).$$

Then the open toric subvariety $Z' \subseteq Z$ given by $\Sigma' \subseteq \Sigma$ is the minimal ambient toric variety of X.

In the following we index the prime divisors with an index matching the numbering of the columns, namely $D_i^X := \overline{\mathbb{T}^3 \cdot z_{\varrho_i}} \cap X$ for $i = 1, 2, 3, 4$.

Remark 1.4.2 (See [36])**.** Let $X \subseteq Z$ arise from Construction 1.4.1. Then the D_i^X intersect as follows



The source and sink each consist of a single elliptic fixed point x^+, x^-. The points x^+ and x^- together with the unique intersection point $x_0 \in D_1^X \cap D_2^X$ form the fixed point set of the $\mathbb{K}^*$-surface X.

Recall that a *del Pezzo surface* is a normal projective surface X admitting an ample anticanonical divisor $-\mathcal{K}_X$.

Proposition 1.4.3 (See also [27, Prop. 3.2]). *Let $X \subseteq Z$ arise from a generator matrix P as in Construction 1.4.1. Then, in homogeneous coordinates on Z, we have the representation*

$$X \ = \ V(T_1T_2 + T_3^{l_1} + T_4^{l_2}) \ \subseteq \ Z.$$

The $\mathbb{K}^*$-surface X is projective, rational, quasismooth, del Pezzo and of Picard number one. With any l_1-th root ζ of -1, the $\mathbb{K}^*$-fixed points of X are

$$x_0 \ = \ [0,0,\zeta,1], \qquad x^+ \ = \ [0,1,0,0], \qquad x^- \ = \ [1,0,0,0].$$

The fixed point x_0 is hyperbolic and x^+, x^- are the elliptic fixed points. There are exactly two non-trivial orbits $\mathbb{K}^* \cdot z_1$ and $\mathbb{K}^* \cdot z_2$ with non-trivial isotropy groups:

$$z_1 = [-1,1,0,1], \quad |\mathbb{K}^*_{z_1}| = l_1, \qquad z_2 = [-1,1,1,0], \quad |\mathbb{K}^*_{z_2}| = l_2.$$

The fake weight vector $w(P) = (w_1, w_2, w_3, w_4)$ *of the ambient fake weighted projective space Z is given explicitly in terms of P as*

$$w(P) \ = \ (-l_1l_2d_0 - l_2d_1 - l_1d_2,\ l_2d_1 + l_1d_2,\ -l_2d_0,\ -l_1d_0) \ \in \ \mathbb{Z}^4_{>0}.$$

Moreover, for the local class group orders and local Gorenstein indices of the three $\mathbb{K}^*$-fixed points $x_0, x^+, x^- \in X$, we obtain

$$\begin{array}{lllll} \mathrm{cl}(x_0) & = & -d_0, & \iota(x_0) & = \ 1, \\ \mathrm{cl}(x^+) & = & w_2, & \iota(x^+) & = \ \frac{w_2}{\gcd(d_1-d_2,l_1+l_2,w_2)}, \\ \mathrm{cl}(x^-) & = & w_1, & \iota(x^-) & = \ \frac{w_1}{\gcd(d_0l_2-d_1+d_2,l_1+l_2,w_1)}. \end{array}$$

Proof of Construction 1.4.1 and Proposition 1.4.3. The assumptions on l_i, d_i made in Construction 1.4.1 ensure that P fits into the setting of Construction 1.2.1. According to Construction 1.2.1 the output X is a normal, rational $\mathbb{K}^*$-surface and X is projective since Z is projective. The statements on the fixed points and the isotropy groups are covered by [16, Prop. 4.8]. Quasismoothness follows from [16, Prop. 4.15] and $\rho(X) = 1$ is a consequence of [16, Prop. 5.1].

The fake weight vector and the local class group orders of the fixed points are given by Moreover, consider the following linear forms

$$u^+ \; = \; \left[\tfrac{d_2-d_1}{w_2},\, \tfrac{d_1-d_2}{w_2},\, \tfrac{l_1+l_2}{w_2}\right], \qquad u^- \; = \; \left[\tfrac{-d_2+d_1-d_0l_2}{w_1},\, \tfrac{d_2-d_1-d_0l_1}{w_1},\, \tfrac{-l_1-l_2}{w_1}\right]$$

and the linear forms u^+, u^- evaluate on the columns v_1, v_2, v_3, v_4 of P as follows:

$$\langle u^+, v_1\rangle = 0,\ \langle u^+, v_3\rangle = 1,\ \langle u^+, v_4\rangle = 1, \qquad \langle u^-, v_2\rangle = 0,\ \langle u^-, v_3\rangle = 1,\ \langle u^-, v_4\rangle = 1.$$

We conclude that $\frac{w_2}{\gcd(d_1-d_2,l_1+l_2,w_2)}\, u^+$ and $\frac{w_1}{\gcd(d_0l_2-d_1+d_2,l_1+l_2,w_1)}\, u^-$ are primitive integral vectors. This yields $\iota(x^+)$ and $\iota(x^-)$. According to [17, Prop. 8.8 (iii)] the local Gorenstein index of the hyperbolic fixed point is given by $\iota(x_0) = 1$. This leads to the desired assertions regarding the Gorenstein indices. ☐

Proposition 1.4.4 (See also [27, Prop. 3.2])**.** *Let $X \subseteq Z$ be as in Construction 1.4.1 and let $w(P) = (w_1, w_2, w_3, w_4)$ be the corresponding fake weight vector. Then the self intersection number of the canonical divisor $\mathcal{K}_X$ on X can be expressed as follows:*

$$\mathcal{K}_X^2 \; = \; \left(\tfrac{1}{w_1} + \tfrac{1}{w_2}\right)\left(2 + \tfrac{l_1}{l_2} + \tfrac{l_2}{l_1}\right) \; = \; \tfrac{\mathrm{cl}(x_0)}{\mathrm{cl}(x^+)\mathrm{cl}(x^-)}(l_1 + l_2)^2.$$

Proof. Using the general formula [17, Prop. 7.9] for rational projective $\mathbb{K}^*$-surfaces, we directly compute

$$\mathcal{K}_X^2 \; = \; \frac{\left(\frac{1}{l_1}+\frac{1}{l_2}\right)^2}{\frac{d_1}{l_1}+\frac{d_2}{l_2}} - \frac{\left(\frac{1}{l_1}+\frac{1}{l_2}\right)^2}{d_0+\frac{d_1}{l_1}+\frac{d_2}{l_2}} \; = \; \left(\frac{1}{\mathrm{cl}(x^+)} \; + \; \frac{1}{\mathrm{cl}(x^-)}\right)\left(2 + \frac{l_1}{l_2} + \frac{l_2}{l_1}\right),$$

where $\mathrm{cl}(x_0), \mathrm{cl}\,(x^{\pm})$ are the local class group orders of the fixed points determined by Proposition 1.4.3. The assertion then follows from $\mathrm{cl}(x^+) + \mathrm{cl}(x^-) = l_1l_2\mathrm{cl}(x_0)$. ☐

Remark 1.4.5. Let $X \subseteq Z$ be as in Construction 1.4.1. Then according to Remark 1.2.10 the anticanonical divisor class w_X on X is given by

$$w_X \; = \; \deg(T_3) + \deg(T_4).$$

Proposition 1.4.6. *Let X be a non-toric, quasismooth, rational, projective $\mathbb{K}^*$-surface of Picard number one. Then X arises from as a surface from a generator matrix P as in Construction 1.4.1.*

Proof. This is a special case of Theorem 1.2.9. ☐

Example 1.4.7. We continue Example 1.1.14. Consider the generator matrix

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

and note that it fits into the framework of Construction 1.4.1. The $\mathbb{K}^*$-surface $X \subseteq Z$, arising from the generator matrix P, is given in homogeneous coordinates on Z as follows

$$X = V\left(T_1T_2 + T_3^2 + T_4^4\right) \subseteq Z.$$

Further, the corresponding fake weight vector of the ambient fake weighted projective space Z is given by $w(P) = (2, 6, 4, 2)$. The local class group orders and local Gorenstein indices of the fixed points of X are

$$\begin{array}{rlrlrl} \mathrm{cl}(x_0) = & 1, & \mathrm{cl}(x^+) & = 6, & \mathrm{cl}(x^-) = & 2, \\ \iota(x_0) = & 1, & \iota(x^+) & = 1, & \iota(x^-) = & 1. \end{array}$$

Moreover, the self intersection number of the canonical divisor $\mathcal{K}_X$ on X is given by $\mathcal{K}_X^2 = 3$.

1.5 Coverings onto fake weighted projective planes

For every quasismooth, rational, projective $\mathbb{K}^*$-surfaces X of Picard number one, we provide finite coverings $X \to Z_1$ and $X \to Z_2$ onto fake weighted projective planes; see Construction 1.5.1. We take a closer look the geometry of the coverings and finally can show that Z_1, Z_2 are the quotients of X by the two non-trivial finite $\mathbb{K}^*$-isotropy groups of X; see Corollary 1.5.11.

Construction 1.5.1. Let $X \subseteq Z$ arise from a generator matrix P as in Construction 1.4.1, that means

$$P := [v_1, v_2, v_3, v_4] := \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix}, \qquad \begin{array}{c} 1 \le d_1 \le l_1 \le l_2, \quad \gcd(l_i, d_i) = 1, \\ d_0 + \frac{d_1}{l_1} + \frac{d_2}{l_2} < 0 < \frac{d_1}{l_1} + \frac{d_2}{l_2}. \end{array}$$

Further, let Σ denote the unique projectice fan of Z in $\mathbb{Z}^3$. Define $\Sigma' \subseteq \Sigma$ to be the subfan with the maximal cones

$$\sigma^+ := \mathrm{cone}(v_1, v_3, v_4), \qquad \sigma^- := \mathrm{cone}(v_2, v_3, v_4), \qquad \tau_0 := \mathrm{cone}(v_1, v_2).$$

Then the open toric subvariety $Z' \subseteq Z$ given by the subfan $\Sigma' \subseteq \Sigma$ satisifies $X \subseteq Z'$. Set $\ell := \gcd(l_1, l_2)$ and $\ell_i := l_i/\ell$. Consider

$$P_1 := \begin{bmatrix} -1 & -1 & \ell_2 \\ 0 & d_0l_1 & \ell_2d_1 + \ell_1d_2 \end{bmatrix}, \qquad P_2 := \begin{bmatrix} -1 & -1 & \ell_1 \\ 0 & d_0l_2 & \ell_2d_1 + \ell_1d_2 \end{bmatrix}.$$

These are generator matrices for fake weighted projective planes Z_1 and Z_2. In terms of the fake weight vector $w(P) = (w_1, w_2, w_3, w_4)$ of Z, we have

$$w(P_1) \;=\; (\ell^{-1}w_1,\, \ell^{-1}w_2,\, w_3), \qquad\qquad w(P_2) \;=\; (\ell^{-1}w_1,\, \ell^{-1}w_2,\, w_4)$$

for the respective fake weight vectors. Let $\psi_i\colon Z' \to Z_i$ be the toric morphisms defined by the homomorphisms $F_i\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ with the representing matrices

$$F_1 \;:=\; \begin{bmatrix} 0 & 1 & 0 \\ -d_1 & d_1 & l_1 \end{bmatrix}, \qquad\qquad F_2 \;:=\; \begin{bmatrix} 1 & 0 & 0 \\ d_2 & -d_2 & l_2 \end{bmatrix}.$$

Restricting to $X \subseteq Z'$ gives a finite covering $\varphi_1\colon X \to Z_1$ of degree l_1 and a finite covering $\varphi_2\colon X \to Z_2$ of degree l_2.

Proof. Everything is basic toric geometry except the statement on $\varphi_i\colon X \to Z_i$. On the acting tori $\mathbb{T}^3 \subseteq Z'$ and $\mathbb{T}^2 \subseteq Z_i$, the map $\psi_2\colon Z' \to Z_2$ is given by

$$\varphi_2(s_1, s_2, s_3) \;=\; \left(s_1, s_2^{-d_2} s_3^{l_2}\right).$$

We derive $X \cap \mathbb{T}^3 = V(1 + S_1 + S_2)$ from Construction 1.2.1. Thus, the points of $X \cap \mathbb{T}^3$ are of the form $\xi = (\xi_1, -1-\xi_1, \xi_2)$ with $\xi_1, \xi_2 \in \mathbb{C}^*$ such that $\xi_1 \neq -1$. For the image and the fibers, we obtain

$$\varphi_2(X \cap \mathbb{T}^3) = \{\eta \in \mathbb{T}^2;\ \eta_1 \neq -1\}, \quad \varphi_2^{-1}(\varphi_2(\xi)) = \{(\xi_1, -1-\xi_1, \zeta\xi_2);\ \zeta^{l_2} = 1\}.$$

Consequently, φ_2 is dominant, hence surjective and its general fiber contains precisely l_2 points. With the coordinate divisors $C_1, C_2, C_3 \subseteq Z_2$, we have

$$Z_2 \setminus \varphi_2(X \cap \mathbb{T}^3) \;=\; C_1 \cup C_2 \cup C_3 \cup C_4, \qquad C_4 \;:=\; \overline{\{\eta \in \mathbb{T}^2;\ \eta_1 = -1\}} \;\subseteq\; Z_2.$$

Let $D_i^X \subseteq X$ be the prime divisors obtained by cutting down the coordinate divisors of Z; see Proposition 1.2.4. Using surjectivity of φ_2, we see

$$Z_2 \setminus \varphi_2(X \cap \mathbb{T}^3) \;=\; \varphi_2(X \setminus \mathbb{T}^3) \;=\; \varphi_2(D_1^X) \cup \ldots \cup \varphi_2(D_4^X).$$

Thus, $\varphi_2\colon X \to Z_2$ must have finite fibers, proving everything we need. The map $\varphi_1\colon X \to Z_1$ can be treated in an analogous manner. ☐

Definition 1.5.2. We write $C(l)$ for the groups of l-th roots of units of the ground field $\mathbb{K}$. Moreover, we define one-parameter subgroups $\lambda\colon \mathbb{K}^* \to \mathbb{Z}^3$, $t \mapsto (1, 1, t)$ as well as

$$\lambda_1\colon \mathbb{K}^* \longrightarrow \mathbb{T}^3, \quad s \mapsto \left(s^{l_1}, 1, s^{d_1}\right), \qquad \lambda_2\colon \mathbb{K}^* \longrightarrow \mathbb{T}^3, \quad s \mapsto \left(1, s^{l_2}, s^{d_2}\right).$$

Proposition 1.5.3. *The morphisms $\varphi_i\colon Z' \to Z_i$ from Construction 1.5.1 are good quotients for the $\mathbb{K}^*$-actions on Z' given by the $\lambda_i\colon \mathbb{K}^* \longrightarrow \mathbb{T}^3$ from Definition 1.5.2.*

Lemma 1.5.4. *Consider P and Σ' as in Construction 1.5.1. Then for $\delta = \sigma^+, \sigma^-, \tau_0$ we have $(\sigma^- + \mathrm{lin}(v_4)) \cap \delta \subseteq \sigma^-$.*

Proof. The inclusion $(\sigma^- + \operatorname{lin}(v_4)) \cap \sigma^- \subseteq \sigma^-$ is clear. Let $v \in (\sigma^- + \operatorname{lin}(v_4)) \cap \sigma^+$. Then we have the two presentations

$$v \;=\; \alpha v_2 + \beta v_3 + \gamma v_4 \;=\; x v_1 + y v_3 + z v_4$$

with $\alpha, \beta, x, y, z \geq 0$ and $\gamma \in \mathbb{Q}$. Assume $\gamma < 0$. Then $v - \gamma v_4$ is represented by the following two positive combinations

$$\alpha v_2 + \beta v_3 \;=\; x v_1 + y v_3 + (z - \gamma)\, v_4 \;\in\; \operatorname{cone}(v_2, v_3) \cap \sigma^+ \;=\; \operatorname{cone}(v_3).$$

The latter equality holds because the involved cones belong to Σ'. As a consequence, we obtain

$$\alpha \;=\; x \;=\; 0, \qquad \beta \;=\; y, \qquad \gamma \;=\; z$$

where $\gamma = z$ contradicts $z \geq 0$ and the assumption $\gamma < 0$. Consequently, we have $\gamma \geq 0$ and $v \in \sigma^- \cap \sigma^+$, i.e.

$$(\sigma^- + \operatorname{lin}(v_4)) \cap \sigma^+ \;\subseteq\; \sigma^-.$$

Now let $v \in (\sigma^- + \operatorname{lin}(v_4)) \cap \tau_0$. Then v can be represented as

$$v \;=\; \alpha v_2 + \beta v_3 + \gamma v_4 \;=\; x v_1 + y v_2$$

with $\alpha, \beta, x, y \geq 0$ and $\gamma \in \mathbb{Q}$. We assume $\gamma < 0$ and get

$$\alpha v_2 + \beta v_3 \;=\; x v_1 + y v_2 - \gamma v_4 \;\in\; \operatorname{cone}(v_2, v_3) \cap \operatorname{cone}(v_1, v_2, v_4) \;=\; \operatorname{cone}(v_2),$$

where we work in the larger fan Σ having P as generator matrix to observe the last identity. This yields

$$\alpha \;=\; y, \qquad \beta \;=\; \gamma \;=\; x \;=\; 0$$

and this is a contradiction to $\gamma < 0$. We conclude $\gamma \geq 0$ and $v \in \sigma^- \cap \tau_0$. Altogether, one obtains

$$(\sigma^- + \operatorname{lin}(v_4)) \cap \tau_0 \;\subseteq\; \sigma^-.$$

□

Lemma 1.5.5. *Consider P and Σ' as in Construction 1.5.1. Then for $\delta = \sigma^+, \sigma^-, \tau_0$ we have $\big(\sigma^+ + \operatorname{lin}(v_4)\big) \cap \delta \subseteq \sigma^+$.*

Proof. For the inclusion $\big(\sigma^+ + \operatorname{lin}(v_4)\big) \cap \sigma^+ \subseteq \sigma^+$ there is nothing to show. Consider $v \in \big(\sigma^+ + \operatorname{lin}(v_4)\big) \cap \sigma^-$. Then v has the two representations

$$v \;=\; \alpha v_1 + \beta v_3 + \gamma v_4 \;=\; x v_2 + y v_3 + z v_4$$

with $\alpha, \beta, x, y, z \geq 0$ and $\gamma \in \mathbb{Q}$. We assume $\gamma < 0$. This implies

$$\alpha v_1 + \beta v_3 \;=\; x v_2 + y v_3 + (z - \gamma)\, v_4 \;\in\; \operatorname{cone}(v_1, v_3) \cap \sigma^- \;=\; \operatorname{cone}(v_3),$$

where the identity regarding the cones holds because they belong to Σ'. This implies

$$\alpha \;=\; x \;=\; 0, \qquad \beta \;=\; y, \qquad \gamma \;=\; z.$$

This is a contradiction to $z \geq 0$ and $\gamma < 0$. As a consequence, we obtain $\gamma \geq 0$ and $v \in \sigma^+ \cap \sigma^-$. That means

$$\Big(\sigma^+ + \operatorname{lin}(v_4)\Big) \cap \sigma^- \;\subseteq\; \sigma^+.$$

Next, let $v \in (\sigma^+ + \operatorname{lin}(v_4)) \cap \tau_0$. Then v can be represented as

$$v \;=\; \alpha v_1 + \beta v_3 + \gamma v_4 \;=\; x v_1 + y v_2$$

with $\alpha, \beta, x, y \geq 0$ and $\gamma \in \mathbb{Q}$. We assume $\gamma < 0$. This yields

$$\alpha v_1 + \beta v_3 \;=\; x v_1 + y v_2 - \gamma v_4 \;\in\; \operatorname{cone}(v_1, v_3) \cap \operatorname{cone}(v_1, v_2, v_4) \;=\; \operatorname{cone}(v_1),$$

where the latter equality follows by working in the larger fan Σ. Thus, we get

$$\alpha = x, \qquad y = 0, \qquad \gamma = 0.$$

This contradicts $\gamma < 0$. Consequently, we have $v \in \sigma^- \cap \tau_0$ and

$$\Big(\sigma^+ + \operatorname{lin}(v_4)\Big) \cap \tau_0 \;\subseteq\; \sigma^+.$$

□

Lemma 1.5.6. *Consider P and Σ' as in Construction 1.5.1. Then for $\delta = \sigma^+, \sigma^-, \tau_0$ we have $(\tau_0 + \operatorname{lin}(v_4)) \cap \delta \subseteq \tau_0$.*

Proof. The inclusion $(\tau_0 + \operatorname{lin}(v_4)) \cap \tau_0 \subseteq \tau_0$ is obvious. Let $v \in (\tau_0 + \operatorname{lin}(v_4)) \cap \sigma^+$. In other words, v can be represented by two linear combinations

$$v \;=\; \alpha v_1 + \beta v_2 + \gamma v_4 \;=\; x v_1 + y v_3 + z v_4$$

with $\alpha, \beta, x, y, z \geq 0$ and $\gamma \in \mathbb{Q}$. First, assume $\gamma < 0$. Then we obtain a positive linear combination

$$\alpha v_1 + \beta v_2 \;=\; x v_1 + y v_3 + (z - \gamma)\, v_4 \;\in\; \tau_0 \cap \sigma^+ \;=\; \operatorname{cone}(v_1).$$

The equality regarding the cones applies because they belong to Σ'. We conclude

$$\alpha \;=\; x, \qquad \beta \;=\; y \;=\; 0, \qquad \gamma \;=\; z.$$

This is a contradiction to $z \geq 0$ and $\gamma < 0$. Hence, we get $\gamma \geq 0$ and $v \in \tau_0 \cap \sigma^+$, i.e.

$$(\tau_0 + \operatorname{lin}(v_4)) \cap \sigma^+ \;\subseteq\; \tau_0.$$

Now consider $v \in (\tau_0 + \mathrm{lin}(v_4)) \cap \sigma^-$, i.e.

$$v \;=\; \alpha v_1 + \beta v_2 + \gamma v_4 \;=\; x v_2 + y v_3 + z v_4$$

with $\alpha, \beta, x, y, z \geq 0$ and $\gamma \in \mathbb{Q}$, First, we treat the case $\gamma < 0$. Then we get

$$\alpha v_1 + \beta v_2 \;=\; x v_2 + y v_3 + (z - \gamma)\, v_4 \;\in\; \tau_0 \cap \sigma^- \;=\; \mathrm{cone}(v_2).$$

The latter equation holds since the cones belong to Σ'. Consequently, we obtain

$$\alpha \;=\; 0, \qquad \beta \;=\; x, \qquad \gamma \;=\; z.$$

This contradicts $\gamma < 0$ and implies $\gamma \geq 0$ and $v \in \sigma^- \cap \tau_0$. Thus, we get

$$\Big(\sigma^+ + \mathrm{lin}(v_4)\Big) \cap \tau_0 \;\subseteq\; \sigma^+.$$

□

Proof of Proposition 1.5.3. We exemplarily treat the case $i = 2$. First, we observe $\ker(F_2) = \mathrm{lin}(v_4)$ and compute

$$F_2 \cdot P \;=\; \begin{bmatrix} -1 & -1 & l_1 & 0 \\ 0 & d_0 l_2 & d_1 l_2 + d_2 l_1 & 0 \end{bmatrix}.$$

By the latter, the maximal cones of the fan with generator matrix P_2 are given as

$$\tilde{\sigma}_1 \;=\; F_2(\sigma^-), \qquad \tilde{\sigma}_2 \;=\; F_2(\sigma^+), \qquad \tilde{\sigma}_3 \;=\; F_2(\tau_0).$$

According to [3, Prop. 2.3.1.6], characterising good toric quotients, it suffices to verify

$$F_2^{-1}(F_2(\delta)) \cap \mathrm{Supp}(\Sigma') \;=\; \delta, \qquad \delta \;=\; \sigma^+, \sigma^-, \tau_0.$$

Recall that $\mathrm{Supp}\,(\Sigma') = \sigma^+ \cup \sigma^- \cup \tau_0$. Because of $\ker(F_2) = \mathrm{lin}(v_4)$, the inverse images $F_2^{-1}(F_2(\delta))$ are given by

$$F_2^{-1}(\tilde{\sigma}_1) \;=\; \sigma^- + \mathrm{lin}(v_4), \quad F_2^{-1}(\tilde{\sigma}_2) \;=\; \sigma^+ + \mathrm{lin}(v_4), \quad F_2^{-1}(\tilde{\sigma}_3) \;=\; \tau_0 + \mathrm{lin}(v_4).$$

Finally, Lemma 1.5.4, 1.5.5 and 1.5.6 directly show that $F_2^{-1}(F_2(\delta))$ intersect with the support of the fan Σ' as needed. □

Recall that, given an algebraic action of an algebraic group G on a variety X, the *stabilizer* of a closed subvariety $Y \subseteq X$ is the subgroup $G_Y = \{g \in G;\; g \cdot Y = Y\}$ of G. An element $g \in G$ belongs to G_Y as soon as $g \cdot Y \subseteq Y$.

Proposition 1.5.7. *Let $X \subseteq Z'$ be as in Construction 1.5.1. Then the stabilizer C_i of X with respect to the $\mathbb{K}^*$-actions $s \cdot z = \lambda_i(s) \cdot z$ on Z' from Proposition 1.5.3 is*

$$C_i \;=\; C(l_i) \;=\; \Big\{\zeta \in \mathbb{K}^*;\; \zeta^{l_i} = 1\Big\} \;\subseteq\; \lambda_i(\mathbb{K}^*) \cap \lambda(\mathbb{K}^*),$$

with λ is as in Definition 1.5.2. Moreover, the restriction $\varphi_i\colon X \to Z_i$ of $\psi_i\colon Z' \to Z_i$ is the quotient for the action of C_i on X. In particular, $Z_i = X/C_i$.

Lemma 1.5.8. *Consider Z' from Construction 1.5.1. Then the $\mathbb{K}^*$-actions on Z' induced by the $\lambda_i\colon \mathbb{K}^* \to \mathbb{T}^3$ from Proposition 1.5.3 are given in Cox coordinates by*

$$\lambda_1(s) \cdot [z_1, z_2, z_3, z_4] \;=\; [z_1, z_2, sz_3, z_4], \tag{1.5.8.1}$$

$$\lambda_2(s) \cdot [z_1, z_2, z_3, z_4] \;=\; [z_1, z_2, z_3, sz_4]. \tag{1.5.8.2}$$

Proof. Let $p\colon \hat{X} \to X$ be the characteristic space from Proposition 1.2.4. Then p is given on the acting tori $\mathbb{T}^4 \subseteq \mathbb{K}^4$ and $\mathbb{T}^3 \subseteq Z'$ as

$$p\colon \mathbb{T}^4 \longrightarrow \mathbb{T}^3, \qquad t \longmapsto \left(\frac{t_3^{l_1}}{t_1 t_2}, \frac{t_4^{l_2}}{t_1 t_2}, t_2^{d_0} t_3^{d_1} t_4^{d_2}\right).$$

In particular, we observe $p(1,1,s,1) = \lambda_1(s)$ and $p(1,1,1,s) = \lambda_2(s)$. This yields the assertion. ☐

Lemma 1.5.9. *Consider the toric morphism $\psi_1\colon Z' \to Z_1$ from Construction 1.5.1. Then the following statements hold:*

(i) *The fibers of ψ_1 over the toric fixed points of Z_1 are given by*

$$\begin{aligned}
\psi_1^{-1}([1,0,0]) &= \{[1,0,0,0]\} \cup \{[1,0,t,0];\ t \in \mathbb{K}^*\},\\
\psi_1^{-1}([0,1,0]) &= \{[0,1,0,0]\} \cup \{[0,1,t,0];\ t \in \mathbb{K}^*\},\\
\psi_1^{-1}([0,0,1]) &= \{[0,0,t,1];\ t \in \mathbb{K}^*\}.
\end{aligned}$$

(ii) *Apart from the toric fixed points, the fibers of ψ_1 over $D_0^{Z_1}, D_1^{Z_1}, D_2^{Z_1}$ are*

$$\begin{aligned}
\psi_1^{-1}([1,z,0]) &= \{[1,z,0,0]\} \cup \{[1,z,t,0];\ t \in \mathbb{K}^*\},\\
\psi_1^{-1}([1,0,z]) &= \{[1,0,0,z]\} \cup \{[1,0,t,z];\ t \in \mathbb{K}^*\},\\
\psi_1^{-1}([0,1,z]) &= \{[0,1,0,z]\} \cup \{[0,1,t,z];\ t \in \mathbb{K}^*\}.
\end{aligned}$$

(iii) *Over the acting torus $\mathbb{T}^2 \subseteq Z_1$ the fibers of ψ_1 look as follows*

$$\psi_1^{-1}([1,z_2,z_3]) \;=\; \{[1,z_2,0,z_3]\} \cup \{[1,z_2,t,z_3];\ t \in \mathbb{K}^*\}.$$

Proof. The generator matrix $P_1 = [\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]$ gives rise to the rays $\tilde{\varrho}_i = \operatorname{cone}(\tilde{v}_i)$ and the cones $\tilde{\sigma}_j \coloneqq \operatorname{cone}(\tilde{v}_i;\ i \neq j)$ forming the fan of Z_1. We directly compute

$$\begin{array}{lll}
F_1(\sigma^+)^\circ, & F_1(\operatorname{cone}(v_1,v_4))^\circ & \subseteq\ \tilde{\sigma}_1^\circ,\\
F_1(\sigma^-)^\circ, & F_1(\operatorname{cone}(v_2,v_4))^\circ & \subseteq\ \tilde{\sigma}_0^\circ,\\
F_1(\tau_0)^\circ & & \subseteq\ \tilde{\sigma}_2^\circ,\\
F_1(\operatorname{cone}(v_1,v_3))^\circ, & F_1(\varrho_1)^\circ & \subseteq\ \tilde{\varrho}_0^\circ,\\
F_1(\operatorname{cone}(v_2,v_3))^\circ, & F_1(\varrho_2)^\circ & \subseteq\ \tilde{\varrho}_1^\circ,\\
F_1(\operatorname{cone}(v_3,v_4))^\circ, & F_1(\varrho_4)^\circ & \subseteq\ \tilde{\varrho}_2^\circ,\\
F_1(\{0\}), & F_1(\varrho_3)^\circ & \subseteq\ \{0\}.
\end{array}$$

As a consequence, we can figure out the toric limit points of Z' mapped by ψ_1 to the toric limit points of Z_1:

$$\begin{array}{rlcl}
\tilde{\sigma}_1: & [0,1,0,0],\ [0,1,1,0] & \in & \psi_1^{-1}([0,1,0]),\\
\tilde{\sigma}_0: & [1,0,0,0],\ [1,0,1,0] & \in & \psi_1^{-1}([1,0,0]),\\
\tilde{\sigma}_2: & [0,0,1,1] & \in & \psi_1^{-1}([0,0,1]),\\
\tilde{\varrho}_0: & [0,1,0,1],\ [0,1,1,1] & \in & \psi_1^{-1}([0,1,1]),\\
\tilde{\varrho}_1: & [1,0,0,1],\ [1,0,1,1] & \in & \psi_1^{-1}([1,0,1]),\\
\tilde{\varrho}_2: & [1,1,0,0],\ [1,1,1,0] & \in & \psi_1^{-1}([1,1,0]),\\
\{0\}: & [1,1,1,1],\ [1,1,0,1] & \in & \psi_1^{-1}([1,1,1]).
\end{array}$$

To obtain the whole ψ_1-fibers over the toric limit points of Z_1 we multiply the l.h.s. limit points by $\lambda_1(\mathbb{K}^*)$, using Lemma 1.5.8. Adjusting with the characteristic quasitorus, we arrive at the assertion. ☐

Lemma 1.5.10. *Consider the toric morphism $\psi_2\colon Z' \to Z_2$ from Construction 1.5.1. Then the following statements hold:*

(i) *The fibers of ψ_2 over the toric fixed points of Z_2 are given by*

$$\begin{array}{rcl}
\psi_2^{-1}([1,0,0]) & = & \{[1,0,0,0]\} \cup \{[1,0,0,t];\ t \in \mathbb{K}^*\},\\
\psi_2^{-1}([0,1,0]) & = & \{[0,1,0,0]\} \cup \{[0,1,0,t];\ t \in \mathbb{K}^*\},\\
\psi_2^{-1}([0,0,1]) & = & \{[0,0,1,t];\ t \in \mathbb{K}^*\}.
\end{array}$$

(ii) *Apart from the toric fixed points, the fibers of ψ_2 over $D_0^{Z_2}, D_1^{Z_2}, D_2^{Z_2}$ without the fixed points are given by*

$$\begin{array}{rcl}
\psi_2^{-1}([1,z,0]) & = & \{[1,z,0,0]\} \cup \{[1,z,0,t];\ t \in \mathbb{K}^*\},\\
\psi_2^{-1}([1,0,z]) & = & \{[1,0,z,0]\} \cup \{[1,0,z,t];\ t \in \mathbb{K}^*\},\\
\psi_2^{-1}([0,1,z]) & = & \{[0,1,z,0]\} \cup \{[0,1,z,t];\ t \in \mathbb{K}^*\}.
\end{array}$$

(iii) *Over the acting torus $\mathbb{T}^2 \subseteq Z_2$ the fibers of ψ_2 look as follows*

$$\psi_2^{-1}([1,z_2,z_3]) = \{[1,z_2,z_3,0]\} \cup \{[1,z_2,z_3,t];\ t \in \mathbb{K}^*\}.$$

Proof. Consider the generator matrix $P_2 = [\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]$. Then the fan of Z_2 consists of $\{0\}$, the rays $\tilde{\varrho}_i = \operatorname{cone}(\tilde{v}_i)$ and the cones $\tilde{\sigma}_j := \operatorname{cone}(\tilde{v}_i;\ i \neq j)$. We check

$$\begin{array}{llcl}
F_2(\sigma^+)^\circ, & F_2(\operatorname{cone}(v_1,v_3))^\circ & \subseteq & \tilde{\sigma}_1^\circ,\\
F_2(\sigma^-)^\circ, & F_2(\operatorname{cone}(v_2,v_3))^\circ & \subseteq & \tilde{\sigma}_0^\circ,\\
F_2(\tau_0)^\circ & & \subseteq & \tilde{\sigma}_2^\circ,\\
F_2(\operatorname{cone}(v_1,v_4))^\circ, & F_2(\varrho_1)^\circ & \subseteq & \tilde{\varrho}_0^\circ,\\
F_2(\operatorname{cone}(v_2,v_4))^\circ, & F_2(\varrho_2)^\circ & \subseteq & \tilde{\varrho}_1^\circ,\\
F_2(\operatorname{cone}(v_3,v_4))^\circ, & F_2(\varrho_3)^\circ & \subseteq & \tilde{\varrho}_2^\circ,\\
F_2(\{0\}), & F_2(\varrho_4)^\circ & \subseteq & \{0\}.
\end{array}$$

Consequently, the corresponding limit points in Z_2 and in their fibers are

$$\begin{array}{rlcl}
\tilde{\sigma}_1: & [0,1,0,0],\ [0,1,0,1] & \in & \psi_2^{-1}([0,1,0]),\\
\tilde{\sigma}_0: & [1,0,0,0],\ [1,0,0,1] & \in & \psi_2^{-1}([1,0,0]),\\
\tilde{\sigma}_2: & [0,0,1,1] & \in & \psi_2^{-1}([0,0,1]),\\
\tilde{\varrho}_0: & [0,1,1,0],\ [0,1,1,1] & \in & \psi_2^{-1}([0,1,1]),\\
\tilde{\varrho}_1: & [1,0,1,0],\ [1,0,1,1] & \in & \psi_2^{-1}([1,0,1]),\\
\tilde{\varrho}_2: & [1,1,0,0],\ [1,1,0,1] & \in & \psi_2^{-1}([1,1,0]),\\
\{0\}: & [1,1,1,1],\ [1,1,1,0] & \in & \psi_2^{-1}([1,1,1]).
\end{array}$$

By multiplying the l.h.s. limit points by $\lambda_2(\mathbb{K}^*)$ as given in Lemma 1.5.8, we obtain the entire ψ_2-fibers over the toric limit points of Z_2. By adjusting with the characteristic quasitorus, the claim follows. ☐

Proof of Proposition 1.5.7. We exemplarily treat the case $i = 2$. We show $C_2 \subseteq C(l_2)$. So, consider $s \in \mathbb{K}^*$ with $X = s \cdot X$. Then $X \cap \mathbb{T}^3 = s \cdot (X \cap \mathbb{T}^3)$. Thus, for any $x \in X \cap \mathbb{T}^3$ we have $s \cdot x \in X \cap \mathbb{T}^3$. By Construction 1.2.1 that means

$$1 + x_1 + s^{l_2} x_2 \ = \ 0 \ = \ 1 + x_1 + x_2.$$

Choosing $x_1 = -2, x_2 = 1$ we end up with $s^{l_2} = 1$. Conversely, we see that any $s \in C(l_2)$ stabilizes $X \cap \mathbb{T}^3$, using again the above equation. We conclude that $s \in C(l_2)$ stabilizes also X.

We prove that $\varphi_2 \colon X \to Z_2$ is the quotient for the action of $C_2 = C(l_2)$ on X. To this end, we show that every fiber of φ_2 is a C_2-orbit. For every $[x_1, x_2, x_3, x_4] \in X$ we have

$$x_1 x_2 + x_3^{l_1} + x_4^{l_2} \ = \ 0. \tag{1.5.10.1}$$

Hence, for all $t \in \mathbb{K}^*$ we obtain

$$[1,0,0,t],\ [0,1,0,t],\ [1,t,0,0],\ [1,0,t,0],\ [0,1,t,0] \ \notin \ X. \tag{1.5.10.2}$$

By combining this with Lemma 1.5.10 and using Lemma 1.5.8 we get the fibers over the toric fixed points $z(0), z(1) \in Z_2$:

$$\begin{array}{lclclcl}
\varphi_2^{-1}([1,0,0]) & = & \psi_2^{-1}([1,0,0]) \cap X & = & \{[1,0,0,0]\} & = & C_2 \cdot [1,0,0,0],\\
\varphi_2^{-1}([0,1,0]) & = & \psi_2^{-1}([0,1,0]) \cap X & = & \{[0,1,0,0]\} & = & C_2 \cdot [0,1,0,0].
\end{array}$$

Now we look at $z(2) \in Z_2$. Using Lemma 1.5.8, we see that $\mathbb{T}^3 \cdot z_{\tau_0} \cap X$ consists of the hyperbolic fixed point $x_0 = [0,0,1,\zeta(-1)]$ from Proposition 1.4.3. By Lemma 1.5.10 and (1.5.10.1) we obtain

$$\varphi_2^{-1}([0,0,1]) \ = \ \psi_2^{-1}([0,0,1]) \cap X \ = \ \{[0,0,1,t];\ t \in \mathbb{K}^*\} \cap X \ = \ \{[0,0,1,\sqrt[l_2]{-1}]\}.$$

Similarily, we infer from Lemma 1.5.10, (1.5.10.1) and (1.5.10.2) that the fibers on the divisors $D_0^{Z_2}, D_1^{Z_2}, D_2^{Z_2}$ apart from the fixed points are given by

$$\begin{array}{lclcl}
\varphi_2^{-1}\left([1,z,0]\right) & = & \{[1,z,0,t];\, t\in\mathbb{K}^*\}\cap X & = & C_2\cdot[1,z,0,\sqrt[l_2]{-z}],\\
\varphi_2^{-1}\left([1,0,z]\right) & = & \{[1,0,z,t];\, t\in\mathbb{K}^*\}\cap X & = & C_2\cdot[1,0,z,\sqrt[l_2]{-z^{l_1}}],\\
\varphi_2^{-1}\left([0,1,z]\right) & = & \{[0,1,z,t];\, t\in\mathbb{K}^*\}\cap X & = & C_2\cdot[0,1,z,\sqrt[l_2]{-z^{l_1}}].
\end{array}$$

Now, let $z_2, z_3 \in \mathbb{K}^*$ be fixed. Because of (1.5.10.1) and Lemma 1.5.10 we have either

$$\varphi_2^{-1}\left([1,z_2,z_3]\right) \;=\; \{[1,z_2,z_3,0]\} \;=\; C_2\cdot[1,z_2,z_3,0]$$

or

$$\varphi_2^{-1}\left([1,z_2,z_3]\right) = \{[1,z_2,z_3,t];\, t\in\mathbb{K}^*\}\cap X = C_2\cdot\left\{\left[1,z_2,z_3,\sqrt[l_2]{-z_2-z_3^{l_1}}\right]\right\}.$$

$\square$

Corollary 1.5.11. *Consider $X \subseteq Z'$ as in Construction 1.5.1, the stabilizers C_1, C_2 of $X \subseteq Z'$ from Proposition 1.5.3 and the isotropy groups $G_3, G_4 \subseteq \mathbb{K}^*$ of the nontrivial $\mathbb{K}^*$-orbits in $D_3^X, D_4^X \subseteq X$. Then we have*

$$C_1 \;=\; G_3 \;\subseteq\; \mathrm{Aut}(X), \qquad C_2 \;=\; G_4 \;\subseteq\; \mathrm{Aut}(X).$$

In particular, $X^{C_1} = X^{G_3}$ and $X^{C_2} = X^{G_4}$. Moreover, $C_1 = C(l_1)$ has general isotropy group $C(\gcd(l_1, l_2))$ along D_4^X and $C_2 = C(l_2)$ has general isotropy group $C(\gcd(l_1, l_2))$ along D_3^X.

Proof. The involved $\mathbb{K}^*$-actions, i.e., those from Definition 1.5.2 arise from one-parameter subgroups $\lambda, \lambda_1, \lambda_2$ of the acting torus $\mathbb{T}^3 \subseteq Z'$. Moreover, Proposition 1.5.3 yields $C_i = C(l_i)$. We directly check

$$\lambda(\mathbb{K}^*)\cap\lambda_i(\mathbb{K}^*) \;=\; \lambda(C(l_i)), \qquad \lambda_1(\mathbb{K}^*)\cap\lambda_2(\mathbb{K}^*) \;=\; \lambda(C(\gcd(l_1,l_2))$$

on the torus $\mathbb{T}^3$, using that the entries l_1, d_1 from the third column of P as well as l_2, d_2 from the fourth one are pairwise coprime. Thus, the displayed identities of the assertion hold in $\mathbb{T}^3 \subseteq \mathrm{Aut}(Z')$ and hence in $\mathbb{K}^* \subseteq \mathrm{Aut}(X)$. $\square$

CHAPTER

TWO

$\mathbb{K}^*$-SURFACES OF PICARD NUMBER ONE AND INTEGRAL DEGREE

This chapter provides an explicit description of all quasismooth, rational, projective surfaces of Picard number one that admit a non-trivial torus action and have an integral canonical self intersection number. The Construction 2.5.1 of toric degenerations appears in the joint article [27] with Milena Wrobel and Jürgen Hausen and the results of Section 2.1 to 2.6 have been made public in the joint article [26] with Jürgen Hausen.

2.1 Squared Markov type equations

Given any number $a \in \mathbb{Z}_{>0}$, the associated *squared Markov type equation* in the variables w_0, w_1, w_2 is

$$(w_0 + w_1 + w_2)^2 \ = \ a w_0 w_1 w_2.$$

By a *solution* we mean a triple $u = (u_0, u_1, u_2) \in \mathbb{Z}^3_{>0}$ satisfying this equation. We denote by $S(a) \subseteq \mathbb{Z}^3_{>0}$ the set of all solutions.

The set $S(a)$ is in bijection with the solution set of the corresponding usual Markov type equation, studied exhaustively by Hurwitz [32]; see Remark 2.1.10. We expect everything of this section to be known; see e.g. [14, Thm. 11] for the case $a = 9$. However, we couldn't find appropriate references and hence allow ourselves to provide an elementary self-contained treatment according to our needs.

Lemma 2.1.1. *Let $a \in \mathbb{Z}_{>0}$. Then we obtain an involution on the set $S(a)$ of solutions of the squared Markov type equation by*

$$\lambda\colon S(a) \ \to \ S(a), \quad u \ \mapsto \ \left(u_0, u_1, \tfrac{(u_0+u_1)^2}{u_2}\right) \ = \ (u_0, u_1, a u_0 u_1 - 2u_0 - 2u_1 - u_2)\,.$$

Proof. Let $u \in S(a)$. We show that the two representations of $\lambda(u)$ coincide. This merely means to compare the third components:

$$au_0u_1 - 2u_0 - 2u_1 - u_2 = \tfrac{(u_0+u_1+u_2)^2}{u_2} - 2u_0 - 2u_1 - u_2 = \tfrac{(u_0+u_1)^2}{u_2}.$$

In particular, λ maps solutions to positive integer tuples. We prove that $u' := \lambda(u)$ is a solution. With $\hat{u} := u_0 + u_1$, we obtain

$$a = \tfrac{(u_0+u_1+u_2)^2}{u_0u_1u_2} = \tfrac{(\hat{u}+u_2)^2}{u_0u_1u_2} = \tfrac{\left(\hat{u}+\frac{\hat{u}^2}{u_2}\right)^2}{u_0u_1\frac{\hat{u}^2}{u_2}} = \tfrac{(u_0'+u_1'+u_2')^2}{u_0'u_1'u_2'}.$$

□

A *one-step mutation* of a triple $u \in S(a)$ is a permutation of its entries followed by the operation λ. A *mutation* of u is a composition of one-step mutations.

Theorem 2.1.2. *Fix $a \in \mathbb{Z}_{>0}$ such that the equation $(w_0 + w_1 + w_2)^2 = aw_0w_1w_2$ admits a solution. Then we have*

$$a \ \in \ \{1, 2, 3, 4, 5, 6, 8, 9\}.$$

For these a, the solutions of the above equation are precisely the mutations of the following triples

$$\left(\tfrac{9}{a}, \tfrac{9}{a}, \tfrac{9}{a}\right), \quad a = 1, 3, 9, \qquad \left(\tfrac{8}{a}, \tfrac{8}{a}, \tfrac{16}{a}\right), \quad a = 1, 2, 4, 8,$$

$$\left(\tfrac{6}{a}, \tfrac{12}{a}, \tfrac{18}{a}\right), \quad a = 1, 2, 3, 6, \qquad \left(\tfrac{5}{a}, \tfrac{20}{a}, \tfrac{25}{a}\right), \quad a = 1, 5.$$

By the *norm* of a triple $u \in \mathbb{Z}^3_{>0}$, we mean the number $\nu(u) := u_0+u_1+u_2$. Obviously the norm is invariant under coordinate permutations. We call a triple $u \in S(a)$ *initial* if $u_0 \le u_1 \le u_2$ and $u_2 \le u_0 + u_1$.

Lemma 2.1.3. *Consider a solution $u \in S(a)$. Set $u' := \lambda(u)$. Then we have the following equivalences:*

$$\begin{aligned} \nu(u) = \nu(u') &\iff u_2 = u_2' \iff u_2 = u_0 + u_1, \\ \nu(u) < \nu(u') &\iff u_2 < u_2' \iff u_2 < u_0 + u_1. \end{aligned}$$

In particular, an ascendingly ordered $u \in S(a)$ is initial if and only if $\nu(u) \le \nu(\tilde{u})$ for any one-step mutation $\tilde{u}$ of u.

Proof. By the definition of λ, the vectors u and u' differ only in the last coordinate. This gives the first equivalence in each row. The remaining two equivalences follow from $u_2u_2' = (u_0 + u_1)^2$. □

Lemma 2.1.4. *Consider a solution $u \in S(a)$.*

(i) *We have $\frac{1}{u_0} + \frac{1}{u_1} \le \frac{a}{4}$.*

(ii) *If u is initial, then* $\frac{a}{4} \le \frac{3}{u_2} + \frac{1}{u_0}$.

Proof. For (i), write the squared Markov type equation as $f(w_2) = 0$ with a polynomial $f \in \mathbb{K}[w_0, w_1][w_2]$. Then f is of degree two in w_2 with discriminant

$$\Delta(f) \;=\; a^2 w_0^2 w_1^2 - 4a w_0^2 w_1 - 4a w_0 w_1^2 \;=\; 4a w_0^2 w_1^2 \left(\tfrac{a}{4} - \tfrac{1}{w_0} - \tfrac{1}{w_1}\right).$$

Since $u \in S(a)$ has positive integer entries, $\Delta(f)$ evaluates non-negatively at u. The assertion follows. We turn to (ii). The definition of an initial triple gives

$$u_2 \le u_0 + u_1, \quad u_0 + u_1 + u_2 \le 2(u_0 + u_1), \quad u_0 \le u_1 \le u_2.$$

We conclude

$$\tfrac{a}{4} \;=\; \tfrac{1}{4}\tfrac{(u_0+u_1+u_2)^2}{u_0u_1u_2} \;\le\; \tfrac{(u_0+u_1)^2}{u_0u_1u_2} \;=\; \tfrac{u_0}{u_1u_2} + \tfrac{2}{u_2} + \tfrac{u_1}{u_0u_2} \;\le\; \tfrac{3}{u_2} + \tfrac{1}{u_0}.$$

□

Proof of Theorem 2.1.2. By Lemma 2.1.3 every solution is a mutation of an initial one. So, it suffices to classify the initial solutions. For $a \ge 5$, any initial triple $u \in S(a)$ satisfies the inequality

$$u_2 \;\le\; \tfrac{12}{a-4},$$

use Lemma 2.1.4 (ii) and $1/u_0 \le 1$. This effectively bounds the largest entry u_2 of the initial triples $u \in S(a)$ for $a \ge 5$. Concretely, we arrive at

a	≥ 9	8	7	6	5
u_2	≤ 2	≤ 3	≤ 4	≤ 5	≤ 6

In each case, we check the (finitely many) ascendingly ordered $u \in \mathbb{Z}^3_{>0}$ satisfying the respective bound on u_2 for initial triples. For $a \ge 10$ and $a = 7$, there are no initial triples and in each of the other cases, there is exactly one:

a	9	8	6	5
u	$(1,1,1)$	$(1,1,2)$	$(1,2,3)$	$(1,4,5)$

Case $a = 4$. Here, Lemma 2.1.4 (i) rules out the case $u_0 = 1$. Thus, $u_0 \ge 2$ and we obtain $u_2 \le 6$, because otherwise Lemma 2.1.4 (ii) would give

$$1 = \tfrac{a}{4} \le \tfrac{3}{u_2} + \tfrac{1}{u_0} \le \tfrac{3}{7} + \tfrac{1}{2}.$$

Going through the ascendingly ordered $u \in \mathbb{Z}^3_{>0}$ with $u_0 \ge 2$ and $u_2 \le 6$, we end up with $(2, 2, 4)$ as the only initial triple in $S(6)$.

Case $a = 3$. Again Lemma 2.1.4 (i) rules out the case $u_0 = 1$. Thus, $u_0 \geq 2$, forcing $u_2 \leq$ 12, because otherwise Lemma 2.1.4 (ii) would give

$$\tfrac{3}{4} = \tfrac{a}{4} \leq \tfrac{3}{u_2} + \tfrac{1}{u_0} \leq \tfrac{3}{13} + \tfrac{1}{2}.$$

The search over all ascendingly ordered $u \in \mathbb{Z}^3_{>0}$ with $u_0 \geq 2$ and $u_2 \leq 12$ produces exactly two initial triples in $S(3)$, namely $(2, 4, 6)$ and $(3, 3, 3)$.

Case $a = 2$. Here, Lemma 2.1.4 (i) rules out $u_0 = 1$ and $u_0 = 2$. Thus, $u_0 \geq 3$ and $u_2 \leq 18$, because otherwise Lemma 2.1.4 (ii) would imply

$$\tfrac{1}{2} \;=\; \tfrac{a}{4} \leq \tfrac{3}{u_2} + \tfrac{1}{u_0} \;\leq\; \tfrac{3}{19} + \tfrac{1}{3}.$$

Inside the set of all ascendingly ordered $u \in \mathbb{Z}^3_{>0}$ with $u_0 \geq 3$ and $u_2 \leq 18$, we find $(4, 4, 8)$ and $(3, 6, 9)$ as the only two initial triples belonging to $S(2)$.

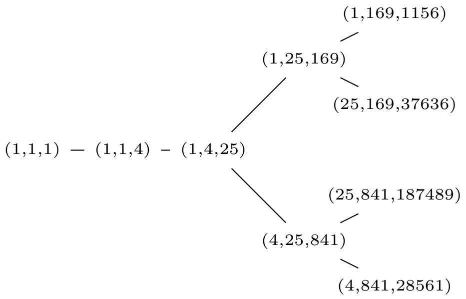
Case $a = 1$. Using Lemma 2.1.4 (i), we can exclude $u_0 = 1, 2, 3, 4$. Thus, $u_0 \geq 5$ and we derive $u_2 \leq 60$, as otherwise Lemma 2.1.4 (ii) would claim

$$\tfrac{1}{4} = \tfrac{a}{4} \leq \tfrac{3}{u_2} + \tfrac{1}{u_0} \leq \tfrac{3}{61} + \tfrac{1}{5}.$$

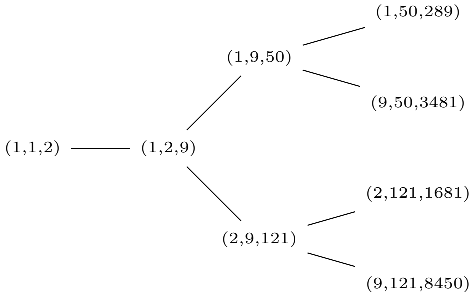
The search in the ascendingly ordered $u \in \mathbb{Z}^3_{>0}$ with $5 \leq u_0$ and $u_2 \leq 60$ yields exactly four initial triples in $S(1)$, namely $(9, 9, 9)$, $(8, 8, 16)$, $(6, 12, 18)$ and $(5, 20, 25)$. $\square$

Remark 2.1.5. Given a, let $T(a) \subseteq S(a)$ denote the subset of all ascendingly ordered solution triples. For $a \in \{9, 8, 6, 5\}$, we can regard $T(a)$ as the vertex set of a tree, where we join two triples $u, u' \in T(a)$ by an edge if they are distinct and arise from each other by a one-step mutation.

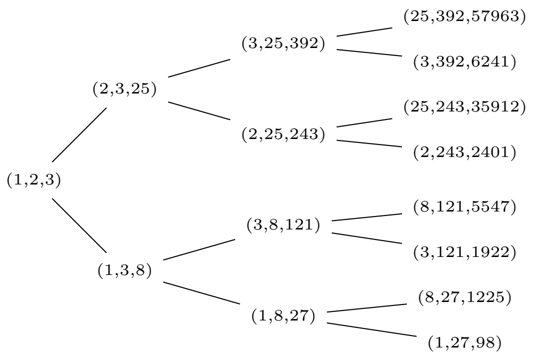
$T(9)$



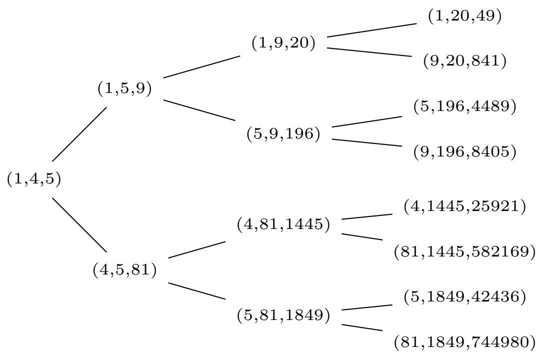
$T(8)$



$T(6)$



$T(5)$



Proposition 2.1.6. *Take $a \in 1, 2, 3, 4, 5, 6, 8, 9$, let $a = ba'$ a factorization into positive integers and consider the set of scaled triples*

$$bS(a) := \{(bu_0, bu_1, bu_2);\ u \in S(a)\}.$$

Then we have $bS(a) \subseteq S(a')$. Moreover, we can express $S(a')$ for $a' = 4, 3, 2, 1$ in terms of the $S(a)$ for $a = 9, 8, 6, 5$ as

$$S(4) = 2S(8), \qquad S(3) = 3S(9) \cup 2S(6), \qquad S(2) = 4S(8) \cup 3S(6),$$

$$S(1) = 9S(9) \cup 8S(8) \cup 6S(6) \cup 5S(5).$$

In particular, $T(4)$ is a tree, each of $T(3)$ and $T(2)$ is a union of two disjoint trees and $T(1)$ is a union of four disjoint trees.

Lemma 2.1.7. *Consider a solution $u \in S(a)$ of the squared Markov type equation and a mutation $\tilde{u}$ of u. Then $\gcd(u_0, u_1, u_2) = \gcd(\tilde{u}_0, \tilde{u}_1, \tilde{u}_2)$.*

Proof. It suffices to treat the case of an ascendingly ordered u and $\tilde{u} = \lambda(u)$. Then $\tilde{u}_i = u_i$ for $i = 0, 1$ and $\tilde{u}_2 = au_0u_1 - 2u_0 - 2u_1 - u_2$. The assertion directly follows. $\square$

Proof of Proposition 2.1.6. In order to check $bS(a) \subseteq S(a')$, let $u \in S(a)$. Then the scaled solution bu satisifies

$$\tfrac{(bu_0+bu_1+bu_2)^2}{bu_0bu_1bu_2} = \tfrac{1}{b}\tfrac{(u_0+u_1+u_2)^2}{u_0u_1u_2} = \tfrac{a}{b} = a'.$$

Thus, $bu \in S(a')$. This verifies in particular all inclusions "$\supseteq$" of the displayed equations in the proposition. We show exemplarily $S(4) = 2S(8)$. Let $u' \in S(4)$. Then Theorem 2.1.2 tells us that u' is a mutation of $(2,2,4)$. Thus, Lemma 2.1.7 ensures $u' = 2u$ with $u \in \mathbb{Z}^3_{>0}$ and the above calculation shows $u \in S(8)$. ☐

Theorem 2.1.8. *Let $a \in \{1,2,3,4,5,6,8,9\}$ and $u \in S(a)$. Then there exist integers $x_0, x_1, x_2 \in \mathbb{Z}_{>0}$ such that, up to permuting the entries, u is of the form*

$$\tfrac{9}{a}\left(x_0^2, x_1^2, x_2^2\right), \quad a = 1,3,9, \qquad \tfrac{8}{a}\left(x_0^2, x_1^2, 2x_2^2\right), \quad a = 1,2,4,8,$$

$$\tfrac{6}{a}\left(x_0^2, 2x_1^2, 3x_2^2\right), \quad a = 1,2,3,6, \qquad \tfrac{5}{a}\left(x_0^2, x_1^2, 5x_2^2\right), \quad a = 1,5.$$

Moreover, for $a = 5,6,8,9$ the entries of u are pairwise coprime and for any a the above numbers x_0, x_1, x_2 satisfy the equation

$$\xi_0x_0^2 + \xi_1x_1^2 + \xi_2x_2^2 = \sqrt{a\xi_0\xi_1\xi_2}x_0x_1x_2, \quad (\xi_0,\xi_1,\xi_2) = \begin{cases} \left(\frac{9}{a}, \frac{9}{a}, \frac{9}{a}\right), & a = 1,3,9, \\ \left(\frac{8}{a}, \frac{8}{a}, \frac{16}{a}\right), & a = 1,2,4,8, \\ \left(\frac{6}{a}, \frac{12}{a}, \frac{18}{a}\right), & a = 1,2,3,6, \\ \left(\frac{5}{a}, \frac{5}{a}, \frac{25}{a}\right), & a = 1,5. \end{cases}$$

Lemma 2.1.9. *Let $a \in \mathbb{Z}_{>0}$ and $u \in S(a)$. Then $\gcd(u_0,u_1) = \gcd(u_0,u_1,u_2)$ holds.*

Proof. First recall that $u \in S(a)$ means $(u_0+u_1+u_2)^2 = au_0u_1u_2$. We rewrite this equation as

$$u_2^2 + (2u_0 + 2u_1 - au_0u_1)u_2 + (u_0+u_1)^2 \ = \ 0.$$

Now, set $q := \gcd(u_0,u_1)$. The task is to show that q divides u_2. Dividing the above equation by q^2 yields

$$\left(\tfrac{u_2}{q}\right)^2 + \left(2\tfrac{u_0+u_1}{q} - a\tfrac{u_0u_1}{q}\right)\tfrac{u_2}{q} + \left(\tfrac{u_0+u_1}{q}\right)^2 \ = \ 0.$$

Since q divides $u_0 + u_1$ and u_0u_1, this is an equation of integral dependence for the fraction u_2/q. As a factorial ring $\mathbb{Z}$ is normal and we can conclude $u_2/q \in \mathbb{Z}$. ☐

Proof of Theorem 2.1.8. Consider any solution $u \in S(a)$ of the squared Markov type equation, where $a = 9,8,6,5$. Theorem 2.1.2 and Lemma 2.1.9 tell us $\gcd(u_0,u_1,u_2) = 1$. Let ξ_i be the minimal divisors of u_i such that $a\xi_0\xi_1\xi_2$ is a square number and set $x_i := u_i/\xi_i$. Then we can write

$$(u_0+u_1+u_2)^2 \ = \ au_0u_1u_2 \ = \ a\xi_0\xi_1\xi_2x_0x_1x_2,$$

where we may assume $\xi_0 \le \xi_1 \le \xi_2$. By Lemma 2.1.7, the u_i are pairwise coprime. Thus, the x_i are pairwise coprime as well and, consequently, must be square numbers. We claim that, according to the value of a, the vector $\xi = (\xi_0, \xi_1, \xi_2)$ is given as follows:

a	9	8	6	5
ξ	$(1,1,1)$	$(1,1,2)$	$(1,2,3)$	$(1,1,5)$

This is obvious for $a = 9, 8, 5$. For $a = 6$, Lemma 2.1.9 and Lemma 2.1.7 yield that precisely one of the entries of u is divisible by 2 and precisely one by 3. For the initial triple $(1, 2, 3)$, no entry is divisible by 6. Applying stepwise Lemma 2.1.9 and Lemma 2.1.7, we see that the latter holds for any mutation of $(1, 2, 3)$. This excludes $\xi = (1, 1, 6)$.

Thus, the assertion is verified for $a = 9, 8, 6, 5$. For $a = 4, 3, 2, 1$, we stress again Theorem 2.1.2 and Lemma 2.1.7 and see that in this case the solutions all are obtained by scaling solutions of the cases $a = 9, 8, 6, 5$ with suitable integers. The assertion follows. ☐

Remark 2.1.10. Theorem 2.1.8 shows in particular that the solutions of the *squared* Markov type equations can be obtained from the solutions of the usual Markov type equations considered in [32], see also [34, Sec. 3].

2.2 More about fake weighted projective spaces

First introductory background on fake weighted projective planes has been given in Section 1.1. Here we gather the more specific topics needed in the subsequent sections of this chapter.

Proposition 2.2.1. *Consider a fake weighted projective space $Z = Z(P)$ as in Construction 1.1.7, its fake weight vector w and a splitting $K := \mathbb{Z}^r/\mathrm{im}(P^*) = \mathbb{Z} \oplus \Gamma$ as in Remark 1.1.13. Then the divisor class group and any degree matrix of Z satisfy*

$$\mathrm{Cl}(Z) \;\cong\; K, \qquad Q \;=\; \begin{bmatrix} u_0 & \dots & u_n \\ \eta_0 & \dots & \eta_n \end{bmatrix}, \qquad \begin{array}{l} \mu := \gcd(w_0, \dots, w_n), \\ u := \mu^{-1} \cdot w. \end{array}$$

The torsion part Γ of $\mathrm{Cl}(Z)$ is of order μ. The local class groups of the toric fixed points and their orders are given by

$$\mathrm{Cl}(Z, z(i)) \;=\; K/\mathbb{Z} \cdot q_i, \qquad \mathrm{cl}(Z, z(i)) \;=\; [K : \mathbb{Z} \cdot q_i] \;=\; w_i,$$

where $q_i = (u_i, \eta_i)$ are the columns of the degree matrix Q. Finally, the Cox ring of Z together with its $\mathrm{Cl}(Z)$-grading is given by

$$\mathcal{R}(Z) \;=\; \mathbb{K}[T_0, \dots, T_n], \qquad \deg(T_i) \;=\; q_i \;\in\; K.$$

Proof. For the shape of Q, note that P annihilates the fake weight vector w and thus u generates the kernel of P. For the remaining statements, see Proposition 1.1.2, Proposition 1.1.5 and Remark 1.1.8. □

Corollary 2.2.2. *For any n-dimensional fake weighted projective space, the torsion part of $\mathrm{Cl}(Z)$ is generated by at most $n-1$ elements.*

Remark 2.2.3. Consider a fake weighted projetive plane $Z = Z(P)$. If the associated fake weight vector w is primitive, then $Q = [w_0, \dots, w_n]$ and $Z = \mathbb{P}(w_0, \dots, w_n)$ is an ordinary weighted projective space.

Proposition 2.2.4 (Cf. [27, Prop. 2.5])**.** *Consider a fake weighted projective plane $Z = Z(P)$. An anticanonical divisor on Z is given by*

$$-\mathcal{K}_Z \;=\; D_0 + D_1 + D_2, \qquad D_i \coloneqq V(T_i) \subseteq Z.$$

Moreover, the canonical self intersection number can be expressed in terms of the fake weight vector $w = (w_0, w_1, w_2)$ as

$$\mathcal{K}_Z^2 \;=\; \tfrac{(w_0+w_1+w_2)^2}{w_0 w_1 w_2}.$$

Proof. The first statement is standard toric geometry; see for example [10, Thm. 8.2.3]. For the second one, we may assume that the generator matrix P of Z is of the form

$$P \;=\; \begin{bmatrix} l_0 & l_1 & l_2 \\ d_0 & d_1 & d_2 \end{bmatrix}, \qquad l_i \neq 0, \; i = 0, 1, 2.$$

For the claim on the intersection number we recall the following from [10, Prop. 6.3.8]. For any two distinct columns v_i, v_j of P in positive orientation, the intersection number of the associated divisors D_i, D_j is given as

$$D_i \cdot D_j \;=\; \det(v_i, v_j)^{-1}.$$

Moreover, we can compute the self intersection number of a divisor D_j. Taking all three columns v_i, v_j, v_k of P in positive orientation, we have

$$D_j^2 \;=\; -\tfrac{\det(v_i,v_k)}{\det(v_i,v_j)\det(v_j,v_k)}.$$

Together with the representation of the w_i from Remark 1.1.8, this yields the claim. □

The following construction shows how to gain fake weighted projective planes Z directly from degree matrices and it allows us to switch between the presentations of a given Z in terms of a degree matrix or a corresponding degree matrix.

Construction 2.2.5. Let $K = \mathbb{Z} \oplus \Gamma$ with Γ finite abelian and let $Q = [q_0, \ldots, q_n]$ with $q_i \in \mathbb{Z}_{>0} \oplus \Gamma$ a degree matrix in K. The quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ has K as its character group and acts on $\mathbb{K}^{n+1}$ via

$$h \cdot z \;=\; (\chi^{q_0}(h)z_0, \ldots, \chi^{q_n}(h)z_n).$$

The orbit space $Z(Q) := \mathbb{K}^{n+1} \setminus \{0\}$ is a fake weighted projective space. More precisely, starting with Q, we can produce the pair of mutually dual exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}^{n+1} \arrow[r, "P"] & \mathbb{Z}^n & \\
0 & K \arrow[l] & \mathbb{Z}^{n+1} \arrow[l, "Q"'] & \mathbb{Z}^n \arrow[l, "P^*"'] & 0, \arrow[l]
\end{tikzcd}$$

by choosing the columns of P^* as the members of a $\mathbb{Z}$-basis for the kernel of Q, regarded as the map $\mathbb{Z}^{n+1} \to K$, $e_i \to q_i$. Then the transpose P is a projective $n \times (n+1)$ generator matrix with Q as corresponding degree matrix and we have

$$Z(Q) \;\cong\; Z(P).$$

Example 2.2.6. Consider a 2×3 degree matrix Q in K. Then, thanks to Corollary 2.2.2, we have $K = \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$, that means

$$Q \;=\; \begin{bmatrix} u_0 & u_1 & u_2 \\ \bar{\eta}_0 & \bar{\eta}_1 & \bar{\eta}_2 \end{bmatrix}, \qquad u_0, u_1, u_2 \in \mathbb{Z}_{>0}, \quad \bar{\eta}_0, \bar{\eta}_1, \bar{\eta}_2 \in \mathbb{Z}/\mu\mathbb{Z}.$$

Moreover, the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ is explicitly given as $H = \mathbb{K}^* \times C(\mu)$, where $C(\mu) \subseteq \mathbb{K}^*$ is the group of μ-th roots of unity and H acts on $\mathbb{K}^3$ via

$$(t, \zeta) \cdot (z_0, z_1, z_2) \;=\; (t^{u_0}\zeta^{\eta_0} z_0,\, t^{u_1}\zeta^{\eta_1} z_1,\, t^{u_2}\zeta^{\eta_2} z_2).$$

Proposition 2.2.7. *Let Q, Q' be degree matrices in $K = \mathbb{Z} \oplus \Gamma$ and Z, Z' the associated weighted projective spaces. Then the following statements are equivalent:*

- (i) *We have an isomorphism of the varieties $Z \cong Z'$.*
- (ii) *We have an isomorphism of graded algebras $\mathcal{R}(Z) \cong \mathcal{R}(Z')$.*
- (iii) *$Q' = [\imath(q_0), \ldots, \imath(q_n)] \cdot B$ with $\imath \in \operatorname{Aut}(K)$ and a permutation matrix B.*

Proof. The equivalence of the first two statements holds more generally for any $\mathbb{Q}$-factorial Mori dream space of Picard number one; see [3, Rem. 3.3.4.2]. The equivalence of the second and the third and statement is clear by Proposition 2.2.1 and the definition of an isomorphism of graded algebras. ☐

Proposition 2.2.7 becomes an efficient classification tool in the case of a cyclic torsion part when we combine it with the following.

Lemma 2.2.8. *Consider the abelian group* $G := \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$*. Then any choice of* $\epsilon = \pm 1$*,* $\bar{a} \in \mathbb{Z}/\mu\mathbb{Z}$ *and* $\bar{c} \in (\mathbb{Z}/\mu\mathbb{Z})^*$ *defines an automorphism*

$$\varphi_{\epsilon,\bar{a},\bar{c}} \colon G \ \to \ G, \qquad (k, \bar{m}) \ \mapsto \ (\epsilon k,\, \bar{a}\bar{k} + \bar{c}\bar{m}).$$

Conversely, every automorphism $\varphi \colon G \to G$ *is of this form. In particular,* $\mathrm{Aut}(G)$ *is of order* $2\mu\Phi(\mu)$*, where* Φ *denotes Euler's totient function.*

Proof. One directly checks that $\varphi_{\epsilon,\bar{a},\bar{c}}$ is an automorphism of G. Let $\varphi \colon G \to G$ be any automorphism. We claim

$$\varphi(1, \bar{0}) \ = \ (\epsilon, \bar{a}), \quad \epsilon = \pm 1,\ \bar{a} \in \mathbb{Z}/\mu\mathbb{Z} \qquad \varphi(0, \bar{1}) \ = \ (0, \bar{c}), \quad \bar{c} \in (\mathbb{Z}/\mu\mathbb{Z})^*.$$

Indeed, $(0, \bar{1})$ generates the torsion part $\mathbb{Z}/\mu\mathbb{Z}$, which in turn is fixed by φ, hence generated by $\varphi(0, \bar{1})$. This gives the second identity. So far, we have

$$\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z} \ = \ \mathbb{Z} \cdot \varphi(1, \bar{0}) \oplus \mathbb{Z} \cdot \varphi(0, \bar{1}) \ = \ \mathbb{Z} \cdot (\epsilon, \bar{a}) \oplus \mathbb{Z} \cdot (0, \bar{c})$$

with some $\epsilon \in \mathbb{Z}$ and $\bar{a} \in \mathbb{Z}/\mu\mathbb{Z}$. This forces $\epsilon = \pm 1$, verifying the claim. Now, comparing the values on $(1, \bar{0})$ and $(0, \bar{1})$ shows $\varphi = \varphi_{\epsilon,\bar{a},\bar{c}}$. ☐

Remark 2.2.9. Consider $K = \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ and a degree matrix $Q = [q_0, \ldots, q_n]$ in K, where $q_i = (u_i, \bar{\eta}_i)$. Only the automorphims of the form $\varphi_{1,\bar{a},\bar{c}}$ from Lemma 2.2.8 respect positivity in the first line of Q. Moreover, observe

$$\left[\begin{array}{ccc} \varphi_{1,\bar{a},\bar{c}}(q_0) & \ldots & \varphi_{1,\bar{a},\bar{c}}(q_n) \end{array}\right] \ = \ \left[\begin{array}{cc} 1 & 0 \\ a & c \end{array}\right] \cdot \left[\begin{array}{ccc} u_0 & \ldots & u_n \\ \bar{\eta}_0 & \ldots & \bar{\eta}_n \end{array}\right].$$

That means that we can realize in practice the application of an automorphism $\varphi_{1,\bar{a},\bar{c}}$ by matrix multiplication or, equivalently, by stepwise adding multiples of the upper to the lower row and scaling the lower row by units.

Remark 2.2.10. Let $Q = [q_0, \ldots, q_n]$ be a degree matrix in $K = \mathbb{Z} \oplus \Gamma$. Then the fake weight vector of $Z(Q)$ can be written as

$$w \ = \ w(Q) \ = \ (w_0, \ldots, w_n), \qquad w_i \ = \ [K : \mathbb{Z} \cdot q_i].$$

We say that Q is *associated* with $w = w(Q)$. Note that $w(Q) = w(Q')$ may happen even if $Z(Q)$ and $Z(Q')$ are not isomorphic to each other.

Definition 2.2.11. Let $a \in \{1, 2, 3, 4, 5, 6, 8, 9\}$, $w \in S(a)$, $\mu := \gcd(w_0, w_1, w_2)$ and $u := \mu^{-1} \cdot w$. We call w *adjusted* if one of the following holds

- $a = 9$ and $w_0 \le w_1 \le w_2$,
- $a = 8$, $w_0 \le w_1$ and $2 \mid w_2$,
- $a = 6$, $2 \mid w_1$ and $3 \mid w_2$,
- $a = 5$, $w_0 \le w_1$ and $5 \mid w_2$,

- $a \leq 4$ and u is adjusted.

Moreover, we call a degree matrix Q associated with $w \in S(a)$ *adjusted* if w is adjusted and for $a \leq 4$, the matrix Q is of the form

$$Q = \begin{bmatrix} u_0 & u_1 & u_2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad 1 \leq \eta < \mu, \quad \bar{\eta} = \begin{cases} \bar{2}, & a = 1,\ u = (1,1,1), \\ \bar{2}, \bar{5}, & a = 1,\ u = (1,1,4), \\ \bar{1}, \bar{3}, \bar{5}, & a = 2,\ u = (1,1,2). \end{cases}$$

Proposition 2.2.12. *Let $a \in \{1,2,3,4,5,6,8,9\}$, $w \in S(a)$ adjusted and Q, Q' adjusted degree matrices associated with w. Then we have*

$$Z(Q) \cong Z(Q') \iff Q = Q'.$$

Proof. The assertion follows from Proposition 2.2.7. □

Example 2.2.13. For $a = 1,2,3,4,5,6,8,9$, we list divisor class group, corresponding generator and adjusted degree matrices for the fake weighted projective planes having an initial triple as fake weight vector:

$$\begin{array}{cc} a = 9 & \mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 8 & \mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 6 & \mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} a = 5 & \mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 4 & \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 3 & \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} a = 3 & \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -6 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 2 & \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -8 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 2 & \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \\ \begin{bmatrix} 2 & 2 & -3 \\ 1 & -3 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} a = 2 & \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -9 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 2 & \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \\ \begin{bmatrix} 3 & 3 & -3 \\ 1 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \\ \begin{bmatrix} 3 & 3 & -6 \\ 1 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -16 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \\ \begin{bmatrix} 4 & 4 & -4 \\ 1 & -3 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \\ \begin{bmatrix} 2 & 2 & -2 \\ 0 & -7 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -18 & 12 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \\ \begin{bmatrix} 3 & 3 & -3 \\ 2 & -4 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & -25 & 20 \end{bmatrix} & \begin{bmatrix} 1 & 4 & 5 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \\ \begin{bmatrix} 5 & 5 & -5 \\ 3 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 4 & 5 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \\ \begin{bmatrix} 5 & 5 & -5 \\ 1 & -4 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 4 & 5 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix} \end{array} \qquad \begin{array}{cc} a = 1 & \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \\ \begin{bmatrix} 5 & 5 & -5 \\ 2 & -3 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 4 & 5 \\ \bar{0} & \bar{1} & \bar{4} \end{bmatrix} \end{array}$$

2.3 Fake weighted projective planes of integral degree

We classify the fake weighted projective planes Z of integral degree $\mathcal{K}_Z^2$. Proposition 2.2.4 shows that the potential fake weight vectors of such Z are the solution triples of the squared Markov type equations. It turns out that in fact every solution triple occurs as a fake weight vector. We determine and verify all associated adjusted degree matrices and list them in a redundance free manner.

Proposition 2.3.1. *Let Z be a projective toric surface of Picard number one of degree $a := \mathcal{K}_Z^2 \in \mathbb{Z}$. Then $a \in \{1,2,3,4,5,6,8,9\}$.*

Proof. We may assume $Z = Z(P)$. Then Proposition 2.2.4 yields that the fake weight vector of Z is a solution triple of a squared Markov type equation. Thus, Theorem 2.1.2 yields the assertion. □

Proposition 2.3.2. *Let Z be a projective toric surface of Picard number one of degree $a \in \{5,6,8,9\}$. Then Z is isomorphic to a weighted projective space $\mathbb{P}(w_0,w_1,w_2)$ for exactly one adjusted $w \in S(a)$.*

Proof. We may assume $Z = Z(P)$. By Proposition 2.2.4, the fake weight vector w of Z satisfies $w \in S(a)$. By Theorem 2.1.8, the entries w_0, w_1, w_2 of w are pairwise coprime. Thus, $Z \cong \mathbb{P}(w_0,w_1,w_2)$ is an ordinary weighted projective plane; see Remark 2.2.3. Taking adjusted solution triples, makes the presentation redundance free. □

For the fake weighted projective planes Z of degree $a \in \{1,2,3,4\}$, we will encounter torsion in the divisor class group. We will list the resulting Z in terms of adjusted degree matrices in the sense of Definition 2.2.11. Whereas the procedure of finding and verifying those follows a common pattern for all $q = 1,2,3,4$, the detailed arguing relies on the specific divisibility properties of the solution triples from $S(a)$ in each case.

Lemma 2.3.3. *Let $Q = [q_0,q_1,q_2]$ be a degree matrix in $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$, write $q_i = (u_i, \bar{\eta}_i)$ and assume $\gcd(u_0,\mu) = 1$. Then we find an automorphism of $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ which, applied to columns, turns Q into*

$$Q' = \begin{bmatrix} u_0 & u_1 & u_2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \bar{\eta} \in (\mathbb{Z}/\mu\mathbb{Z})^*.$$

We have $Z(Q) \cong Z(Q')$ and Q, Q' share the same corresponding generator matrices. Moreover, the local class group of $z(i) \in Z(Q)$ is cyclic of order μu_i and the fake weight vector is given by $w(Q) = \mu \cdot (u_0,u_1,u_2)$.

Proof. Proposition 2.2.7 and Lemma 2.2.8 tell us that suffices to apply suitable operations from Remark 2.2.9. Thanks to $\gcd(u_0,\mu) = 1$, we have $\bar{u}_0 \in (\mathbb{Z}/\mu\mathbb{Z})^*$. Hence $\bar{\eta}_0 = \bar{a}\bar{u}_0$ for some $a \in \mathbb{Z}$. Subtracting $\bar{a} \cdot (\bar{u}_0, \bar{u}_1, \bar{u}_2)$ from the second row, we achieve $\bar{\eta}_0 = \bar{0}$. As each of the pairs q_0, q_1 and q_0, q_2 generates $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ as a group, we can conclude $\bar{\eta}_1, \bar{\eta}_2 \in (\mathbb{Z}/\mu\mathbb{Z})^*$. Scaling the second row of Q with the multiplicative inverse on $\bar{\eta}_1$ finally turns Q into Q'.

By Construction 2.2.5 and Proposition 2.2.1, the local class groups $K_i := \mathrm{Cl}(Z, z_i)$ are given as $K_i = K/\mathbb{Z} \cdot q'_i$, where $K = \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ and q'_i is the i-th column of Q'. Clearly K_i is generated by the classes of $(1, \bar{0})$ and $(0, \bar{1})$. We treat K_0. Let $\zeta, \kappa \in \mathbb{Z}$ with $\zeta\mu + \kappa u_0 = 1$. Then we observe

$$(1, \bar{0}) \ = \ \zeta\mu \cdot (1, \bar{1}) + \kappa \cdot (u_0, \bar{0}), \qquad (0, \bar{1}) \ = \ (1, \bar{1}) - (1, \bar{0}),$$

and conclude that the class of $(1, \bar{1})$ generates K_0. Moreover, due to the coprimeness of u_0 and μ, the class $(1, \bar{1})$ in K_0 of order μu_0. Now consider K_i for $i = 1, 2$. Set $\eta_1 := 1$ and $\eta_2 := \eta$. Choosing $\zeta_i \in \mathbb{Z}$ with $\bar{\zeta}_i \bar{\eta}_i = -\bar{1}$ in $\mathbb{Z}/\mu\mathbb{Z}$. Then

$$(0, \bar{1}) \ = \ \zeta_i u_i \cdot (1, \bar{0}) - \zeta_i \cdot (u_i, \bar{\eta}_i),$$

and thus the class of $(1, \bar{0})$ generates K_i, where $i = 1, 2$. Consequently, each of the local class groups $\mathrm{Cl}(Z, z(i)) = K_i$ is cyclic of order μu_i. The statement on the fake weight vector is clear by Proposition 2.2.1. □

Proposition 2.3.4. *Let Z be a fake weighted projective plane of degree 4. Then $Z \cong Z(Q)$ with $w = w(Q)$ adjusted and there are $x_0, x_1, x_2 \in \mathbb{Z}_{>0}$ such that*

$$(x_0^2, x_1^2, 2x_2^2) \ \in \ S(8), \qquad w \ = 2 \cdot (x_0^2, x_1^2, 2x_2^2).$$

Moreover, the divisor class group is $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the degree matrix Q can be choosen as

$$Q(w) \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}.$$

Any such matrix is a degree matrix of a fake weighted projective plane of degree 4 and distinct matrices represent non-isomorphic fake weighted projective planes.

Proof. We may assume $Z = Z(P)$ with a generator matrix P. Let Q be a corresponding degree matrix. Suitably renumbering the columns of both, we achieve that $w = w(P)$ is adjusted. By Proposition 2.2.4, we have $w \in S(4)$. Proposition 2.1.6 says $w = 2 \cdot u$ with $u \in S(8)$. From Theorem 2.1.8 we infer $u = (x_0^2, x_1^2, 2x_2^2)$ and pairwise coprimeness of the entries of u. Proposition 2.2.1 shows $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and establishes the first row of Q. Since x_0 is coprime to $2x_2^2$, it must be odd. Thus, Lemma 2.3.3 allows us to assume $Q = Q(w)$ as claimed.

Now consider any $Q(w)$ as in the assertion, that means a matrix with first row $u = (x_0^2, x_1^2, 2x_2^2)$ from $S(8)$ and second row $(\bar{0}, \bar{1}, \bar{1})$ with entries in $\mathbb{Z}/2\mathbb{Z}$. We verify that $Q(w)$ is a degree matrix of a fake weighted projective plane with the claimed properties. Given two columns q_i, q_j of Q we have to show that these generate $K = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as a group. Since u_i, u_j are coprime, we find $(1, \bar{0})$ or $(1, \bar{1})$ in the span of q_i, q_j. In the first case, we are done. In the second case, observe that we find also an element $(b, \bar{0})$ with b

odd in the span of q_i, q_j. Subtracting the $(b-1)$-fold of $(1, \bar{1})$ gives $(1, \bar{0})$ and we are done as well. Thus, $Q(w)$ is a degree matrix in K. Set $Z = Z(Q(w))$. Lemma 2.3.3 ensures that Z has fake weight vector $2 \cdot u \in S(4)$ and we can apply Proposition 2.2.4 to see that Z is of degree 4. Finally, Proposition 2.2.12 ensures that distinct matrices $Q(w)$ define non-isomorphic fake weighted projective planes. □

Proposition 2.3.5. *Let Z be a fake weighted projective plane of degree 3. Then $Z \cong Z(Q)$ with $w = w(Q)$ adjusted and there are $x_0, x_1, x_2 \in \mathbb{Z}_{>0}$ such that we are in one of the following situations:*

(i) *we have $(x_0^2, x_1^2, x_2^2) \in S(9)$, $w = 3 \cdot (x_0^2, x_1^2, x_2^2)$, the divisor class group is $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, and the degree matrix Q can be chosen as*

$$Q(w) \;=\; \begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix},$$

(ii) *we have $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$, $w = 2 \cdot (x_0^2, 2x_1^2, 3x_2^2)$, the divisor class group is $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the degree matrix Q can be chosen as*

$$Q(w) \;=\; \begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}.$$

Any matrix as above is a degree matrix of a fake weighted projective plane of degree 3 and distinct matrices represent non-isomorphic fake weighted projective planes.

Lemma 2.3.6. *Let $u \in S(9)$. Then $\bar{u}_i = \bar{1} \in \mathbb{Z}/3\mathbb{Z}$ holds for $i = 0, 1, 2$. In particular, $3 \nmid u_i$ for $i = 0, 1, 2$.*

Proof. Clearly the assertion holds for $u = (1, 1, 1)$. By Theorem 2.1.2, we only have to show that, if $u \in S(9)$ satisfies the assertion, then $u' = \lambda(u)$ does so. Recall

$$u_0' \;=\; u_0, \quad u_1' \;=\; u_1, \quad u_2' \;=\; 9u_0u_1 - 2u_0 - 2u_1 - u_2$$

from the construction of λ given in Lemma 2.1.1. For u_0' and u_1' is nothing to show and, passing to the classes in $\mathbb{Z}/3\mathbb{Z}$, we directly see $\bar{u}_2' = -\bar{5} = \bar{1}$. □

Proof of Proposition 2.3.5. We may assume $Z = Z(P)$ and $w = w(P)$ being adjusted. By Proposition 2.2.4, we have $w \in S(3)$. According to Proposition 2.1.6, we either have $w = 3 \cdot u$ with $u \in S(9)$ or $w = 2 \cdot u$ with $u \in S(6)$. We go through each of these two cases and establish the claimed shape for a corresponding degree matrix Q.

Case (i): $w = 3 \cdot u$ with $u \in S(9)$. Theorem 2.1.8 provides us with $u = (x_0^2, x_1^2, x_2^2)$ and Proposition 2.2.1 yields that the torsion part of $\mathrm{Cl}(Z)$ is of order three. Moreover, Lemma 2.3.6 says in particular $\gcd(x_0^2, 3) = 1$ and thus Lemma 2.3.3 shows that Q can be choosen to be of the shape

$$Q \;=\; \begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \eta \in (\mathbb{Z}/3\mathbb{Z})^*.$$

We show $\bar{\eta} \neq \bar{1}$. Otherwise, due to Lemma 2.3.6, adding $\bar{2} \cdot (\bar{x}_0^2, \bar{x}_1^2, \bar{x}_2^2)$ turns the second row of Q into $(\bar{2}, \bar{0}, \bar{0})$, which contradicts to Remark 2.2.9, guaranteeing that this operation respects the properties of a degree matrix.

We check that each of the matrices $Q(w)$ from Case (i) is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ defining a fake weighted projective plane Z as claimed. Given two columns q_i, q_j of $Q(w)$, we use Lemma 2.3.6 to observe

$$u_j q_i - u_i q_j \ = \ (0, \bar{\eta}_i + \bar{2}\bar{\eta}_j) \ \in \ \{(0, \bar{1}), \, (0, \bar{2})\},$$

where $q_k = (u_k, \bar{\eta}_k)$. Since u_i and u_j are coprime, $(1, \bar{\zeta})$ lies in the span of q_i, q_j for some $\zeta \in \mathbb{Z}/3\mathbb{Z}$. We conclude that q_i, q_j generate K as a group. Lemma 2.3.3 ensures that Z has fake weight vector $3 \cdot (x_0^2, x_1^2, x_2^2) \in S(3)$. Thus, Proposition 2.2.4 yields that Z is of degree 3.

Case (ii): $w = 2 \cdot u$ with $u \in S(6)$. By Theorem 2.1.8 we know $u = (x_0^2, 2x_1^2, 3x_2^2)$ and that u has pairwise coprime entries. In particular, x_0^2 is odd. Proposition 2.2.1 gives $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and by Lemma 2.3.3, the degree matrix can be choosen as

$$Q \ = \ \begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}.$$

We check that each of the matrices $Q(w)$ from Case (ii) is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ defining a fake weighted projective plane Z as claimed. As $u = (x_0^2, 2x_1^2, 3x_2^2)$ has pairwise coprime entries, x_0 and x_2 are odd. Thus, for any two columns q_i, q_j of $Q(w)$, we obtain

$$u_j q_i - u_i q_j \ = \ (0, \bar{1}).$$

Moreover, since u_i and u_j are coprime, $(1, \bar{\zeta})$ lies in the span of q_i, q_j for $\zeta \in \mathbb{Z}/2\mathbb{Z}$. We conclude that q_i, q_j generate K as a group. Lemma 2.3.3 ensures that Z has fake weight vector $2 \cdot u \in S(3)$ and Proposition 2.2.4 guarantees that Z is of degree 3.

Finally, in each of the Cases (i) and (ii), Proposition 2.2.12 tells us that distinct matrices $Q(w)$ define non-isomorphic fake weighted projective planes. □

Proposition 2.3.7. *Let Z be a fake weighted projective plane of degree 2. Then $Z \cong Z(Q)$ with $w = w(Q)$ adjusted and there are $x_0, x_1, x_2 \in \mathbb{Z}_{>0}$ such that we are in one of the following situations:*

(i) *we have $(x_0^2, x_1^2, x_2^2) \in S(8)$, $w = 4 \cdot (x_0^2, x_1^2, 2x_2^2)$, the divisor class group is $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and the degree matrix Q can be chosen as*

$$Q(w, \bar{1}) \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}, \qquad Q(w, \bar{3}) \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix},$$

(ii) *we have $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$, $w = 3 \cdot (x_0^2, 2x_1^2, 3x_2^2)$, the divisor class group is $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and the degree matrix Q can be chosen as*

$$Q(w, \bar{1}) \ = \ \begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}, \qquad Q(w, \bar{2}) \ = \ \begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}.$$

Any matrix as above is a degree matrix of a fake weighted projective plane of degree 2 *and distinct matrices represent non-isomorphic fake weighted projective planes.*

Lemma 2.3.8. *Let* $u \in S(8)$ *be adjusted. Then* $(\bar{u}_0, \bar{u}_1, \bar{u}_2) = (\bar{1}, \bar{1}, \bar{2})$ *holds in* $\mathbb{Z}/4\mathbb{Z}$.

Proof. The claim holds for $u = (1, 1, 2)$. By Theorem 2.1.2 it suffices to show that if $u \in S(8)$ fulfills the claim after adjusting, then $u' := \lambda(u)$ does so. Lemma 2.1.1 says

$$u'_0 \ = \ u_0, \quad u'_1 \ = \ u_1, \quad u'_2 \ = \ 8u_0u_1 - 2u_0 - 2u_1 - u_2.$$

If $\bar{u}_2 = \bar{2}$ holds, then $\bar{u}_0 = \bar{u}_1 = \bar{1}$ and we compute $\bar{u}'_2 = -\bar{u}_2 = \bar{2}$. If $\bar{u}_2 = \bar{1}$ holds, then one of $\bar{u}_0, \bar{u}_1$ equals $\bar{1}$ and the other equals $\bar{2}$. We obtain $\bar{u}'_2 = -\bar{3} = \bar{1}$. □

Lemma 2.3.9. *Let* $u \in S(6)$ *be adjusted. Then* $(\bar{u}_0, \bar{u}_1, \bar{u}_2) = (\bar{1}, \bar{2}, \bar{0})$ *holds in* $\mathbb{Z}/3\mathbb{Z}$.

Proof. The assertion is true for $u = (1, 2, 3)$. By Theorem 2.1.2, we only have to show that, if $u \in S(6)$ fulfills the claim after adjusting, then $u' = \lambda(u)$ does so. According to Lemma 2.1.1, we have

$$u'_0 \ = \ u_0, \quad u'_1 \ = \ u_1, \quad u'_2 \ = \ 6u_0u_1 - 2u_0 - 2u_1 - u_2.$$

If $\bar{u}_2 = \bar{0}$, then we may assume $\bar{u}_0 = \bar{1}$ and $\bar{u}_1 = \bar{2}$ and compute $\bar{u}'_2 = \bar{0}$. If $\bar{u}_2 = \bar{1}$, then we may assume $\bar{u}_0 = \bar{0}$ and $\bar{u}_1 = \bar{2}$ and compute $\bar{u}'_2 = -\bar{5} = \bar{1}$. If $\bar{u}_2 = \bar{2}$, then we may assume $\bar{u}_0 = \bar{0}$ and $\bar{u}_1 = \bar{1}$ and compute $\bar{u}'_2 = -\bar{4} = \bar{2}$. □

Proof of Proposition 2.3.7. We may assume $Z = Z(P)$ and that $w = w(P)$ is adjusted. Let Q be a degree matrix corresponding to P. Proposition 2.2.4 yields $w \in S(2)$. According to Proposition 2.1.6, we either have $w = 4 \cdot u$ with $u \in S(8)$ or $w = 3 \cdot u$ with $u \in S(6)$.

Case (i): $w = 4 \cdot u$ with $u \in S(8)$. Then $u = (x_0^2, x_1^2, 2x_2^2)$ and u has pairwise coprime entries; see Theorem 2.1.8. Moreover, from Proposition 2.2.1 and Corollary 2.2.2 we infer that the toriosn part of $\mathrm{Cl}(Z)$ is cyclic of order four. Coprimeness of x_0^2 and $2x_2^2$ forces x_0^2 to be odd and thus Lemma 2.3.3 allows us to assume

$$Q \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \bar{\eta} \in (\mathbb{Z}/4\mathbb{Z})^* = \{\bar{1}, \bar{3}\}.$$

We verify that each of the matrices $Q(w, \bar{\eta})$ from Case (i) is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, defining a fake weighted projective plane Z as claimed. Lemma 2.3.8 delivers $(\bar{u}_0, \bar{u}_1, \bar{u}_2) = (\bar{1}, \bar{1}, \bar{2})$ for $u = (x_0^2, x_1^2, 2x_2^2)$. Thus, given two columns q_i, q_j of $Q(w, \bar{\eta})$, we can compute

$$u_j q_i - u_i q_j \ \in \ \{(0, \bar{1}), (0, \bar{3})\}.$$

Moreover, since u_i and u_j are coprime, $(1, \bar{\zeta})$ lies in the span of q_i, q_j for some $\zeta \in \mathbb{Z}/4\mathbb{Z}$. We conclude that q_i, q_j generate K as a group. Lemma 2.3.3 establishes the fake weight vector $w = 4 \cdot u \in S(2)$ and Proposition 2.2.4 says that Z is of degree 2.

Case (ii): $w = 3 \cdot u$ with $u \in S(6)$. Then $u = (x_0^2, 2x_1^2, 3x_2^2)$ and u has pairwise coprime entries as provided by Theorem 2.1.8. Proposition 2.2.1 shows that $\mathrm{Cl}(Z)$ has torsion part of order three. Moreover, $\gcd(x_0^2, 3) = 1$ brings Lemma 2.3.3 into the game and we can assume

$$Q \;=\; \begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \bar{\eta} \in (\mathbb{Z}/3\mathbb{Z})^*.$$

We check that each of the matrices $Q(w, \bar{\eta})$ from Case (ii) is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and defines a fake weighted projective plane Z as claimed. From Lemma 2.3.9 we know

$$(\bar{x}_0^2, \bar{2}\bar{x}_1^2, \bar{3}\bar{x}_2^2) \;=\; (\bar{1}, \bar{2}, \bar{0}) \;\in\; \mathbb{Z}/3\mathbb{Z}.$$

Thus, given two columns q_i, q_j of Q, we see that $u_j q_i - u_i q_j$ equals either $(0, \bar{1})$ or $(0, \bar{2})$, where $u = (x_0^2, 2x_1^2, 3x_2^2)$. Since u_i and u_j are coprime, we find additionally $(1, \bar{\zeta})$ in the span of q_i, q_j for some $\zeta \in \mathbb{Z}/3\mathbb{Z}$. Thus, q_i, q_j generate K as a group. Lemma 2.3.3 tells us that the fake weight vector of Z is $w = 3 \cdot u \in S(2)$. Thus, Proposition 2.2.4 ensures that Z is of degree 2.

As in the preceding proofs, Proposition 2.2.12 guarantees that in all the cases just treated, distinct matrices define non-isomorphic varieties. ☐

Proposition 2.3.10. *Let Z be a fake weighted projective plane of degree 1. Then $Z \cong Z(Q)$ with $w = w(Q)$ adjusted and there are $x_0, x_1, x_2 \in \mathbb{Z}_{>0}$ such that we are in one of the following situations:*

(i) *we have $(x_0^2, x_1^2, x_2^2) \in S(9)$, $w = 9 \cdot (x_0^2, x_1^2, x_2^2)$, the divisor class group is given by $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ and the degree matrix Q can be chosen as*

$$Q(w, \bar{\eta}) \;=\; \begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \bar{\eta} = \bar{2}, \bar{5}, \bar{8}.$$

(ii) *we have $(x_0^2, x_1^2, 2x_2^2) \in S(8)$, $w = 8 \cdot (x_0^2, x_1^2, 2x_2^2)$, the divisor class group is $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ and the degree matrix Q can be chosen as*

$$Q(w, \bar{\eta}) \;=\; \begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \bar{\eta} = \bar{1}, \bar{3}, \bar{5}, \bar{7}.$$

(iii) *we have $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$, $w = 6 \cdot (x_0^2, 2x_1^2, 3x_2^2)$, the divisor class group is $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and the degree matrix Q can be chosen as*

$$Q(w, \bar{\eta}) \;=\; \begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \bar{\eta} = \bar{1}, \bar{5}.$$

(iv) *we have* $(x_0^2, x_1^2, 5x_2^2) \in S(5)$, $w = 5 \cdot (x_0^2, x_1^2, 5x_2^2)$, *the divisor class group is* $\mathrm{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ *and the degree matrix* Q *can be chosen as*

$$Q(w, \bar\eta) \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar 0 & \bar 1 & \bar\eta \end{bmatrix}, \qquad \bar\eta = \bar 1, \bar 2, \bar 3, \bar 4.$$

Any $Q(w, \bar\eta)$ *as above is a degree matrix of a fake weighted projective plane of degree* 1. *Two distinct* $Q(w, \bar\eta)$ *define non-isomorphic fake weighted projective planes, except they stem from (1-9-*$*$*) or (1-8-*$*$*) and are both taken from one of the sets*

$$\left\{ \begin{bmatrix} 1 & 1 & 1 \\ \bar 0 & \bar 1 & \bar 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ \bar 0 & \bar 1 & \bar 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ \bar 0 & \bar 1 & \bar 8 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 & 1 & 4 \\ \bar 0 & \bar 1 & \bar 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 4 \\ \bar 0 & \bar 1 & \bar 8 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 1 & 1 & 2 \\ \bar 0 & \bar 1 & \bar 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ \bar 0 & \bar 1 & \bar 7 \end{bmatrix} \right\}.$$

In these cases, for any pair of degree matrices stemming from a common set, the associated fake weighted projective planes are in fact isomorphic to each other.

Lemma 2.3.11. *Let* $u \in S(8)$ *be adjusted. Then* $(\bar u_0, \bar u_1, \bar u_2) = (\bar 1, \bar 1, \bar 2)$ *in* $\mathbb{Z}/8\mathbb{Z}$.

Proof. The claim holds for $u = (1, 1, 2)$. By Theorem 2.1.2 it suffices to show that if $u \in S(8)$ fulfills the claim after adjusting, then $u' := \lambda(u)$ does so. Lemma 2.1.1 says

$$u'_0 \ = \ u_0, \quad u'_1 \ = \ u_1, \quad u'_2 \ = \ 8u_0u_1 - 2u_0 - 2u_1 - u_2.$$

If $\bar u_2 = \bar 2$ holds, then $\bar u_0 = \bar u_1 = \bar 1$ and we compute $\bar u'_2 = -\bar 6 = \bar 2$. If $\bar u_2 = \bar 1$ holds, then one of $\bar u_0, \bar u_1$ equals $\bar 1$ and the other equals $\bar 2$. We obtain $\bar u'_2 = -\bar 7 = \bar 1$. □

Lemma 2.3.12. *Let* $u \in S(6)$ *be adjusted. Then* $(\bar u_0, \bar u_1, \bar u_2) = (\bar 1, \bar 2, \bar 3)$ *in* $\mathbb{Z}/6\mathbb{Z}$.

Proof. The assertion is valid for $u = (1, 2, 3)$. By Theorem 2.1.2, we only have to show that, if $u \in S(6)$ fulfills the claim after adjusting, then $u' = \lambda(u)$ does as well. According to Lemma 2.1.1, we have

$$u'_0 \ = \ u_0, \quad u'_1 \ = \ u_1, \quad u'_2 \ = \ 6u_0u_1 - 2u_0 - 2u_1 - u_2.$$

If $\bar u_2 = \bar 3$, then we may assume $\bar u_0 = \bar 1$ and $\bar u_1 = \bar 2$ and compute $\bar u'_2 = -\bar 9 = \bar 3$. If $\bar u_2 = \bar 1$, then we may assume $\bar u_0 = \bar 2$ and $\bar u_1 = \bar 3$ and compute $\bar u'_2 = -\overline{11} = \bar 1$. If $\bar u_2 = \bar 2$, then we may assume $\bar u_0 = \bar 1$ and $\bar u_1 = \bar 3$ and compute $\bar u'_2 = -\overline{10} = \bar 2$. □

Lemma 2.3.13. *Let* $u \in S(5)$ *be adjusted. Then* $(\bar u_0, \bar u_1, \bar u_2) = (\bar 1, \bar 4, \bar 0)$, $(\bar 4, \bar 1, \bar 0)$, *in* $\mathbb{Z}/5\mathbb{Z}$.

Proof. The assertion is true for $u = (1, 4, 5)$. Due to Theorem 2.1.2, it is enough to show that, if $u \in S(5)$ fulfills the claim after adjusting, then $u' = \lambda(u)$ does so. By Lemma 2.1.1, we have

$$u'_0 \ = \ u_0, \quad u'_1 \ = \ u_1, \quad u'_2 \ = \ 5u_0u_1 - 2u_0 - 2u_1 - u_2.$$

If $\bar u_2 = \bar 0$, then up to switching $\bar u_0 = \bar 1$ and $\bar u_1 = \bar 4$ and we obtain $\bar u'_2 = -\overline{10} = \bar 0$. If $\bar u_2 = \bar 1$, then up to switching $\bar u_0 = \bar 0$ and $\bar u_1 = \bar 4$ and we obtain $\bar u'_2 = -\bar 9 = \bar 1$. If $\bar u_2 = \bar 4$, then up to switching $\bar u_0 = \bar 0$ and $\bar u_1 = \bar 1$ and we obtain $\bar u'_2 = -\bar 6 = \bar 4$. □

Proof of Proposition 2.3.10. We may assume $Z = Z(P)$ and that $w = w(P)$ is an adjusted fake weight vector. Let Q be a degree matrix corresponding to P. Proposition 2.2.4 yields $w \in S(1)$. By Proposition 2.1.6, one of the following holds

$$w = 9 \cdot u,\ u \in S(9),\ \ w = 8 \cdot u,\ u \in S(8),\ \ w = 6 \cdot u,\ u \in S(6),\ \ w = 5 \cdot u,\ u \in S(5).$$

Case (i): $w = 9 \cdot u$ with $u \in S(9)$. Theorem 2.1.8 provides us with $u = (x_0^2, x_1^2, x_2^2)$, having pairwise coprime entries. Due to Proposition 2.2.1 and Corollary 2.2.2, the torsion part of $\mathrm{Cl}(Z)$ is cyclic of order nine. Lemma 2.3.3 and Lemma 2.3.6 allow us to assume

$$Q \ = \ \begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad \bar{\eta} \in (\mathbb{Z}/9\mathbb{Z})^* = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}.$$

We exclude $\bar{\eta} = \bar{1}, \bar{4}, \bar{7}$. Otherwise, Q would be a degree matrix for one of these values. Then $(0, \bar{1})$ must lie in the span of the last two columns of Q. Since x_1, x_2 are coprime, this means $\bar{x}_2^2 = \bar{x}_1^2 \bar{\eta} + \bar{1}$ in $\mathbb{Z}/9\mathbb{Z}$. Inserting the values of $\bar{\eta}$ gives

$$\bar{x}_2^2 \ = \ \bar{x}_1^2 + \bar{1}, \qquad \bar{x}_2^2 \ = \ \bar{4}\bar{x}_1^2 + \bar{1}, \qquad \bar{x}_2^2 \ = \ \bar{7}\bar{x}_1^2 + \bar{1},$$

as identities in $\mathbb{Z}/9\mathbb{Z}$. Moreover, Lemma 2.3.6 says that $\bar{x}_1^2, \bar{x}_2^2$ are taken from $\{\bar{1}, \bar{4}, \bar{7}\}$ as well. A direct computation shows that none of the above identities can be satisfied this way. Thus, $\bar{\eta}$ must be one of $\bar{2}, \bar{5}, \bar{8}$.

We show that $Q(w, \bar{\eta})$ for $\bar{\eta} = \bar{2}, \bar{5}, \bar{8}$ is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$. Since $u = (x_0^2, x_1^2, x_2^2)$ has are pairwise coprime entries, we find an element of the form $(1, \bar{\zeta})$ with $\bar{\zeta} \in \mathbb{Z}/9\mathbb{Z}$ in the span of any two columns of Q. Moreover, $(0, \bar{\kappa}_{ij})$ lies the span of the columns q_i, q_j of $Q(w, \bar{\eta})$ for

$$\bar{\kappa}_{ij} \ := \ \bar{x}_j^2 \bar{\eta}_i - \bar{x}_i^2 \bar{\eta}_j,$$

where we write $q_k = (x_k^2, \bar{\eta}_k)$. Recall that due to Lemma 2.3.6 the classes $\bar{x}_i^2, \bar{x}_j^2$ belong to $\{\bar{1}, \bar{4}, \bar{7}\}$. Thus, for $i = 0$ and $j = 1, 2$ we see due to $\bar{\eta}_0 = \bar{0}$ that $\bar{\kappa}_{ij} = \bar{x}_0^2 \bar{\eta}_j$ is a unit in $\mathbb{Z}/9\mathbb{Z}$. Moreover,

$$\bar{\kappa}_{12} = \bar{x}_2^2 - \bar{\eta}_2 \bar{x}_1^2 \in (\mathbb{Z}/9\mathbb{Z})^*$$

is checked via directly by inserting all possible values $\bar{1}, \bar{4}, \bar{7}$ for the $\bar{x}_i$ and $\bar{2}, \bar{5}, \bar{8}$ for $\bar{\eta}_2$. Lemma 2.3.3 shows that the fake weight vector of the fake weighted projective plane Z associated with $Q(w, \bar{\eta})$ is $w = 9 \cdot u \in S(1)$ for $u = (x_0^2, x_1^2, x_2^2)$. Thus, Proposition 2.2.4 yields that Z is of degree 1.

Case (ii): $w = 8 \cdot u$ with $u \in S(8)$. Then $u = (x_0^2, x_1^2, 2x_2^2)$ and u has pairwise coprime entries; see Theorem 2.1.8. Proposition 2.2.1 and Corollary 2.2.2 ensure that $\mathrm{Cl}(Z)$ has cyclic torsion part of order eight. As x_0 is odd by coprimeness of $x_0, 2x_2$, Lemma 2.3.3 allows us to assume

$$Q \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{\eta}_0 & \bar{\eta}_1 & \bar{\eta}_2 \end{bmatrix}, \qquad \bar{0}, \bar{1}, \bar{\eta} \in \mathbb{Z}/8\mathbb{Z} = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}.$$

Let us see why each $Q(w, \bar\eta)$ is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, defining a fake weighted projective plane Z as claimed. Lemma 2.3.11 delivers $(\bar u_0, \bar u_1, \bar u_2) = (\bar 1, \bar 1, \bar 2)$ for $u = (x_0^2, x_1^2, 2x_2^2)$. Using this, we derive for any two columns q_i, q_j of $Q(w, \bar\eta)$ the following:

$$u_j q_i - u_i q_j \ \in \ \{(0, \bar\zeta);\ \zeta \in (\mathbb{Z}/8\mathbb{Z})^*\}.$$

Indeed, for q_0, q_j this obvious and for q_1, q_2, we arrive at $\bar\zeta = \bar 2 - \bar\eta$, which is a unit for $\eta = \bar 1, \bar 3, \bar 5, \bar 7$. Moreover, since u_i and u_j are coprime, $(1, \bar\zeta)$ belongs to the span of q_i, q_j. We conclude that q_i, q_j generate K as a group. Lemma 2.3.3 delivers the fake weight vector $w = 8 \cdot u \in S(1)$ with $u = (x_0^2, x_1^2, 2x_2^2)$. So, Proposition 2.2.4 yields $\mathcal{K}_Z^2 = 1$.

Case (iii): $w = 6 \cdot u$ with $u \in S(6)$. Then $u = (x_0^2, 2x_1^2, 3x_2^2)$ and u has pairwise coprime entries by Theorem 2.1.8. Proposition 2.2.1 yields that the torsion part of $\mathrm{Cl}(Z)$ is $\mathbb{Z}/6\mathbb{Z}$. Moreover, x_0^2 is coprime to 6 and thus Lemma 2.3.3 says that we may assume

$$Q \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar 0 & \bar 1 & \bar\eta \end{bmatrix}, \qquad \bar\eta \in (\mathbb{Z}/6\mathbb{Z})^* = \{\bar 1, \bar 5\}.$$

Let us see that each of the matrices $Q(w, \bar\eta)$ is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and defines the desired fake weighted projective plane Z. For $u = (x_0^2, 2x_1^2, 3x_2^2)$, Lemma 2.3.12 delivers $(\bar x_0^2, \bar 2\bar x_1^2, \bar 3\bar x_2^2) = (\bar 1, \bar 2, \bar 3)$. Using this, we obtain that any two columns q_i, q_j of $Q(w, \bar\eta)$, satisfy

$$u_j q_i - u_i q_j \ \in \ \{(0, \bar 1), (0, \bar 5)\}.$$

For q_0, q_j this is obvious. Moreover, one computes $u_2 q_1 - u_1 q_2 = (0, \bar 3 - \bar 2\bar\eta)$. By coprimeness of u_i and u_j, the span of q_i, q_j contains $(1, \bar\zeta)$ for some $\zeta \in \mathbb{Z}/6\mathbb{Z}$. Hence, q_i, q_j generate K as a group. Lemma 2.3.3 provides us with the fake weight vector $w = 6 \cdot u \in S(1)$. Using Proposition 2.2.4, we get $\mathcal{K}_Z^2 = 1$.

Case (iv): $w = 5 \cdot u$ with $u \in S(5)$. Then $u = (x_0^2, x_1^2, 5x_2^2)$ and u has pairwise coprime entries by Theorem 2.1.8. According to Proposition 2.2.1, the torsion part of $\mathrm{Cl}(Z)$ is cyclic of order five and by Lemma 2.3.3, we can choose

$$Q \ = \ \begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar 0 & \bar 1 & \bar\eta \end{bmatrix}, \qquad \bar\eta \in (\mathbb{Z}/5\mathbb{Z})^*.$$

We check that each of the $Q(w, \bar\eta)$ is a degree matrix in $K := \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, defining a fake weighted projective plane Z as claimed. We know $(\bar x_0^2, \bar x_1^2, \bar 5\bar x_2^2) = (\bar 1, \bar 4, \bar 0)$ from Lemma 2.3.13. Set $u = (x_0^2, x_1^2, 5x_2^2)$. Then, given two columns q_i, q_j of $Q(w, \bar\eta_2)$, we observe

$$u_j q_i - u_i q_j \ \in \ \{(0, \bar\zeta);\ \bar\zeta \in (\mathbb{Z}/5\mathbb{Z})^*\}.$$

This is obvious for q_0, q_j and $u_2 q_1 - u_1 q_2$ evaluates to $(0, -\bar 4\bar\eta)$. Moreover, since u_i and u_j are coprime, some $(1, \bar\zeta)$ lies in the span of q_i, q_j. Lemma 2.3.3 gives the fake weight vector $w = 5 \cdot u \in S(1)$. Proposition 2.2.4 yields that Z is of degree 1.

Once more, Proposition 2.2.12 guarantees that in all the cases, apart from the exceptions listed in the assertion, distinct matrices define non-isomorphic varieties. See Example 2.6.3, 2.6.4 and 2.6.6 for the verifications of the isomorphies in the exceptional cases. ☐

2.4 Local Gorenstein indices and T-singularities

We investigate the singularities of the fake weighted projective planes of integral degree, classified in the preceding section. Recall that the only possible singular points of a fake weighted projective plane $Z = Z(P)$ are the toric fixed points $z(k) \in Z$. Moreover, $z(k) \in Z$ is singular, unless the entries of the fake weight vector $w = w(P)$ are coprime and we have $w_k = 1$; in this case, Z must be an ordinary weighted projective plane.

Let us recall the notion of a T-singularity; see also [15, Sections 2, 4]. Let k, p be coprime positive integers, denote by $C(dk^2) \subseteq \mathbb{K}^*$ the group of dk^2-th roots of unity and consider the action

$$C(dk^2) \times \mathbb{K}^2 \ \to \ \mathbb{K}^2, \qquad \zeta \cdot z \ = \ (\zeta z_1, \, \zeta^{dpk-1} z_2).$$

Then $U := \mathbb{K}^2 / C(dk^2)$ is an affine toric surface and the image $u \in U$ of $0 \in \mathbb{K}^2$ is singular as soon as $k > 1$. A *(cyclic) T-singularity*, also called a *singularity of type*

$$\tfrac{1}{dk^2}(1, dpk - 1),$$

is a surface singularity isomorphic to $u \in U$ as above. The *local Gorenstein index* $\iota(z)$ of a point z in a normal variety Z is the order of the canonical divisor class in the local class group $\mathrm{Cl}(Z, z)$.

Remark 2.4.1 (See [10, Thm. 4.2.8]; also [17, Rem. 3.7])**.** Let $v = (a, c)$ and $v' = (b, d)$ be primitive vectors in $\mathbb{Z}^2$ generating a two-dimensional cone in $\mathbb{Q}^2$ and let Z be the associated affine toric variety. Then, the local Gorenstein index of the toric fixed point $z \in Z$ is given by

$$\iota(z) \ = \ \tfrac{|ad-bc|}{\gcd(c-d,\, b-a)}.$$

Lemma 2.4.2 (See also [27, Lem. 2.10])**.** *Let U be an affine toric surface with fixed point $u \in U$. Then the following statements are equivalent:*

(i) $u \in U$ *is of type* $\frac{1}{dk^2}(1, dpk-1)$.
(ii) $\iota(u)^2$ *divides* $\mathrm{cl}(u)$.

If these statements hold, then $d = \mathrm{cl}(u)/\iota(u)^2$ and $k = \iota(u)$ and there exists $b \in \mathbb{Z}$ such that U has generator matrix

$$P \ = \ \begin{bmatrix} \iota(u) & \iota(u) \\ d\iota(u) + b & b \end{bmatrix}.$$

Proof. Let $u \in U$ be of type $\frac{1}{dk^2}(1, dpk-1)$. Choose $a, b \in \mathbb{Z}$ such that $ak - bp = 1$. Consider the affine toric surface U' given by the generator matrix

$$P = \begin{bmatrix} k & k \\ dk+b & b \end{bmatrix}.$$

Proposition 1.1.3 provides us with the homomorphism $\mathbb{T}^2 \to \mathbb{T}^2$, $t \mapsto (t_1^k t_2^k, t_1^{dk+b} t_2^b)$ associated with the matrix P, extending to a toric morphism $\pi\colon \mathbb{K}^2 \to U'$. Moreover, we obtain an injective morphism

$$\nu\colon C(dk^2) \to \ker(\pi), \qquad \zeta \mapsto (\zeta, \zeta^{dpk-1}).$$

Thus, $|\mathrm{im}(\nu)| = dk^2$ and $|\ker(\pi)| = \det(P) = dk^2$ implies the surjectivity. We conclude $U' \cong \mathbb{K}^2/C(dk^2)$ with $C(dk^2)$ acting as needed for type $\frac{1}{dk^2}(1, dpk-1)$. Thus, $U' \cong U$ and using Remark 2.4.1, we obtain

$$\mathrm{cl}(u) = |\det(P)| = dk^2, \qquad \iota(u) = k.$$

For the other implication, assume that there exists a $d \in \mathbb{Z}$ such that $\mathrm{cl}(u) = d\iota(u)^2$. The affine toric surface U is given by a generator matrix P. With $k := \iota(u)$, a suitable unimodular transformation turns P into

$$P = \begin{bmatrix} k & k \\ c & b \end{bmatrix}, \qquad \gcd(c, k) = \gcd(b, k) = 1.$$

By assumption $\mathrm{cl}(u) = |\det(P)|$ equals $d\iota(u)^2 = dk^2$. Thus, we may assume $c = dk + b$. Take $a, p \in \mathbb{Z}$ with $ak - bp = 1$ and $p \geq 1$. Then we have an action

$$C(dk^2) \times \mathbb{K}^2 \to \mathbb{K}^2, \qquad \zeta \cdot z = (\zeta z_1, \zeta^{dpk-1} z_2).$$

With similar arguments as above, we verify that U is the quotient $\mathbb{K}^2/C(dk^2)$ for this action and thus see that $u \in U$ is of type $\frac{1}{dk^2}(1, dpk-1)$. □

Example 2.4.3. Take integers c, c_0, l with $c_0 < 0$ and $l > 0$ such that l and c are coprime and consider the affine toric surface U with the generator matrix

$$P = \begin{bmatrix} l & l \\ c & c + lc_0 \end{bmatrix}.$$

The toric fixed point $u \in U$ has local class group order $\mathrm{cl}(u) = -c_0 l^2$ and local Gorenstein index $\iota(u) = l$; see Remark 2.4.1. Thus, U is at most T-singular.

The aim of the section is to determine the local Gorenstein indices and the T-singularities for all fake weighted projective planes of integral degree. We begin with the following auxiliary observations.

Lemma 2.4.4. *Let* $w \in S(a)$, *set* $\mu := \gcd(w_0, w_1, w_2)$ *and consider a degree matrix in* $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ *associated with* w *of the following shape (last row omitted for* $a \geq 5$*):*

$$Q \;=\; [q_0, q_1, q_2] \;=\; \begin{bmatrix} x_0^2 & \xi_1 x_1^2 & \xi_2 x_2^2 \\ \bar{0} & \bar{1} & \bar{\eta} \end{bmatrix}, \qquad (\bar{\eta} \in \mathbb{Z}/\mu\mathbb{Z})^*,$$

where $(x_0^2, \xi_1 x_1^2, \xi_2 x_2^2) \in S(\mu a)$ *is as in Theorem 2.1.8. Then the anticanonical divisor class of the (fake) weighted projective plane associated with* Q *is*

$$w_Z \;=\; (\sqrt{\mu a \xi_1 \xi_2}\, x_0 x_1 x_2,\ \bar{1} + \bar{\eta}) \;\in\; \mathrm{Cl}(Z) \;=\; \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}.$$

Moreover, the local Gorenstein index ι_k *of the toric fixed point* $z(k) \in Z(Q)$ *equals the minimal positive multiple* νx_k *such that* $\nu x_k w_Z \in \mathbb{Z} \cdot q_k$.

Proof. Proposition 2.2.4 says that $D_0 + D_1 + D_2$ is an anticanonical divisor on Z for $D_k = V(T_k)$. Moreover, the divisor class of D_k equals the column q_k. Thus, the anticanonical divisor class is given by

$$w_Z \;=\; q_0 + q_1 + q_2 \;=\; (x_0^2 + \xi_1 x_1^2 + \xi_2 x_2^2,\ \bar{1} + \bar{\eta}) \;=\; (\sqrt{\mu a \xi_1 \xi_2}\, x_0 x_1 x_2,\ \bar{1} + \bar{\eta}),$$

where we use Theorem 2.1.8 for the last equality. The local Gorenstein index ι_k is the order of w_Z in the local class group of $z(k)$, which according to Proposition 2.2.1 is given as

$$\mathrm{Cl}(Z, z(k)) \;=\; (\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z})/\mathbb{Z} \cdot q_k.$$

Hence, the local Gorenstein index of $z(k)$ is the minimal positive integer ι_k satisfying $\iota_k w_Z \in \mathbb{Z} \cdot q_k$. By the above description of w_Z, the latter implies

$$\sqrt{\mu a \xi_1 \xi_2}\, x_0 x_1 x_2 \iota_k \;\in\; \xi_k x_k^2 \mathbb{Z}.$$

According to Theorem 2.1.8 we distinguish the following cases.

Case 1: $\sqrt{\mu a \xi_1 \xi_2} = 3$. Then $\xi_1 = \xi_2 = 1$ and $3 x_0 x_1 x_2 \iota_k$ is a multiple of $\xi_k x_k^2$. Moreover, $3, x_0, x_1, x_2$ are pairwise coprime by Lemma 2.3.6. We conclude $x_k \mid \iota_k$.

Case 2: $\sqrt{\mu a \xi_1 \xi_2} = 4$. Then $\xi_1 = 1$, $\xi_2 = 2$ and $4 x_0 x_1 x_2 \iota_k$ is a multiple of $\xi_k x_k^2$. Pairwise coprimeness of $x_0, x_1, 2x_2$ yields $x_0, x_1 \mid \iota_0$. Further, x_2 is odd due to Lemma 2.3.11. Thus, $x_2 \mid \iota_2$.

Case 3: $\sqrt{\mu a \xi_1 \xi_2} = 5$. Then $\xi_1 = 1$, $\xi_2 = 5$ and $5 x_0 x_1 x_2 \iota_k$ is a multiple of $\xi_k x_k^2$. Pairwise coprimeness of $x_0, x_1, 5x_2$ directly gives $x_k \mid \iota_k$.

Case 4: $\sqrt{a \mu \xi_1 \xi_2} = 6$. Then $\xi_1 = 2$, $\xi_2 = 3$ and $6 x_0 x_1 x_2 \iota_k$ is a multiple of $\xi_k x_k^2$. Again, pairwise coprimeness of $x_0, 2x_1, 3x_2$ leads to $x_k \mid \iota_k$. ☐

Proposition 2.4.5. *The following table lists the divisor class group, the adjusted degree matrix Q, the anticanonical class w_Z, the constellation of local Gorenstein indices $\iota_k = \iota(z(k))$ and the constellation of T-singularities for the fake weighted projective planes $Z = Z(Q)$ of degrees $9, 8, 6, 5$:*

ID	Cl(Z)	Q	w_Z	$(\iota_0, \iota_1, \iota_2)$	$(\pm, \pm, \pm)$
9-1-0	$\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \end{bmatrix}$ $(x_0^2, x_1^2, x_2^2) \in S(9)$	$\begin{bmatrix} 3x_0x_1x_2 \end{bmatrix}$	(x_0, x_1, x_2)	$(+, +, +)$
8-1-0	$\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \end{bmatrix}$	(x_0, x_1, x_2)	$(+, +, +)$
6-1-0	$\mathbb{Z}$	$\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \end{bmatrix}$ $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$	$\begin{bmatrix} 6x_0x_1x_2 \end{bmatrix}$	(x_0, x_1, x_2)	$(+, +, +)$
5-1-0	$\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \end{bmatrix}$ $(x_0^2, x_1^2, 5x_2^2) \in S(5)$	$\begin{bmatrix} 5x_0x_1x_2 \end{bmatrix}$	(x_0, x_1, x_2)	$(+, +, +)$

Proof. This is classically known, compare [15]. In our setting, one can proceed as follows. Observe that $x_k w_Z \in \mathbb{Z} w_k$ holds in all cases. Thus, Lemma 2.4.4 establishes the claim on the local Gorenstein indices. Obviously, $x_k^2 \mid w_k$ holds in all cases. Thus, Lemma 2.4.2 shows that the $z(k)$ are at most T-singular. □

Proposition 2.4.6. *The following table lists the divisor class group, the adjusted degree matrix Q, the anticanonical class w_Z, the constellation of local Gorenstein indices $\iota_k = \iota(z(k))$ and the constellation of T-singularities for the fake weighted projective planes $Z = Z(Q)$ of degree 4:*

ID	Cl(Z)	Q	w_Z	$(\iota_0, \iota_1, \iota_2)$	$(\pm, \pm, \pm)$
4-2-1	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	(x_0, x_1, x_2)	$(+, +, +)$

Proof. The ID, the divisor class group and the degree matrix are taken from Proposition 2.3.4. We enter Lemma 2.4.4 with

$$\mu = 2, \quad a = 4, \quad \xi_1 = 1, \quad \xi_2 = 2.$$

As a direct outcome, we obtain the desired representation of the anticanonical divisor class $w_Z \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This in turn gives us

$$x_0 w_Z \;=\; 4x_1x_2 \cdot (x_0^2, \bar{0}), \quad x_1 w_Z \;=\; 4x_0x_2 \cdot (x_1^2, \bar{0}), \quad x_2 w_Z \;=\; 2x_0x_1 \cdot (2x_2^2, \bar{0}).$$

Consequently, $x_k w_Z \in \mathbb{Z} \cdot q_k$ for $k = 0, 1, 2$. Using Lemma 2.4.4 again we see $\iota_k = x_k$. Lemma 2.3.3 provides us with the local class group orders of the fixed points:

$$\mathrm{cl}(z(0)) \;=\; 2x_0^2, \qquad \mathrm{cl}(z(1)) \;=\; 2x_1^2, \qquad \mathrm{cl}(z(2)) \;=\; 4x_2^2.$$

Altogether, we conclude $\iota_k^2 \mid \mathrm{cl}(z(k))$. Now, Lemma 2.4.2 shows that each of the toric fixed points $z(k)$ is a T-singularity. ☐

Proposition 2.4.7. *The following table lists the divisor class group, the adjusted degree matrix Q, the anticanonical class w_Z, the constellation of local Gorenstein indices $\iota_k = \iota(z(k))$ and the constellation of T-singularities for all fake weighted projective planes $Z = Z(Q)$ of degree 3:*

ID	$\mathrm{Cl}(Z)$	Q	w_Z	$(\iota_0, \iota_1, \iota_2)$	$(\pm, \pm, \pm)$
3-3-2	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $(x_0^2,x_1^2,x_2^2) \in S(9)$	$\begin{bmatrix} 3x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	(x_0, x_1, x_2)	$(+,+,+)$
3-2-1	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $(x_0^2,2x_1^2,3x_2^2) \in S(6)$	$\begin{bmatrix} 6x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	$(x_0, 2x_1, x_2)$	$(+,+,+)$

Proof. The first three columns of the table stem from Proposition 2.3.4. We treat the ID 3-3-2. As before, Lemma 2.4.4 delivers the description of the anticanonical class $w_Z \in \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Next, observe

$$x_0 w_Z = 3x_1x_2 \cdot (x_0^2, \bar{0}), \quad x_1 w_Z = 3x_0x_2 \cdot (x_1^2, \bar{1}), \quad x_2 w_Z = 3x_0x_1 \cdot (x_2^2, \bar{2}).$$

This means $x_k w_z \in \mathbb{Z} \cdot q_k$ and thus Lemma 2.4.4 ensures $\iota_k = x_k$ for $k = 0, 1, 2$. Lemma 2.3.3 yields the local class group orders $\mathrm{cl}(z(k)) = 3x_k^2$ for $k = 0, 1, 2$. By Lemma 2.4.2, all $z(k)$ are T-singular.

We turn to the ID 3-2-1. Again, the anticanonical class $w_z \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is provided by Lemma 2.4.4. Moreover, we have

$$x_0 w_Z = 6x_1x_2 \cdot (x_0^2, \bar{0}), \qquad\qquad x_2 w_Z = 2x_0x_2 \cdot (3x_2^2, \bar{1}),$$

Hence, $x_k w_z \in \mathbb{Z} \cdot q_k$ and $\iota_k = x_k$ for $k = 0, 2$. For the case $k = 1$, recall from Theorem 2.1.8 that $x_0, 2x_1, 3x_2$ are pairwise coprime. This implies

$$x_1 w_Z \ = \ (6x_0x_1^2x_2, \bar{0}) \ \notin \ \mathbb{Z} \cdot (2x_1^2, \bar{1}),$$

as otherwise $x_1 w_z = \kappa \cdot (2x_1^2, \bar{1})$, where $\kappa = 3x_0x_2$ must be even, which contradicts to the above coprimeness condition. However,

$$2x_1 w_Z \ = \ (12x_0x_1^2x_2, \bar{0}) \ = \ 6x_0x_2 \cdot (2x_1^2, \bar{1}).$$

Thus, $2x_1 w_z \in \mathbb{Z} \cdot q_1$ and, using Lemma 2.4.4, one obtains $\iota_1 = 2x_1$ for the local Gorenstein index. Moreover, Lemma 2.3.3 tells us

$$\mathrm{cl}(z(0)) \ = \ 2x_0^2, \qquad \mathrm{cl}(z(1)) \ = \ 4x_1^2, \qquad \mathrm{cl}(z(2)) \ = \ 6x_2^2.$$

Consequently, $\iota_k^2 \mid \mathrm{cl}(z(k))$ for $k = 0, 1, 2$. Lemma 2.4.2 yields that each of the toric fixed points $z(k)$ is a T-singularity. □

Proposition 2.4.8. *The following table lists the divisor class group, the adjusted degree matrix Q, the anticanonical class w_Z, the possible constellations of local Gorenstein indices $\iota_k = \iota(z(k))$ and the constellation of T-singularities for all fake weighted projective planes $Z = Z(Q)$ of degree 2:*

ID	$\mathrm{Cl}(Z)$	Q	w_Z	$(\iota_0, \iota_1, \iota_2)$	$(\pm, \pm, \pm)$
2-4-1	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \\ \bar{2} \end{bmatrix}$	$(2x_0, 2x_1, x_2)$	$(+, +, +)$
2-4-3	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	$(x_0, x_1, 2x_2)$	$(+, +, +)$
2-3-1	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$	$\begin{bmatrix} 6x_0x_1x_2 \\ \bar{2} \end{bmatrix}$	$(3x_0, 3x_1, 3x_2)$ $(3x_0, 3x_1, x_2)$	$(-, -, +)$
2-3-2	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$	$\begin{bmatrix} 6x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	$(x_0, x_1, 3x_2)$	$(+, +, +)$

Proof. The ID, the divisor class group and the degree matrix are taken from Proposition 2.3.7. For the ID 2-4-1, we have

$$\mu \ = \ 4, \quad a \ = \ 2, \quad \xi_1 \ = \ 1, \quad \xi_2 \ = \ 2.$$

Lemma 2.4.4 gives the desired representation of $w_Z \in \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. From Lemma 2.3.8 we infer $\bar{x}_k = \bar{1}$ or $\bar{x}_k = \bar{3}$ for $k = 0, 1, 2$. Using this, we obtain

$$x_0 w_Z = (4x_0^2 x_1 x_2, \bar{2}\bar{x}_0) \notin \mathbb{Z} \cdot (x_0^2, \bar{0}), \qquad x_1 w_Z = (4x_0 x_1^2 x_2, \bar{2}\bar{x}_1) \notin \mathbb{Z} \cdot (x_1^2, \bar{1}).$$

To see the last statement, assume $x_1 w_Z = \kappa \cdot (x_1^2, \bar{1})$ with $\kappa \in \mathbb{Z}$. Then, as x_1 is odd due to Theorem 2.1.8, 4 divides κ and thus $x_1 w_Z = (\kappa x_1^2, \bar{0})$; a contradiction. Moreover,

$$2x_0 w_Z = 8x_1 x_2 \cdot (x_0^2, \bar{0}), \quad 2x_1 w_Z = 8x_0 x_2 \cdot (x_1^2, \bar{1}), \quad x_2 w_Z = 2x_0 x_1 \cdot (2x_2^2, \bar{1}).$$

Thus, Lemma 2.4.4 yields $\iota_k = 2x_k$ for $k = 0, 1$ and $\iota_2 = 2x_2$. Lemma 2.3.3 provides us with the local class group orders

$$\mathrm{cl}(z(0)) \ = \ 4x_0^2, \qquad \mathrm{cl}(z(1)) \ = \ 4x_1^2, \qquad \mathrm{cl}(z(2)) \ = \ 8x_2^2.$$

Altogether, we conclude $\iota_k^2 \mid \mathrm{cl}(z(k))$. Now, Lemma 2.4.2 shows that each of the toric fixed points $z(k)$ is a T-singularity.

Next we treat ID 2-4-3. The infer the shape of $w_Z \in \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ is again provided Lemma 2.4.4. We observe

$$x_0 w_Z = 4x_1 x_2 \cdot (x_0^2, \bar{0}), \qquad x_1 w_Z = 4x_0 x_2 \cdot (x_1^2, \bar{1}),$$

Moreover, Lemma 2.3.8 yields $\bar{x}_k = \bar{1}$ or $\bar{x}_k = \bar{3}$ for $k = 0, 1, 2$. Thus, arguing as in preceding cases, one obtains

$$x_2 w_Z = (4x_0 x_1 x_2^2, \bar{3}\bar{x}_2) \notin \mathbb{Z} \cdot (2x_2^2, \bar{3}), \qquad 2x_1 w_Z = 4x_0 x_1 \cdot (2x_2^2, \bar{3}).$$

Lemma 2.4.4 yields the local Gorenstein indics $x_0, x_2, 2x_2$ and Lemma 2.3.3 local class group orders $4x_0^2, 4x_1^2, 8x_2^2$. Thus all $z(k)$ are T-singular.

Let us take a closer look at the ID 2-3-1, where we expect bad singularities and a new effect for the local Gorenstein indices. We enter Lemma 2.4.4 with

$$\mu \ = \ 3, \quad a \ = \ 2, \quad \xi_1 \ = \ 2, \quad \xi_2 \ = \ 3.$$

Again, the desired representation of $w_Z \in \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is a direct consequence. From Lemma 2.3.9 we infer $\bar{x}_k = \bar{1}$ or $\bar{x}_k = \bar{2}$ for $k = 0, 1$. Using this, we obtain

$$x_0 w_Z = (6x_0^2 x_1 x_2, \bar{2}\bar{x}_0) \notin \mathbb{Z} \cdot (x_0^2, \bar{0}), \qquad 2x_0 w_Z = (12x_0^2 x_1 x_2, \bar{4}\bar{x}_0) \notin \mathbb{Z} \cdot (x_0^2, \bar{0}).$$

Clearly, $3x_0 w_Z = 18x_1 x_2 \cdot (x_0^2, \bar{0})$ holds and thus Lemma 2.4.4 shows $\iota_0 = 3x_0$. In the case $k = 1$, we have

$$x_1 w_Z = (6x_0 x_1^2 x_2, \bar{2}\bar{x}_1) \notin \mathbb{Z} \cdot (x_1^2, \bar{1}), \qquad 2x_1 w_Z = (12x_0^2 x_1 x_2, \bar{4}\bar{x}_1) \notin \mathbb{Z} \cdot (x_1^2, \bar{1}),$$

as otherwise $x_1 w_Z = \kappa \cdot (x_1^2, \bar{1})$ or $2x_1 w_Z = \kappa \cdot (x_1^2, \bar{1})$, hence $3 \mid \kappa$, forcing $\bar{2}\bar{x}_1 = \bar{0}$ or $\bar{4}\bar{x}_1 = \bar{0}$. Again $3x_1 w_Z = 18x_0 x_2 \cdot (x_1^2, \bar{0})$ and thus $\iota_1 = 3x_1$. Consider

$$x_2 w_Z = (6x_0 x_1 x_2^2, \bar{2}\bar{x}_2), \quad 2x_2 w_Z = (12x_0 x_1 x_2^2, \bar{4}\bar{x}_2), \quad 3x_2 w_Z = (18x_0 x_1 x_2^2, \bar{6}\bar{x}_2).$$

For $\bar{x}_2 = \bar{1}, \bar{2}$, we see $x_2 w_2, 2x_2 w_2 \notin \mathbb{Z} \cdot (3x_2^2, \bar{1})$ and $\iota_2 = 3x_2$ as before. For $\bar{x}_2 = \bar{0}$, we have $x_2 w_Z = 2x_0 x_1 \cdot (3x_2^2, \bar{0})$ and thus $\iota_2 = x_2$. Moreover,

$$\mathrm{cl}(z(0)) \;=\; 3x_0^2, \qquad \mathrm{cl}(z(1)) \;=\; 6x_1^2, \qquad \mathrm{cl}(z(2)) \;=\; 9x_2^2,$$

according to Lemma 2.3.3. We arrive at $\iota_k^2 \nmid \mathrm{cl}(z(k))$ for $k = 0, 1$ and $\iota_2^2 \mid \mathrm{cl}(z(2))$. Lemma 2.4.2 ensures that the constellation of T-singularities is as claimed.

Next, we show the statements for the ID 2-3-2. The anticanonical class $w_z \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is provided by Lemma 2.4.4. Moreover, note

$$x_0 w_Z = 6x_1 x_2 \cdot (x_0^2, \bar{0}), \qquad\qquad x_1 w_Z = 3x_0 x_2 \cdot (2x_1^2, \bar{1}).$$

Thus, $\iota_k = x_k$ for $k = 0, 1$. According to Theorem 2.1.8, the numbers $x_0, 2x_1, 3x_2$ are pairwise coprime. This implies

$$x_1 w_Z \;=\; (6x_0 x_1^2 x_2, \bar{0}) \;\notin\; \mathbb{Z} \cdot (3x_2^2, \bar{2}), \quad 2x_1 w_Z \;=\; (12x_0 x_1^2 x_2, \bar{0}) \;\notin\; \mathbb{Z} \cdot (3x_2^2, \bar{2}),$$

as otherwise $x_2 w_z = \kappa \cdot (3x_2^2, \bar{2})$, thus $\bar{\kappa} = \bar{0}$ in $\mathbb{Z}/3\mathbb{Z}$, which contradicts to the above coprimeness condition. Instead, we have

$$3x_2 w_Z \;=\; (18x_0 x_1^2 x_2, \bar{0}) \;=\; 6x_0 x_1 \cdot (3x_2^2, \bar{2}).$$

With the aid of Lemma 2.4.4, we arrive at the local Gorenstein index $\iota_2 = 3x_2$. Moreover, Lemma 2.3.3 tells us

$$\mathrm{cl}(z(0)) \;=\; 3x_0^2, \qquad \mathrm{cl}(z(1)) \;=\; 6x_1^2, \qquad \mathrm{cl}(z(2)) \;=\; 9x_2^2.$$

Consequently, $\iota_k^2 \nmid \mathrm{cl}(z(k))$ for $k = 0, 1$ and $\iota_2^2 \mid \mathrm{cl}(z(2))$. By Lemma 2.4.2, the point $z(2)$ is a T-singularity, whereas $z(0), z(1)$ are not. □

Proposition 2.4.9. *The following table lists the divisor class group, the adjusted degree matrix Q, anticanonical class w_Z, the possible constellations of local Gorenstein indices $\iota_k = \iota(z(k))$ and the constellation of T-singularities for all fake weighted projective planes $Z = Z(Q)$ of degree 1:*

ID	Cl(Z)	Q	w_Z	$(\iota_0, \iota_1, \iota_2)$	$(\pm, \pm, \pm)$
1-9-2	$\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $(x_0^2, x_1^2, x_2^2) \in S(9)$	$\begin{bmatrix} 3x_0x_1x_2 \\ \bar{3} \end{bmatrix}$	$(3x_0, x_1, 3x_2)$ $(3x_0, 3x_1, x_2)$	$(+, +, +)$
1-9-5	$\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}$ $(x_0^2, x_1^2, x_2^2) \in S(9)$	$\begin{bmatrix} 3x_0x_1x_2 \\ \bar{6} \end{bmatrix}$	$(3x_0, x_1, 3x_2)$ $(3x_0, 3x_1, x_2)$	$(+, +, +)$
1-9-8	$\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 \\ \bar{0} & \bar{1} & \bar{8} \end{bmatrix}$ $(x_0^2, x_1^2, x_2^2) \in S(9)$	$\begin{bmatrix} 3x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	$(x_0, 3x_1, 3x_2)$	$(+, +, +)$
1-8-1	$\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \\ \bar{2} \end{bmatrix}$	$(4x_0, 4x_1, x_2)$ $(4x_0, 4x_1, 2x_2)$	$(-, -, +)$
1-8-3	$\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \\ \bar{4} \end{bmatrix}$	$(2x_0, x_1, 4x_2)$	$(+, +, +)$
1-8-5	$\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \\ \bar{6} \end{bmatrix}$	$(4x_0, 4x_1, x_2)$ $(4x_0, 4x_1, 2x_2)$	$(-, -, +)$
1-8-7	$\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 2x_2^2 \\ \bar{0} & \bar{1} & \bar{7} \end{bmatrix}$ $(x_0^2, x_1^2, 2x_2^2) \in S(8)$	$\begin{bmatrix} 4x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	$(x_0, 2x_1, 4x_2)$	$(+, +, +)$
1-6-1	$\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$	$\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$	$\begin{bmatrix} 6x_0x_1x_2 \\ \bar{2} \end{bmatrix}$	$(3x_0, 6x_1, x_2)$ $(3x_0, 6x_1, 3x_2)$	$(-, -, +)$
1-6-5	$\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$	$\begin{bmatrix} x_0^2 & 2x_1^2 & 3x_2^2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}$ $(x_0^2, 2x_1^2, 3x_2^2) \in S(6)$	$\begin{bmatrix} 6x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	$(x_0, 2x_1, 3x_2)$	$(+, +, +)$
1-5-1	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ $(x_0^2, x_1^2, 5x_2^2) \in S(5)$	$\begin{bmatrix} 5x_0x_1x_2 \\ \bar{2} \end{bmatrix}$	$(5x_0, 5x_1, x_2)$ $(5x_0, 5x_1, 5x_2)$	$(-, -, +)$
1-5-2	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$ $(x_0^2, x_1^2, 5x_2^2) \in S(5)$	$\begin{bmatrix} 5x_0x_1x_2 \\ \bar{3} \end{bmatrix}$	$(5x_0, 5x_1, x_2)$ $(5x_0, 5x_1, 5x_2)$	$(-, -, +)$
1-5-3	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}$ $(x_0^2, x_1^2, 5x_2^2) \in S(5)$	$\begin{bmatrix} 5x_0x_1x_2 \\ \bar{4} \end{bmatrix}$	$(5x_0, 5x_1, x_2)$ $(5x_0, 5x_1, 5x_2)$	$(-, -, +)$
1-5-4	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$	$\begin{bmatrix} x_0^2 & x_1^2 & 5x_2^2 \\ \bar{0} & \bar{1} & \bar{4} \end{bmatrix}$ $(x_0^2, x_1^2, 5x_2^2) \in S(5)$	$\begin{bmatrix} 5x_0x_1x_2 \\ \bar{0} \end{bmatrix}$	$(x_0, x_1, 5x_2)$	$(+, +, +)$

Proof. In all cases, ID, the divisor class group and the degree matrix are as in Proposition 2.3.4 and Lemma 2.4.4 delivers the description of the anticanonical class. The silently used tool for determining the local Gorenstein indices is Lemma 2.4.4. The constellation of T-singularities is verified with the aid of Lemma 2.3.3 and Lemma 2.4.2.

*IDs 1-9-**: First note that $3x_kw_Z = (9x_0x_1x_2x_k, \bar{0}) \in \mathbb{Z} \cdot q_k$ holds for $k = 0, 1, 2$. Thus, all the local Gorenstein indices are of the form

$$\iota_k = a_kx_k, \quad 1 \le a_k \le 3.$$

By Lemma 2.3.6, the $\bar{x}_k$ can take only values from the group of multiplicative units $(\mathbb{Z}/9\mathbb{Z})^*$. For the ID 1-9-2, look at

$$x_0w_Z = \begin{bmatrix} 3x_0^2x_1x_2 \\ \bar{3}\bar{x}_0 \end{bmatrix}, \quad 2x_0w_Z = \begin{bmatrix} 6x_0^2x_1x_2 \\ \bar{6}\bar{x}_0 \end{bmatrix}, \quad q_0 = \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$x_1w_Z = \begin{bmatrix} 3x_0x_1^2x_2 \\ \bar{3}\bar{x}_1 \end{bmatrix}, \quad 2x_1w_Z = \begin{bmatrix} 6x_0x_1^2x_2 \\ \bar{6}\bar{x}_1 \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1^2 \\ \bar{1} \end{bmatrix},$$

$$x_2w_Z = \begin{bmatrix} 3x_0x_1x_2^2 \\ \bar{3}\bar{x}_2 \end{bmatrix}, \quad 2x_2w_Z = \begin{bmatrix} 6x_0x_1x_2^2 \\ \bar{6}\bar{x}_2 \end{bmatrix}, \quad q_2 = \begin{bmatrix} x_2^2 \\ \bar{2} \end{bmatrix}.$$

The respective conditions characterizing $x_kw_Z, 2x_kw_Z \in \mathbb{Z} \cdot q_k$, where $k = 0, 1, 2$, are the following:

$$\bar{3}\bar{x}_0 = \bar{0},\ \bar{6}\bar{x}_0 = \bar{0}, \quad \bar{3}\bar{x}_1 = \bar{3}\bar{x}_0\bar{x}_2,\ \bar{6}\bar{x}_1 = \bar{6}\bar{x}_0\bar{x}_2, \quad \bar{3}\bar{x}_2 = \bar{6}\bar{x}_0\bar{x}_1,\ \bar{6}\bar{x}_2 = \bar{3}\bar{x}_0\bar{x}_1.$$

Checking all triples $(\bar{x}_0, \bar{x}_1, \bar{x}_3)$, where $x_k \in (\mathbb{Z}/9\mathbb{Z})^*$, we arrive at the constellations of the assertion. For the ID 1-9-5, look at

$$x_0w_Z = \begin{bmatrix} 3x_0^2x_1x_2 \\ \bar{6}\bar{x}_0 \end{bmatrix}, \quad 2x_0w_Z = \begin{bmatrix} 6x_0^2x_1x_2 \\ \bar{3}\bar{x}_0 \end{bmatrix}, \quad q_0 = \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$x_1w_Z = \begin{bmatrix} 3x_0x_1^2x_2 \\ \bar{6}\bar{x}_1 \end{bmatrix}, \quad 2x_1w_Z = \begin{bmatrix} 6x_0x_1^2x_2 \\ \bar{3}\bar{x}_1 \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1^2 \\ \bar{1} \end{bmatrix},$$

$$x_2w_Z = \begin{bmatrix} 3x_0x_1x_2^2 \\ \bar{6}\bar{x}_2 \end{bmatrix}, \quad 2x_2w_Z = \begin{bmatrix} 6x_0x_1x_2^2 \\ \bar{3}\bar{x}_2 \end{bmatrix}, \quad q_2 = \begin{bmatrix} x_2^2 \\ \bar{5} \end{bmatrix}.$$

The respective conditions characterizing $x_kw_Z, 2x_kw_Z \in \mathbb{Z} \cdot q_k$, where $k = 0, 1, 2$, are the following:

$$\bar{6}\bar{x}_0 = \bar{0},\ \bar{3}\bar{x}_0 = \bar{0}, \quad \bar{6}\bar{x}_1 = \bar{3}\bar{x}_0\bar{x}_2,\ \bar{3}\bar{x}_1 = \bar{6}\bar{x}_0\bar{x}_2, \quad \bar{6}\bar{x}_2 = \bar{6}\bar{x}_0\bar{x}_1,\ \bar{3}\bar{x}_2 = \bar{3}\bar{x}_0\bar{x}_1.$$

Checking all triples $(\bar{x}_0, \bar{x}_1, \bar{x}_3)$, where $x_k \in (\mathbb{Z}/9\mathbb{Z})^*$, we arrive at the constellations of the assertion. For the ID 1-9-8, look at

$$x_0w_Z = \begin{bmatrix} 3x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad 2x_0w_Z = \begin{bmatrix} 6x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad q_0 = \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$x_1 w_Z = \begin{bmatrix} 3x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \quad 2x_1 w_Z = \begin{bmatrix} 6x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1^2 \\ \bar{1} \end{bmatrix},$$

$$x_2 w_Z = \begin{bmatrix} 3x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad 2x_2 w_Z = \begin{bmatrix} 6x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad q_2 = \begin{bmatrix} x_2^2 \\ \bar{8} \end{bmatrix}.$$

The respective characterizing conditions for $x_k w_Z, 2x_k w_Z \in \mathbb{Z} \cdot q_k$, where $k = 0, 1, 2$, are the following:

$$\bar{0} = \bar{0},\ \bar{0} = \bar{0}, \quad \bar{0} = \bar{3}\bar{x}_0\bar{x}_2,\ \bar{0} = \bar{6}\bar{x}_0\bar{x}_2, \quad \bar{0} = \bar{6}\bar{x}_0\bar{x}_1,\ \bar{0} = \bar{3}\bar{x}_0\bar{x}_1.$$

We see $\iota_0 = x_0$ and, using the fact that the $\bar{x}_k$ are units in $\mathbb{Z}/9\mathbb{Z}$, we arrive at $\iota_1 = 3x_1$ and $\iota_2 = 3x_2$.

*IDs 1-8-**: First note that $4x_k w_Z = (16x_0x_1x_2x_k, \bar{0}) \in \mathbb{Z} \cdot q_k$ holds for $k = 0, 1, 2$. Thus, the local Gorenstein indices are of the form

$$\iota_k = c_k x_k, \quad 1 \le c_k \le 4.$$

Lemma 2.3.8 implies that $\bar{x}_k$ can take the values $\bar{1}, \bar{3}, \bar{5}, \bar{7} \in \mathbb{Z}/8\mathbb{Z}$. These are precisely the units of $\mathbb{Z}/8\mathbb{Z}$ and $\bar{4}\bar{x}_k = \bar{4}$ holds for each of them. For the ID 1-8-1, look at

$$x_0 w_Z = \begin{bmatrix} 4x_0^2x_1x_2 \\ \bar{2}\bar{x}_0 \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0^2x_1x_2 \\ \bar{4}\bar{x}_0 \end{bmatrix}, \quad 3x_0 w_Z = \begin{bmatrix} 12x_0^2x_1x_2 \\ \bar{6}\bar{x}_0 \end{bmatrix}, \quad q_0 = \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$x_1 w_Z = \begin{bmatrix} 4x_0x_1^2x_2 \\ \bar{2}\bar{x}_1 \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0x_1^2x_2 \\ \bar{4}\bar{x}_1 \end{bmatrix}, \quad 3x_1 w_Z = \begin{bmatrix} 12x_0x_1^2x_2 \\ \bar{6}\bar{x}_1 \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1^2 \\ \bar{1} \end{bmatrix},$$

$$x_2 w_Z = \begin{bmatrix} 4x_0x_1x_2^2 \\ \bar{2}\bar{x}_2 \end{bmatrix}, \quad 2x_2 w_Z = \begin{bmatrix} 8x_0x_1x_2^2 \\ \bar{4}\bar{x}_2 \end{bmatrix}, \quad 3x_2 w_Z = \begin{bmatrix} 12x_0x_1x_2^2 \\ \bar{6}\bar{x}_2 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 2x_2^2 \\ \bar{1} \end{bmatrix},$$

We list the respective conditions characterizing $x_k w_Z, 2x_k w_Z, 3x_k w_Z \in \mathbb{Z} \cdot q_k$. For $k = 0, 1$, these are the following:

$$\bar{2}\bar{x}_0 = \bar{0},\ \bar{4}\bar{x}_0 = \bar{0},\ \bar{6}\bar{x}_0 = \bar{0}, \quad \bar{2}\bar{x}_1 = \bar{4}\bar{x}_0\bar{x}_2,\ \bar{4}\bar{x}_1 = \bar{0},\ \bar{6}\bar{x}_1 = \bar{4}\bar{x}_0\bar{x}_2,$$

Plugging in the possible values of the $\bar{x}_k$, we see $\iota_0 = 4x_0$ and $\iota_1 = 4x_1$, as claimed. For $k = 2$, we have the conditions

$$\bar{2}\bar{x}_2 = \bar{2}\bar{x}_0\bar{x}_1, \quad \bar{4}\bar{x}_2 = \bar{4}\bar{x}_0\bar{x}_1, \quad \bar{6}\bar{x}_2 = \bar{6}\bar{x}_0\bar{x}_1.$$

Depending on the concrete values of the $\bar{x}_k \in \mathbb{Z}/8\mathbb{Z}$, we have $\iota_2 = x_2$ or $\iota_2 = 2x_2$, as claimed. Note that both cases occur. For the ID 1-8-3, look at

$$x_0 w_Z = \begin{bmatrix} 4x_0^2x_1x_2 \\ \bar{4}\bar{x}_0 \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad 3x_0 w_Z = \begin{bmatrix} 12x_0^2x_1x_2 \\ \bar{4}\bar{x}_0 \end{bmatrix}, \quad q_0 = \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$x_1 w_Z = \begin{bmatrix} 4x_0x_1^2x_2 \\ \bar{4}\bar{x}_1 \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \quad 3x_1 w_Z = \begin{bmatrix} 12x_0x_1^2x_2 \\ \bar{4}\bar{x}_1 \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1^2 \\ \bar{1} \end{bmatrix},$$

$$x_2 w_Z = \begin{bmatrix} 4x_0x_1x_2^2 \\ \bar{2}\bar{x}_2 \end{bmatrix}, \quad 2x_2 w_Z = \begin{bmatrix} 8x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad 3x_2 w_Z = \begin{bmatrix} 12x_0x_1x_2^2 \\ \bar{4}\bar{x}_2 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 2x_2^2 \\ \bar{3} \end{bmatrix},$$

We list the respective conditions characterizing $x_k w_Z, 2x_k w_Z, 3x_k w_Z \in \mathbb{Z}\cdot q_k$ for $k = 0, 1, 2$, where we stop as soon as the condition is fullfilled:

$$\bar{4}\bar{x}_0 = \bar{0},\ \bar{0} = \bar{0}, \qquad \bar{4}\bar{x}_1 = \bar{4}\bar{x}_0\bar{x}_2, \qquad \bar{4}\bar{x}_2 = \bar{6}\bar{x}_0\bar{x}_1,\ \bar{0} = \bar{4}\bar{x}_0\bar{x}_1,\ \bar{4}\bar{x}_2 = \bar{2}\bar{x}_0\bar{x}_1.$$

Thus, we and up with the local Gorenstein indices $\iota_0 = 2x_0$, $\iota_1 = x_1$ and $\iota_2 = 4x_2$, as claimed. For the ID 1-8-5, look at

$$x_0 w_Z = \begin{bmatrix} 4x_0^2x_1x_2 \\ \bar{6}\bar{x}_0 \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0^2x_1x_2 \\ \bar{4}\bar{x}_0 \end{bmatrix}, \quad 3x_0 w_Z = \begin{bmatrix} 12x_0^2x_1x_2 \\ \bar{2}\bar{x}_0 \end{bmatrix}, \quad q_0 = \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$x_1 w_Z = \begin{bmatrix} 4x_0x_1^2x_2 \\ \bar{6}\bar{x}_1 \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0x_1^2x_2 \\ \bar{4}\bar{x}_1 \end{bmatrix}, \quad 3x_1 w_Z = \begin{bmatrix} 12x_0x_1^2x_2 \\ \bar{2}\bar{x}_1 \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1^2 \\ \bar{1} \end{bmatrix},$$

$$x_2 w_Z = \begin{bmatrix} 4x_0x_1x_2^2 \\ \bar{6}\bar{x}_2 \end{bmatrix}, \quad 2x_2 w_Z = \begin{bmatrix} 8x_0x_1x_2^2 \\ \bar{4}\bar{x}_2 \end{bmatrix}, \quad 3x_2 w_Z = \begin{bmatrix} 12x_0x_1x_2^2 \\ \bar{2}\bar{x}_2 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 2x_2^2 \\ \bar{5} \end{bmatrix},$$

We list the respective conditions characterizing $x_k w_Z, 2x_k w_Z, 3x_k w_Z \in \mathbb{Z}\cdot q_k$. For $k = 0, 1$, these are the following:

$$\bar{6}\bar{x}_0 = \bar{0},\ \bar{4}\bar{x}_0 = \bar{0},\ \bar{2}\bar{x}_0 = \bar{0}, \qquad \bar{6}\bar{x}_1 = \bar{4}\bar{x}_0\bar{x}_2,\ \bar{4}\bar{x}_1 = \bar{0},\ \bar{2}\bar{x}_1 = \bar{4}\bar{x}_0\bar{x}_2,$$

Plugging in the possible values of the $\bar{x}_k$, we see $\iota_0 = 4x_0$ and $\iota_1 = 4x_1$, as claimed. For $k = 2$, we have the conditions

$$\bar{6}\bar{x}_2 = \bar{2}\bar{x}_0\bar{x}_1, \quad \bar{4}\bar{x}_2 = \bar{4}\bar{x}_0\bar{x}_1, \quad \bar{2}\bar{x}_2 = \bar{6}\bar{x}_0\bar{x}_1.$$

Depending on the concrete values of the $\bar{x}_k \in \mathbb{Z}/8\mathbb{Z}$, we have $\iota_2 = x_2$ or $\iota_2 = 2x_2$, as claimed. Note that both cases occur. For the ID 1-8-7, look at

$$x_0 w_Z = \begin{bmatrix} 4x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad 3x_0 w_Z = \begin{bmatrix} 12x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad q_0 = \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$x_1 w_Z = \begin{bmatrix} 4x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \quad 2x_0 w_Z = \begin{bmatrix} 8x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \quad 3x_1 w_Z = \begin{bmatrix} 12x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1^2 \\ \bar{1} \end{bmatrix},$$

$$x_2 w_Z = \begin{bmatrix} 4x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad 2x_2 w_Z = \begin{bmatrix} 8x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad 3x_2 w_Z = \begin{bmatrix} 12x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad q_2 = \begin{bmatrix} 2x_2^2 \\ \bar{7} \end{bmatrix}.$$

We list the respective conditions characterizing $x_k w_Z, 2x_k w_Z, 3x_k w_Z \in \mathbb{Z}\cdot q_k$ for $k = 0, 1, 2$, where we stop as soon as the condition is fullfilled:

$$\bar{0} = \bar{0}, \qquad \bar{0} = \bar{4}\bar{x}_0\bar{x}_2,\ \bar{0} = \bar{0}, \qquad \bar{0} = \bar{2}\bar{x}_0\bar{x}_1,\ \bar{0} = \bar{4}\bar{x}_0\bar{x}_1,\ \bar{0} = \bar{6}\bar{x}_0\bar{x}_1.$$

Thus, we end up with the local Gorenstein indices $\iota_0 = x_0$, $\iota_1 = 2x_1$ and $\iota_2 = 4x_2$, as claimed.

IDs 1-6-∗: First note that we only have to check $x_kw_Z, 2x_kw_Z \in \mathbb{Z}\cdot q_k$ for $k = 0, 2$ and $x_1w_Z, \dots, 5x_1w_Z \in \mathbb{Z}\cdot q_1$ due to the identities

$$3x_0w_Z \;=\; 18x_1x_2\cdot q_0, \qquad 6x_1w_Z \;=\; 6x_0x_2\cdot q_1, \qquad 3x_2w_Z \;=\; 6x_1x_2\cdot q_2.$$

According to Lemma 2.3.12 we have $\bar{x}_0 = \bar{1}, \bar{5}$, $\bar{x}_1 \neq \bar{3}$ and $\bar{x}_2 \neq \bar{2}, \bar{4}$ for the values in $\mathbb{Z}/6\mathbb{Z}$. For the ID 1-6-1, look at

$$x_0w_Z \;=\; \begin{bmatrix} 6x_0^2x_1x_2 \\ \bar{2}\bar{x}_0 \end{bmatrix}, \quad 2x_0w_Z \;=\; \begin{bmatrix} 12x_0^2x_1x_2 \\ \bar{4}\bar{x}_0 \end{bmatrix}, \quad q_0 \;=\; \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$\begin{aligned}
x_1w_Z \;&=\; \begin{bmatrix} 6x_0x_1^2x_2 \\ \bar{2}\bar{x}_1 \end{bmatrix}, & 2x_1w_Z \;&=\; \begin{bmatrix} 12x_0x_1^2x_2 \\ \bar{4}\bar{x}_1 \end{bmatrix}, & 3x_1w_Z \;&=\; \begin{bmatrix} 18x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \\
4x_1w_Z \;&=\; \begin{bmatrix} 24x_0x_1^2x_2 \\ \bar{2}\bar{x}_1 \end{bmatrix}, & 5x_1w_Z \;&=\; \begin{bmatrix} 30x_0x_1^2x_2 \\ \bar{4}\bar{x}_1 \end{bmatrix}, & q_1 \;&=\; \begin{bmatrix} 2x_1^2 \\ \bar{1} \end{bmatrix},
\end{aligned}$$

$$x_2w_Z \;=\; \begin{bmatrix} 6x_0x_1x_2^2 \\ \bar{2}\bar{x}_2 \end{bmatrix}, \quad 2x_2w_Z \;=\; \begin{bmatrix} 12x_0x_1x_2^2 \\ \bar{4}\bar{x}_2 \end{bmatrix}, \quad q_2 \;=\; \begin{bmatrix} 3x_2^2 \\ \bar{1} \end{bmatrix}.$$

The respective conditions characterizing $x_kw_Z, 2x_kw_Z \in \mathbb{Z}\cdot q_k$ for $k = 0, 2$ are the following:

$$\bar{2}\bar{x}_0 = \bar{0},\ \bar{4}\bar{x}_0 = \bar{0}, \qquad \bar{2}\bar{x}_2 = \bar{2}\bar{x}_0\bar{x}_1,\ \bar{4}\bar{x}_2 = \bar{4}\bar{x}_0\bar{x}_1.$$

and the conditions characterizing $x_1w_Z, \dots, 5x_1w_Z \in \mathbb{Z}\cdot q_1$ are

$$\bar{2}\bar{x}_1 = \bar{3}\bar{x}_0\bar{x}_2, \quad \bar{4}\bar{x}_1 = \bar{0}, \quad \bar{0} = \bar{3}\bar{x}_0\bar{x}_2, \quad \bar{2}\bar{x}_1 = \bar{0}, \quad \bar{4}\bar{x}_1 = \bar{3}\bar{x}_0\bar{x}_2.$$

The possible values of the $\bar{x}_k$ allow only $\iota_0 = 3x_0$, $\iota_1 = 6x_1$ and $\iota_2 = 2$ is excluded by equivalence of the conditions for $k = 2$. For the ID 1-6-5, look at

$$x_0w_Z \;=\; \begin{bmatrix} 6x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad 2x_0w_Z \;=\; \begin{bmatrix} 12x_0^2x_1x_2 \\ \bar{0} \end{bmatrix}, \quad q_0 \;=\; \begin{bmatrix} x_0^2 \\ \bar{0} \end{bmatrix},$$

$$\begin{aligned}
x_1w_Z \;&=\; \begin{bmatrix} 6x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, & 2x_1w_Z \;&=\; \begin{bmatrix} 12x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, & 3x_1w_Z \;&=\; \begin{bmatrix} 18x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, \\
4x_1w_Z \;&=\; \begin{bmatrix} 24x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, & 5x_1w_Z \;&=\; \begin{bmatrix} 30x_0x_1^2x_2 \\ \bar{0} \end{bmatrix}, & q_1 \;&=\; \begin{bmatrix} 2x_1^2 \\ \bar{1} \end{bmatrix},
\end{aligned}$$

$$x_2w_Z \;=\; \begin{bmatrix} 6x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad 2x_2w_Z \;=\; \begin{bmatrix} 12x_0x_1x_2^2 \\ \bar{0} \end{bmatrix}, \quad q_2 \;=\; \begin{bmatrix} 3x_2^2 \\ \bar{5} \end{bmatrix}.$$

The respective conditions characterizing $x_kw_Z, 2x_kw_Z \in \mathbb{Z}\cdot q_k$ for $k = 0, 2$ are the following:

$$\bar{0} = \bar{0},\ \bar{0} = \bar{0}, \qquad \bar{0} = \bar{2}\bar{x}_0\bar{x}_1,\ \bar{0} = \bar{4}\bar{x}_0\bar{x}_1$$

and the conditions characterizing $x_1w_Z, \dots, 5x_1w_Z \in \mathbb{Z}\cdot q_1$ are

$$\bar{0} = \bar{3}\bar{x}_0\bar{x}_2, \quad \bar{0} = \bar{0}, \quad \bar{0} = \bar{3}\bar{x}_0\bar{x}_2, \quad \bar{0} = \bar{0}, \quad \bar{0} = \bar{3}\bar{x}_0\bar{x}_2.$$

Thus, we obtain $\iota_0 = x_0$, $\iota_1 = 2x_1$ and $\iota_2 = 3x_2$ for the local Gorenstein indices, as claimed.

IDs 1-5-$*$: First note that we only have to check $cx_k w_Z \in \mathbb{Z} \cdot q_k$ for $k = 0, 1, 2$ and $c = 1, 2, 3, 4$ due to the identities

$$5x_k w_Z \ = \ 25\tfrac{x_0x_1x_2}{x_k} \cdot q_k, \quad k = 0, 1, 2.$$

Consider the IDs 1-5-1, 1-5-2, 1-5-3. For the cases $k = 0, 1$, the corresponding multiples of the anticanonical class and the two involved columns are

$$cx_k w_Z = \left[\begin{smallmatrix} 5cx_0x_1x_2x_k \\ \bar{c}(\bar{1}+\bar{s}) \end{smallmatrix}\right], \quad c = 1, 2, 3, 4, \ s = 1, 2, 3, \qquad q_0 = \left[\begin{smallmatrix} x_0^2 \\ \bar{0} \end{smallmatrix}\right], \quad q_1 = \left[\begin{smallmatrix} x_1^2 \\ \bar{1} \end{smallmatrix}\right].$$

The condition characterizing $cx_k w_Z \in \mathbb{Z} \cdot q_k$ is $2\bar{c}(\bar{1} + \bar{s}) = \bar{0}$, fulfilled for none of the $c_k = 1, 2, 3, 4$ and $s = 1, 2, 3$. Thus, $\iota_0 = 5x_0$ and $\iota_1 = 5x_1$. Moreover, look at

$$cx_2 w_Z = \left[\begin{smallmatrix} 5cx_0x_1x_2^2 \\ \bar{c}(\bar{1}+\bar{s}) \end{smallmatrix}\right], \quad q_2 = \left[\begin{smallmatrix} 5x_1^2 \\ \bar{s} \end{smallmatrix}\right] \quad c = 1, 2, 3, 4, \ s = 1, 2, 3.$$

Then $cx_2 w_Z \in \mathbb{Z} \cdot q_2$ happens if and only if $\bar{1} + \bar{s} = \bar{s}\bar{x}_0\bar{x}_1$. Thus $\iota_2 = x_2$ and $\iota_2 = 5x_2$ can (and will) occur. For the ID 1-5-4, look at

$$cx_k w_Z = \left[\begin{smallmatrix} 5cx_0x_1x_2x_k \\ \bar{0} \end{smallmatrix}\right], \qquad q_0 = \left[\begin{smallmatrix} x_0^2 \\ \bar{0} \end{smallmatrix}\right], \quad q_1 = \left[\begin{smallmatrix} x_1^2 \\ \bar{1} \end{smallmatrix}\right], \quad q_2 = \left[\begin{smallmatrix} 5x_2^2 \\ \bar{4} \end{smallmatrix}\right].$$

We directly see $x_k w_Z \in \mathbb{Z} \cdot q_k$ for $k = 0, 1$. Hence $\iota_0 = x_0$ and $\iota_1 = x_1$. Moreover, $cx_k w_Z \in \mathbb{Z} \cdot q_k$ is characterized by $\bar{4}\bar{c}\bar{x}_0\bar{x}_1 = \bar{0}$. Thus, $\iota_2 = 5x_2$. $\square$

2.5 $\mathbb{K}^*$-surfaces of Picard number one and integral degree

We explicitly describe all quasismooth projective rational $\mathbb{K}^*$-surfaces of Picard number one and integral degree; see Theorem 2.5.13 and 2.5.15. An important step on the way is Proposition 2.5.8, showing, roughly speaking, that a fake weighted projective plane is an equivariant limit of a quasismooth $\mathbb{K}^*$-surface of the same Picard number and degree provided that at least one of its toric fixed points is a T-singularity. The second main ingredient is the study of the singularities of the fake weighted projective planes performed in the previous section.

Construction 2.5.1. Let $X \subseteq Z$ arise from a generator matrix P as in Construction 1.4.1, that means

$$P \ := \ [v_1, v_2, v_3, v_4] \ := \ \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix}, \qquad \begin{array}{c} {\scriptstyle 1 \le d_1 \le l_1 \le l_2, \quad \gcd(l_i, d_i) = 1,} \\ {\scriptstyle d_0 + \frac{d_1}{l_1} + \frac{d_2}{l_2} \ < \ 0 \ < \ \frac{d_1}{l_1} + \frac{d_2}{l_2}.} \end{array}$$

Further, set

$$\begin{aligned} \mathcal{X}_1 &\coloneqq V(T_1T_2 + ST_3^{l_1} + T_4^{l_2}) \ \subseteq\ Z \times \mathbb{K}, \\ \mathcal{X}_2 &\coloneqq V(T_1T_2 + T_3^{l_1} + ST_4^{l_2}) \ \subseteq\ Z \times \mathbb{K}, \end{aligned}$$

where the T_i are the homogeneous coordinates on Z and S is the coordinate on $\mathbb{K}$. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ are invariant under the respective $\mathbb{K}^*$-actions on $Z \times \mathbb{K}$ given by

$$\begin{aligned} \vartheta \cdot ([z_1, z_2, z_3, z_4], s) &= ([z_1, z_2, \vartheta^{-1}z_3, z_4], \vartheta s), \\ \vartheta \cdot ([z_1, z_2, z_3, z_4], s) &= ([z_1, z_2, z_3, \vartheta^{-1}z_4], \vartheta s). \end{aligned}$$

Restricting the projection $Z \times \mathbb{K} \to \mathbb{K}$ yields flat families $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$ being compatible with the above $\mathbb{K}^*$-actions and the scalar multiplication on $\mathbb{K}$. Set

$$\tilde{P}_1 \coloneqq \begin{bmatrix} l_1 & l_1 & -l_2 \\ d_1 & d_1 + l_1 d_0 & d_2 \end{bmatrix}, \qquad \tilde{P}_2 \coloneqq \begin{bmatrix} l_2 & l_2 & -l_1 \\ d_2 & d_2 + l_2 d_0 & d_1 \end{bmatrix}$$

and let $\tilde{Z}_1, \tilde{Z}_2$ denote the associated fake weighted projective planes. Then the central fiber $\mathcal{X}_{\kappa,0} = \psi_\kappa^{-1}(0)$ equals $\tilde{Z}_\kappa$ and any other fiber $\psi_\kappa^{-1}(s)$ is isomorphic to X.

Proposition 2.5.2. *Let $X \subseteq Z$, where the fake weighted projective space Z arises from the fan Σ in $\mathbb{Z}^3$, and let the generator matrices $\tilde{P}_\kappa$ be as in Construction 2.5.1. For $\kappa = 1, 2$ consider the fan*

$$\Delta_\kappa^{\mathrm{at}} = \{\eta_\kappa^{-1}(\sigma \cap \operatorname{lin}(e_\kappa, e_3);\ \sigma \in \Sigma\}$$

from Construction 1.3.4, where the maps η_κ are given by

$$\eta_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \eta_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then $\tilde{P}_\kappa$ is the generator matrix of the fan $\Delta_\kappa^{\mathrm{at}}$. The maximal cones are given in terms of the columns $\tilde{v}_i$ of $\tilde{P}_\kappa = [\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]$ as follows

$$\sigma_j \coloneqq \operatorname{cone}(\tilde{v}_i;\ i \neq j), \qquad j = 0, 1, 2.$$

Lemma 2.5.3. *Let P be a generator matrix as in Construction 2.5.1. Then the following statements hold:*

(i) *We have* $\operatorname{cone}(v_1, v_3, v_4) \cap \operatorname{lin}(e_1, e_3) = \operatorname{cone}(v_3,\ (-l_2, 0, d_2))$.
(ii) *We have* $\operatorname{cone}(v_2, v_3, v_4) \cap \operatorname{lin}(e_1, e_3) = \operatorname{cone}(v_3,\ (-l_2, 0, d_0 l_2 + d_2))$.
(iii) *We have* $\operatorname{cone}(v_1, v_2, v_3) \cap \operatorname{lin}(e_1, e_3) = \operatorname{cone}(v_3)$.
(iv) *We have* $\operatorname{cone}(v_1, v_2, v_4) \cap \operatorname{lin}(e_1, e_3) = \operatorname{cone}((-l_2, 0, d_2),\ (-l_2, 0, d_0 l_2 + d_2))$.

Proof. We begin by proving (i). The solution set of the system

$$[v_1, v_4, -e_1, -e_3] \cdot x^\top \;=\; \begin{bmatrix} -1 & 0 & -1 & 0 \\ -1 & l_2 & 0 & 0 \\ 0 & d_2 & 0 & -1 \end{bmatrix} \cdot x^\top \;=\; 0.$$

is given by $\{t(l_2, 1, -l_2, d_2);\ t \in \mathbb{K}\}$. Hence, the intersection of the two planes $\mathrm{lin}(v_1, v_4)$ and $\mathrm{lin}(e_1, e_3)$ is given by $\{t(l_2 \cdot v_1 + 1 \cdot v_4);\ t \in \mathbb{K}\}$. Since $l_2 > 0$, we obtain a positive linear combination $l_2 v_1 + v_4 = (-l_2, 0, d_2) \in \mathrm{cone}(v_1, v_4)$. Together with $v_3 \in \mathrm{lin}(e_1, e_3)$, the claim follows.

We proceed with (ii). The solutions $x \in \mathbb{K}^4$ of the system

$$[v_2, v_4, -e_1, -e_3] \cdot x^\top \;=\; \begin{bmatrix} -1 & 0 & -1 & 0 \\ -1 & l_2 & 0 & 0 \\ d_0 & d_2 & 0 & -1 \end{bmatrix} \cdot x^\top = 0$$

are given by $x \in \{t(l_2, 1, -l_2, d_0 l_2 + d_2);\ t \in \mathbb{K}\}$. This implies that the two planes spanned by $\mathrm{lin}(v_2, v_4)$ and $\mathrm{lin}(e_1, e_3)$ intersect in $\{t(l_2 \cdot v_2 + 1 \cdot v_4);\ t \in \mathbb{K}\}$. Because of $l_2 > 0$ we conclude $l_2 v_2 + v_4 = (-l_2, 0, d_0 l_2 + \ d_2) \in \mathrm{cone}(v_2, v_4)$. By using $v_3 \in \mathrm{lin}(e_1, e_3)$, we get the assertion.

The statement (iii) is clear. For (iv) we note that (i) and (ii) provide us with

$$\begin{aligned} &\mathrm{cone}(v_1, v_4) \cap \mathrm{lin}(e_1, e_3) = \mathrm{cone}((-l_2, 0, d_2)), \\ &\mathrm{cone}(v_2, v_4) \cap \mathrm{lin}(e_1, e_3) = \mathrm{cone}((-l_2, 0, d_0 l_2 + d_2)). \end{aligned}$$

□

Lemma 2.5.4. *Let P be a generator matrix as in Construction 2.5.1. Then the following statements hold:*

(i) *We have* $\mathrm{cone}(v_1, v_3, v_4) \cap \mathrm{lin}(e_2, e_3) = \mathrm{cone}(v_4,\ (0, -l_1, d_1))$.
(ii) *We have* $\mathrm{cone}(v_2, v_3, v_4) \cap \mathrm{lin}(e_2, e_3) = \mathrm{cone}(v_4,\ (0, -l_1, d_0 l_1 + d_1))$.
(iii) *We have* $\mathrm{cone}(v_1, v_2, v_4) \cap \mathrm{lin}(e_2, e_3) = \mathrm{cone}(v_4)$.
(iv) *We have* $\mathrm{cone}(v_1, v_2, v_3) \cap \mathrm{lin}(e_2, e_3) = \mathrm{cone}((0, -l_1, d_0 l_1 + d_1),\ (0, -l_1, d_1))$.

Proof. First, we show (i). The solution set of the following system

$$[v_1, v_3, -e_2, -e_3] \cdot x^\top \;=\; \begin{bmatrix} -1 & l_1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & d_1 & 0 & -1 \end{bmatrix} \cdot x^\top = 0$$

is given by $\{t(l_1, 1, -l_1, d_1);\ t \in \mathbb{Z}\}$. Thus, the intersection line of $\mathrm{lin}(v_1, v_3)$ and $\mathrm{lin}(e_2, e_3)$ is $\{t(l_1 \cdot v_1 + 1 \cdot v_3);\ t \in \mathbb{K}\}$. Due to $l_1 > 0$ we obtain a positive linear combination $l_1 v_1 + v_3 = (0, -l_1, d_1) \in \mathrm{cone}(v_1, v_3)$. Further, we have $v_4 \in \mathrm{lin}(e_2, e_3)$. Then the assertion follows.

We proceed with (ii). The solutions $x \in \mathbb{Z}^4$ of the system

$$[v_2, v_3, -e_2, -e_3] \cdot x^\top \; = \; \begin{bmatrix} -1 & l_1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ d_0 & d_1 & 0 & -1 \end{bmatrix} \cdot x^\top = 0$$

are given by $x \in \{t(l_1, 1, -l_1, d_0 l_1 + d_1);\ t \in \mathbb{Z}\}$. Hence, $v \in \operatorname{cone}(v_2, v_3) \cap \operatorname{lin}(e_2, e_3)$ is equivalent to v being an integer multiple of $l_1 v_2 + 1 \cdot v_3 = (0, -l_1, d_0 l_1 + d_1)$.

Further, (iii) is clear. We derive the statement (iv) from (i) and (ii). That means

$$\begin{aligned} \operatorname{cone}(v_1, v_3) \cap \operatorname{lin}(e_2, e_3) &= \operatorname{cone}((0, -l_1, d_1)), \\ \operatorname{cone}(v_2, v_3) \cap \operatorname{lin}(e_2, e_3) &= \operatorname{cone}((0, -l_1, d_0 l_1 + d_1)). \end{aligned}$$

□

Proof of Proposition 2.5.2. For all maximal cones $\sigma \in \Sigma$, Lemma 2.5.3 provides us with the generators of the cones $\sigma \cap \operatorname{lin}(e_1, e_3)$. We compute the preimages of $\sigma \cap \operatorname{lin}(e_1, e_3)$ as follows

$$\begin{aligned} \eta_1^{-1}(\operatorname{cone}(v_3,\ (-l_2, 0, d_2)) &= \operatorname{cone}((d_1, -l_1),\ (d_2, l_2)) &&= \sigma_2, \\ \eta_1^{-1}(\operatorname{cone}(v_3,\ (-l_2, 0, d_0 l_2 + d_2)) &= \operatorname{cone}((d_1, -l_1),\ (d_0 l_2 + d_2, l_2)) &&= \sigma_1, \\ \eta_1^{-1}(\operatorname{cone}((-l_2, 0, d_2),\ (-l_2, 0, d_0 l_2 + d_2)) &= \operatorname{cone}((d_2, l_2),\ (d_0 l_2 + d_2, l_2)) &&= \sigma_3. \end{aligned}$$

This yields the maximal cones and the generator matrix $\tilde{P}_1$ of the fan Δ_1^{at}.

For the maximal cones $\sigma \in \Sigma$, we derive $\sigma \cap \operatorname{lin}(e_2, e_3)$ from Lemma 2.5.4. Then the preimages of $\sigma \cap \operatorname{lin}(e_2, e_3)$ are given by

$$\begin{aligned} \eta_2^{-1}(\operatorname{cone}(v_4,\ (0, -l_1, d_1)) &= \operatorname{cone}((d_2, -l_2),\ (d_1, l_1)) &&= \sigma_2, \\ \eta_2^{-1}(\operatorname{cone}(v_4,\ (0, -l_1, d_0 l_1 + d_1)) &= \operatorname{cone}((d_2, -l_2),\ (d_0 l_1 + d_1, l_1)) &&= \sigma_1, \\ \eta_2^{-1}(\operatorname{cone}((0, -l_1, d_1),\ (0, -l_1, d_0 l_1 + d_1)) &= \operatorname{cone}((d_1, l_1),\ (d_0 l_1 + d_1, l_1)) &&= \sigma_3. \end{aligned}$$

This provides us with the maximal cones and the generator matrix $\tilde{P}_2$ of the fan Δ_2^{at}. □

Proof of Construction 2.5.1. The families $\psi_\kappa \colon \mathcal{X}_\kappa \to \mathbb{K}$ are those delivered by Construction 1.3.1 for $\kappa = 1, 2$. From Proposition 1.3.7 and Proposition 2.5.2 we derive that the $\tilde{P}_\kappa$ are the generator matrices of the central fibers. Thus, $\psi_\kappa^{-1}(0) = \tilde{Z}_\kappa$. □

Remark 2.5.5 ([27, Rem. 3.7])**.** The flat families $\psi_i \colon \mathcal{X}_i \to \mathbb{K}$ from Construction 2.5.1 are special equivariant test configurations in the sense of [18, Def. 5.2] for the del Pezzo $\mathbb{K}^*$-surface X. Moreover, according to [18, Prop. 5.4], any other special equivariant test configuration of X has limit $\tilde{Z}_1$ or $\tilde{Z}_2$.

Proposition 2.5.6. *Consider $X = X(P)$ in $Z = Z(P)$, the fake weight vector $w(P) = (w_1, w_2, w_3, w_4)$ and $\mathcal{X}_i \to \mathbb{K}$ as in Construction 2.5.1. Then*

$$w(\tilde{P}_1) \ = \ (w_1,\, w_2,\, -d_0 l_1^2), \qquad w(\tilde{P}_2) \ = \ (w_1,\, w_2,\, -d_0 l_2^2)$$

holds for the fake weight vectors of the central fibers $\tilde{Z}_1$ and $\tilde{Z}_2$. The canonical self intersection numbers of $X, \tilde{Z}_1, \tilde{Z}_2$ satisfy

$$\mathcal{K}_X^2 \ = \ \mathcal{K}_{\tilde{Z}_1}^2 \ = \ \mathcal{K}_{\tilde{Z}_2}^2.$$

Furthermore, let $Z^{\pm} \subseteq Z$ be the affine toric subvariety containing $x^{\pm} \in Z$ and define $X^{\pm} := X \cap Z^{\pm}$. Then we have isomorphisms of affine surfaces

$$X^{+} \ \cong \ \tilde{Z}_1 \setminus V(T_1) \ \cong \ \tilde{Z}_2 \setminus V(T_1), \qquad X^{-} \ \cong \ \tilde{Z}_1 \setminus V(T_0) \ \cong \ \tilde{Z}_2 \setminus V(T_0).$$

Finally, let $Z_0 \subseteq Z$ be the affine toric subvariety given by $\operatorname{cone}(v_1, v_2)$ *and set $X_0 := X \cap Z_0$. Then X_0 is the index one cover of $\tilde{Z}_i \setminus V(T_2)$, $i = 1, 2$.*

Lemma 2.5.7. *Let (l_i, d_i), where $i = 1, 2$ be two pairs of coprime integers, let $d_0 \in \mathbb{Z}$ and consider the affine generator matrices*

$$B \ = \ \begin{bmatrix} 1 & l_1 & 0 \\ -1 & 0 & l_2 \\ d_0 & d_1 & d_2 \end{bmatrix}, \qquad \tilde{B} \ := \ \begin{bmatrix} l_1 & -l_2 \\ d_1 + l_1 d_0 & d_2 \end{bmatrix}.$$

Then the associated affine toric varieties share the divisor class groups K and class group order w and these are explicitly given by

$$\begin{aligned} w \ &= \ \det(B) \ = \ \det(\tilde{B}) \ = \ -d_0 l_1 l_2 - d_1 l_2 - d_2 l_1, \\ K \ &= \ \mathbb{Z}^3/\operatorname{im}(B^*) \ = \ \mathbb{Z}^2/\operatorname{im}(\tilde{B}^*) \ = \ \mathbb{Z}/w\mathbb{Z}. \end{aligned}$$

Moreover, let $\tilde{C} = [\bar{c}_1, \bar{c}_2]$ be a degree matrix for $\tilde{B}$ in K. Then $C = [\bar{l}_1 \bar{c}_1, \bar{c}_1, \bar{c}_2]$ is a degree matrix for B in K.

Proof. The statement on the determinants is obvious. To proceed, we turn B by means of unimodular row operations into the matrix

$$B_0 \ = \ \begin{bmatrix} 1 & l_1 & 0 \\ 0 & l_1 & -l_2 \\ 0 & d_1 + d_0 l_1 & d_2 \end{bmatrix}.$$

Then $K = \mathbb{Z}^3/\operatorname{im}(B^*) = \mathbb{Z}^3/\operatorname{im}(B_0^*) = \mathbb{Z}^2/\operatorname{im}(\tilde{B}^*)$. As $\tilde{C}$ is a degree matrix, K is generated by $\bar{c}_1$. Finally, B_0 and B share C as a degree matrix in K. $\square$

Proof of Proposition 2.5.6. The statement on the fake weight vectors is obtained by Remark 1.1.8 and Proposition 1.4.3. Also the identity of the canonical self intersections can be directly verified, using Proposition 1.4.3, Proposition 1.4.4 and Proposition 2.2.4. For the remaining statements, consider the following submatrices

$$P^+ \;=\; \begin{bmatrix} -1 & l_1 & 0 \\ -1 & 0 & l_2 \\ 0 & d_1 & d_2 \end{bmatrix}, \qquad\qquad P^- \;=\; \begin{bmatrix} -1 & l_1 & 0 \\ -1 & 0 & l_2 \\ d_0 & d_1 & d_2 \end{bmatrix}.$$

The cones spanned by the columns of $P^\pm$ define open toric subvarieties $Z^\pm \subseteq Z$. Cutting down to X provides us with affine $\mathbb{K}^*$-invariant open neighbourhoods

$$x^\pm \;\in\; X^\pm \;:=\; X \cap Z^\pm \;\cong\; X(P^\pm),$$

where the affine $\mathbb{K}^*$-surface $X(P^\pm)$ is constructed in an analogous manner to the proceeding in the projective case; see [29]:

$$X(P^\pm) \;=\; \bar{X}^\pm/H^\pm, \qquad \bar{X}_i \;=\; V(T_1 + T_2^{l_1} + T_3^{l_2}) \;\subseteq\; Z_i.$$

Here, respective quasitori $H^\pm \subseteq \mathbb{K}^*$ are the subgroups of $w_{2,1}$-th roots of unity and, due to Lemma 2.5.7, the corresponding degree matrices are given by

$$Q^+ \;=\; [\bar{l}_1\bar{b}_1,\, \bar{b}_1,\, \bar{c}_1], \qquad\qquad Q^- \;=\; [\bar{l}_1\bar{b}_2,\, \bar{b}_2,\, \bar{c}_2],$$

with degree matrices $\tilde{Q}^+ = [\bar{b}_1, \bar{c}_1]$ for $\tilde{Z}_1 \setminus V(T_1)$ and $\tilde{Q}^- = [\bar{b}_2, \bar{c}_2]$ for $\tilde{Z}_1 \setminus V(T_0)$. In particular, we have $H^\pm$-equivariant isomorphisms

$$\varphi^\pm \colon \mathbb{K}^2 \;\to\; \bar{X}^\pm, \qquad z \;\mapsto\; (-z_1^{l_1} - z_2^{l_2}, z_1, z_2).$$

Passing to the induced morphisms of the respective quotients by $H^\pm$ gives the desired isomorphies of affine surfaces

$$\tilde{Z}_1 \setminus V(T_1) = \mathbb{K}^2/H^+ \cong \bar{X}^+/H^+ = X^+, \quad \tilde{Z}_1 \setminus V(T_0) = \mathbb{K}^2/H^- \cong \bar{X}^-/H^- = X^-.$$

Finally, we infer from [3, Prop. 3.4.4.6] that $x_0 \in X_0$ is isomorphic to the toric fixed point in the affine toric surface Z_0 by the generator matrix

$$A \;:=\; \begin{bmatrix} 1 & 1 \\ 0 & d_0 \end{bmatrix}.$$

In terms of generator matrices, one directly checks that Z_0 naturally maps onto $\tilde{Z}_1 \setminus V(T_2)$ and $\tilde{Z}_2 \setminus V(T_2)$ and that these maps are the index one covers:

$$S_i \cdot A \;=\; \begin{bmatrix} l_i & l_i \\ d_i & d_i + d_0 l_i \end{bmatrix}, \qquad S_i \;:=\; \begin{bmatrix} l_i & 0 \\ d_i & l_i \end{bmatrix}.$$

□

Proposition 2.5.8. *Let $\tilde{Z}_1$ be a fake weighted projective plane with fake weight vector $w = (w_0, w_1, w_2)$ and degree matrix $\tilde{Q}_1$ such that $z(2) \in \tilde{Z}_1$ is T-singular. Then $\tilde{Q}_1$ admits precisely one corresponding generator matrix $\tilde{P}_1$ of the form*

$$\tilde{P}_1 \;=\; \begin{bmatrix} l_1 & l_1 & -l_2 \\ d_1 & d_1 + d_0 l_1 & d_2 \end{bmatrix}, \qquad \begin{array}{l} l_1{=}\iota(z(2)),\ d_0{=}{-}w_2/l_1^2,\ 0{\le}d_1{<}l_1, \\ l_2{=}l_1(w_0{+}w_1)/w_2, \\ d_2{=}{-}(d_1w_0{+}d_1w_1{+}d_0l_1w_1)/w_2, \\ d_0l_1l_2{+}d_1l_2{+}l_1d_2{<}0{<}d_1l_2{+}l_1d_2. \end{array}$$

The entries of the matrix $\tilde{P}_1$ give rise to generator matrices P and $\tilde{P}_2$, defining a $\mathbb{K}^*$-surface $X(P)$ and a fake weighted projective plane $\tilde{Z}_2$, namely

$$P \;=\; \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix}, \qquad \tilde{P}_2 \;=\; \begin{bmatrix} l_2 & l_2 & -l_1 \\ d_2 & d_2 + d_0 l_2 & d_1 \end{bmatrix}.$$

The $\tilde{Z}_i$ are the central fibers of the flat families $\mathcal{X}_i \to \mathbb{K}$ from Construction 2.5.1 applied to $X(P)$, their fake weight vectors satisfy

$$w(\tilde{P}_2) \;=\; \lambda(w(\tilde{P}_1)),$$

and the point $z(2) \in \tilde{Z}_2$ is at most T-singular. Moreover, the $\mathbb{K}^*$-surface $X(P)$ is non-toric if and only if l_1 and $l_1(w_0 + w_1)/w_2$ both differ from one.

Proof. Let P_1 be any generator matrix corresponding to $\tilde{Q}_1$. Lemma 2.4.2 tells us that, after multiplying P_1 from the left with a suitable unimodular matrix, the first two columns look as those of $\tilde{P}_1$. Observe that the requirements $d_0 < 0$ and $0 \le d_1 < l_1$ uniquely determine d_1. The third column of P_1 is determined by the fact that P_1 annihilates the fake weight vector $w(\tilde{Q}_1)$. Observe that the conditions on the entries of $\tilde{P}_1$ ensure the P and $\tilde{P}_2$ are indeed projective generator matrices. The remaining statements are clear by Construction 2.5.1, Proposition 2.5.6 and Lemma 2.4.2. ☐

Corollary 2.5.9. *Let Z be a fake weighted projective plane with fake weight vector $w = (w_0, w_1, w_2)$. Then the following two statements are equivalent.*

(i) *Z is the central fiber of an equivariant test configuration of a non-toric, quasismooth, rational, projective $\mathbb{K}^*$-surface X of Picard number one.*

(ii) *At least one of the three toric fixed points of Z, say $z(i)$, is T-singular and satisfies*

$$\iota(z(i)) > 1, \qquad w_0 + w_1 + w_2 > \left(\frac{1}{\iota(z(i))} + 1\right) w_i.$$

Proof. Combine Construction 2.5.1 and Proposition 2.5.8 with [18, Prop. 5.4]. ☐

Remark 2.5.10. Consider the families $\psi_i\colon \mathcal{X}_i \to \mathbb{K}$ from Construction 2.5.1. The fiberwise $\mathbb{K}^*$-actions and the horizontal $\mathbb{K}^*$-action define an action of the two-torus $\mathbb{T}^2$ on the $\mathcal{X}_i$. This allows us to track the three fixed points x_0, x_1, x_2 in fibers $\psi_i^{-1}(1) = X(P)$ in the degeneration process. By properness of ψ_i, the horizontal $\mathbb{K}^*$-orbit through x_k has a limit point in $\psi_k^{-1}(0) = \tilde{Z}_i$. The hyperbolic fixed point x_0 tends to the toric fixed point $z(2)$ and the two elliptic fixed points x_1, x_2 to toric fixed points $z(0)$, $z(1)$.

Definition 2.5.11. We call two generator matrices P_1, P_2 of fake weighted projective planes Z_1, Z_2 *adjacent* if P_1, P_2 arise via Construction 2.5.1 from a common matrix P. In this situation, we call any pair Z_1', Z_2' of fake weighted projective planes with $Z_1' \cong Z_1$ and $Z_2' \cong Z_2$ as well adjacent.

Definition 2.5.12. Let Q_1 and Q_2 be degree matrices of fake weighted projective planes Z_1 and Z_2, respectively. We call Q_1, Q_2 *adjacent* if the following holds:

(i) there exist corresponding generator matrices P_1 for Q_1 and P_2 for Q_2 such that P_1, P_2 are adjacent,

(ii) we have $w_0 \leq w_1$ for the first two (common) components of the fake weight vectors $w(Q_1)$ and $w(Q_2)$,

(iii) up to permuting the columns, each of the Q_1, Q_2 is adjusted in the sense of Definition 2.2.11.

Theorem 2.5.13. *For every fake weighted projective plane Z_1 of integral degree, there is a pair (Q_1, Q_2) of adjacent degree matrices with $Z_1 \cong Z(Q_1)$.*

Proof. Proposition 2.4.5 to 2.4.9 ensure that at least one of the toric fixed points of the fake weighted projective plane Z_1 is at most T-singular. Thus Proposition 2.5.8 yields the assertion. □

Definition 2.5.14. Let Q_1, Q_2 be adjacent degree matrices. Set $Z_i = Z(Q_i)$ and denote by l_i the local Gorenstein indices of $z(2) \in Z_i$. We call (Q_1, Q_2) *ordered* if $l_1 \leq l_2$ and we say that (Q_1, Q_2) is *non-toric* if $l_1, l_2 \geq 2$.

By the definition of adjacency, every non-toric, ordered pair (Q_1, Q_2) of adjacent degree matrices gives rise to a $\mathbb{K}^*$-surface X, degenerating via Construction 2.5.1 to the fake weighted projective planes associated with Q_1, Q_2. Due to Proposition 2.5.8, the surface X is non-toric and uniquely determined up to isomorphism by (Q_1, Q_2). We write $X = X(Q_1, Q_2)$.

Theorem 2.5.15. *Let X be a non-toric, quasismooth, rational, projective $\mathbb{K}^*$-surface of Picard number one with $\mathcal{K}_X^2 \in \mathbb{Z}$. Then $X \cong X(Q_1, Q_2)$ with a non-toric, ordered pair of adjacent degree matrices. Moreover, distinct ordered pairs (Q_1, Q_2) of adjacent degree matrices yield non-isomorphic $\mathbb{K}^*$-surfaces.*

Proof. Let X be a quasismooth, rational, projective $\mathbb{K}^*$-surface of Picard number one and integral degree. Then $X \cong X(P)$ with a generator matrix P as introduced in Section 2.5; see also [27, Constr. 3.1]. According to Construction 2.5.1, we have $X(P) = X(Q_1, Q_2)$, where Q_1, Q_2 are adjacent degree matrices corresponding to the generator matrices P_1, P_2 given there, and we may assume that (Q_1, Q_2) is ordered. Moreover, Proposition 2.5.8 tells us that (Q_1, Q_2) is non-toric. Finally, the definition of a non-toric, ordered pair of adjacent degree matrices ensures that distinct pairs define non-isomorphic $\mathbb{K}^*$-surfaces. □

2.6 Examples and observations

In a first example series, we check the isomorphies for members inside one of the series (1-9-$*$) or (1-8-$*$) in the case of a small fake weight vector and we discuss the first fake weight vector, where any two different values of the entry $\bar{\eta}$ in the adjusted degree matrix lead to non-isomorphic fake weighted projective planes. We begin by presenting the tacitly used tool box.

Lemma 2.6.1. *Let $\varphi\colon \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ be a surjective group homomorphism. Then φ is an isomorphism.*

Proof. We first show that only torsion elements map to torsion elements. Consider $x, y \in \mathbb{Z}$ with $\varphi(x, \bar{y}) \in \{0\} \oplus \mathbb{Z}/\mu\mathbb{Z}$. Then we have

$$\varphi(\mu x, \bar{0}) \ = \ \mu\varphi(x, \bar{y}) \ = \ (0, \bar{0}).$$

Consequently, $\mu x\mathbb{Z} \oplus \{\bar{0}\} \subseteq \ker(\varphi)$. The homomorphism theorem yields a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z} \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z} \\
& \mathbb{Z}/\mu x\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z} \arrow[ur, "\bar{\varphi}"'] &
\end{tikzcd}$$

As φ is surjective, also $\bar{\varphi}$ is so. By group order reasons, $x = 0$. The claim is verified. As a consequence, we obtain isomorphisms

$$\begin{tikzcd}
\{0\} \oplus \mathbb{Z}/\mu\mathbb{Z} \arrow[r, equal] & \varphi^{-1}(\{0\} \oplus \mathbb{Z}/\mu\mathbb{Z}) \arrow[r, "\cong", "\varphi"'] & \{0\} \oplus \mathbb{Z}/\mu\mathbb{Z}.
\end{tikzcd}$$

In particular the kernel of $\varphi\colon \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$, being contained in the torsion part, must be trivial. □

Remark 2.6.2. Consider a degree matrix Q in $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ and a 2×3 projective generator matrix P sharing the same fake weight vector. Then Lemma 2.6.1 says that Q and P correspond to each other if and only if Q annihilates the rows of P.

Example 2.6.3. In the series (1-9-$*$), consider the degree matrices sharing the fake weight vector $(9, 9, 9)$:

$$\begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{8} \end{bmatrix}.$$

As corresponding generator matrices we obtain the following ones, all defining isomorphic fake weighted projective planes:

$$\begin{bmatrix} 3 & 3 & -6 \\ 1 & -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & -2 \\ 0 & -9 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 3 & -6 \\ 2 & -1 & -1 \end{bmatrix}.$$

For the toric fixed points $z(0)$, $z(1)$, $z(2)$, the local Gorenstein indices, T-singularity identifier and numbers of exceptional curves of the minimal resolution are

$$\begin{array}{ccc} (3,1,3), & (3,3,1), & (1,3,3), \\ (+,+,+), & (+,+,+), & (+,+,+), \\ (2,8,2), & (2,2,8), & (8,2,2). \end{array}$$

Example 2.6.4. In the series (1-9-$*$), consider the degree matrices sharing the fake weight vector $(9, 9, 36)$:

$$\begin{bmatrix} 1 & 1 & 4 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 4 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 4 \\ \bar{0} & \bar{1} & \bar{8} \end{bmatrix}.$$

As corresponding generator matrices we obtain the following ones, where the last two define isomorphic fake weighted projective planes:

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & -17 & 4 \end{bmatrix}, \quad \begin{bmatrix} 6 & 6 & -3 \\ 1 & -5 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 6 & -3 \\ 5 & -1 & -1 \end{bmatrix}.$$

For the toric fixed points $z(0)$, $z(1)$, $z(2)$, the local Gorenstein indices, T-singularity identifier and numbers of exceptional curves of the minimal resolution are

$$\begin{array}{ccc} (3,3,2), & (3,1,6), & (1,3,6), \\ (+,+,+), & (+,+,+), & (+,+,+), \\ (2,2,9), & (2,8,5), & (8,2,5). \end{array}$$

Example 2.6.5. In the series (1-9-$*$), consider the degree matrices sharing the fake weight vector $(9, 36, 225)$:

$$\begin{bmatrix} 1 & 4 & 25 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 4 & 25 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 4 & 25 \\ \bar{0} & \bar{1} & \bar{8} \end{bmatrix}.$$

As corresponding generator matrices we obtain the following ones, defining pairwise non-isomorphic fake weighted projective planes:

$$\begin{bmatrix} 15 & 15 & -3 \\ 2 & -13 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 5 & 5 & -1 \\ 1 & -44 & 7 \end{bmatrix}, \qquad \begin{bmatrix} 15 & 15 & -3 \\ 7 & -8 & 1 \end{bmatrix}.$$

For the toric fixed points $z(0)$, $z(1)$, $z(2)$, the local Gorenstein indices, T-singularity identifier and numbers of exceptional curves of the minimal resolution are

$$\begin{array}{ccc} (3,2,15), & (3,6,5), & (1,6,15), \\ (+,+,+), & (+,+,+), & (+,+,+), \\ (2,9,8), & (2,5,12), & (8,5,8). \end{array}$$

Example 2.6.6. In the series (1-8-$*$), consider the degree matrices $Q_1, \ldots, Q_4$, sharing the fake weight vector $(8, 8, 16)$:

$$\begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 2 \\ \bar{0} & \bar{1} & \bar{7} \end{bmatrix}.$$

As corresponding generator matrices $P_1, \ldots, P_4$, we obtain the following, showing in particular $Z(P_2) \cong Z(P_4)$:

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -16 & 8 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & -4 \\ 1 & -3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & -2 \\ 1 & -7 & 3 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & -4 \\ 3 & -1 & -1 \end{bmatrix}.$$

For the toric fixed points $z(0)$, $z(1)$, $z(2)$, the local Gorenstein indices, T-singularity identifier and numbers of exceptional curves of the minimal resolution are

$$\begin{array}{cccc} (4,4,1), & (2,1,4), & (4,4,2), & (1,2,4), \\ (-,-,+), & (+,+,+), & (-,-,+), & (+,+,+), \\ (1,1,15), & (2,7,3), & (3,3,4), & (7,2,3). \end{array}$$

Example 2.6.7. In the series (1-8-$*$), consider the degree matrices $Q_1, \ldots, Q_4$, sharing the fake weight vector $(8, 16, 72)$:

$$\begin{bmatrix} 1 & 9 & 2 \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 9 & 2 \\ \bar{0} & \bar{1} & \bar{3} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 9 & 2 \\ \bar{0} & \bar{1} & \bar{5} \end{bmatrix}, \qquad \begin{bmatrix} 1 & 9 & 2 \\ \bar{0} & \bar{1} & \bar{7} \end{bmatrix}.$$

As corresponding generator matrices $P_1, \ldots, P_4$, we obtain the following, no two of them defining isomorphic fake weighted projective planes:

$$\begin{bmatrix} 2 & 2 & -10 \\ 1 & -7 & 31 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & -20 \\ 3 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & -5 \\ 0 & -16 & 72 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 & -20 \\ 1 & -3 & 13 \end{bmatrix}.$$

For the toric fixed points $z(0)$, $z(1)$, $z(2)$, the local Gorenstein indices, T-singularity identifier and numbers of exceptional curves of the minimal resolution are

$$\begin{array}{cccc} (4,12,2), & (2,3,4), & (4,12,1), & (1,6,4), \\ (-,-,+), & (+,+,+), & (-,-,+), & (+,+,+), \\ (1,11,4), & (2,9,3), & (3,3,15), & (7,6,3). \end{array}$$

We take a closer look at the adjacency relation on the class of fake weighted projective planes, as introduced in Definition 2.5.11. The aim is to indicate, how adjacency can be read off from adjusted degree matrices. In the case of at most T-singular fake weighted projective planes, the situation is completely described by the trees $T(a)$ of the Markov type equations.

Construction 2.6.8. Let $T(a,\mu)$ denote the set of isomorphy classes of all at most T-singular fake weighted projective planes of degree $a \in \mathbb{Z}_{>0}$ and multiplicity μ. We join two distinct classes $\mathcal{Z}_1, \mathcal{Z}_2 \in T(a,\mu)$ by an edge, if there are representatives $Z_i \in \mathcal{Z}_i$ such that Z_1 and Z_2 are adjacent.

Proposition 2.6.9. *For $a = 9, 8, 6, 5, 4, 3$, each of the graphs $T(a,\mu)$ is connected. Moreover, we have canonical isomophisms of graphs:*

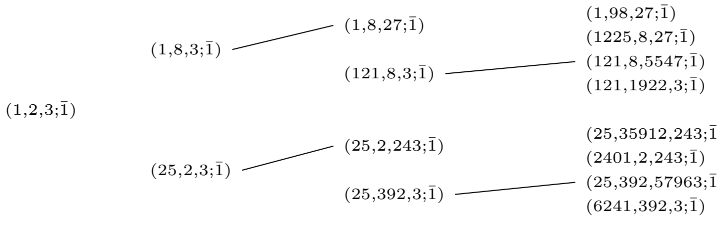
$$T(9) \cong T(9,1) \cong T(3,3), \qquad T(8) \cong T(8,1) \cong T(4,2),$$
$$T(6) \cong T(6,1) \cong T(3,2) \cong T(2,3) \cong T(1,6), \qquad T(5) \cong T(5,1) \cong T(1,5),$$

where $T(a)$ denotes the tree of ascendingly ordered solution triples of the squared Markov equation from Remark 2.1.5.

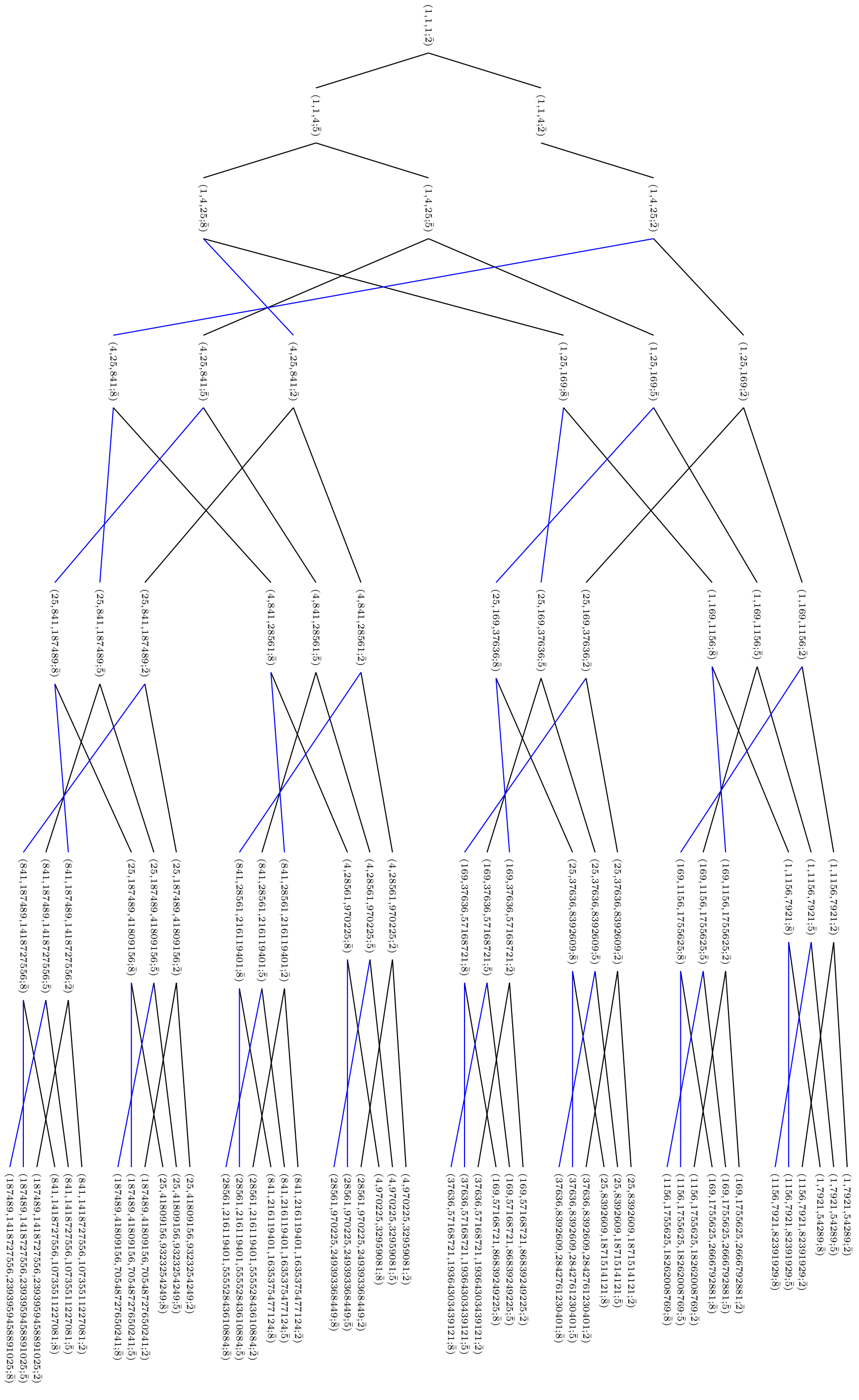
We discuss the remaining cases. For the subsequent examples, we explicitly computed the data of Proposition 2.5.8 for the first adjusted degree matrices in the series provided by our classification results. We omit the computation and just present the resulting graphs, where the vertices, that means the isomorphy classes of fake weighted projective planes, are specified by $(u_0, u_1, u_2; \bar{\eta})$ with the first row $u = (u_0, u_1, u_2)$ of the corresponding adjusted degree matrix Q and the last entry $\bar{\eta}$ of the second row of Q.

Remark 2.6.10. The graph $T(2,4)$ has the series (2-4-1) and (2-4-3) as its connected components, where each of them is isomorphic to $S(8)$ as a graph.

Remark 2.6.11. The series (2-3-1) is not connected to $T(2,3)$ by adjacency. Inside (2-3-1), we observe a simple pattern for the adjacencies:

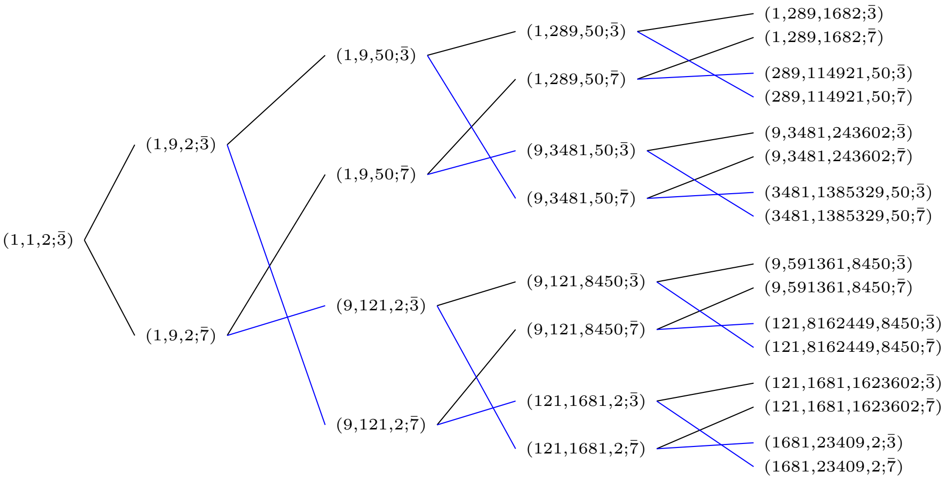


Remark 2.6.12. We take a look at the connected graph $T(1,9)$, where the vertices stem from the three series (1-9-2), (1-9-5) and (1-9-8):



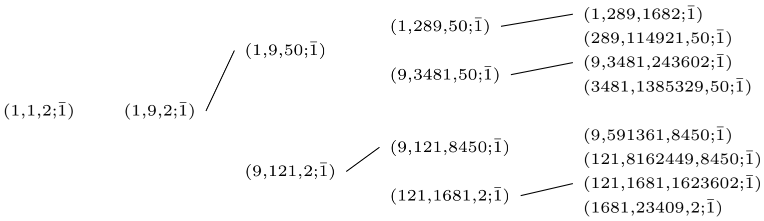
Here, the blue edges indicate the jumps between the series (1-9-2), (1-9-5), (1-9-8) for the first seven levels. We expect the pattern to continue.

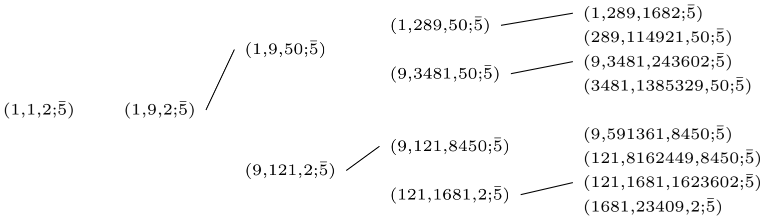
Remark 2.6.13. The connected graph $T(1,8)$ takes its vertices from the two series (1-8-3) and (1-8-7) and we have the following adjacencies:



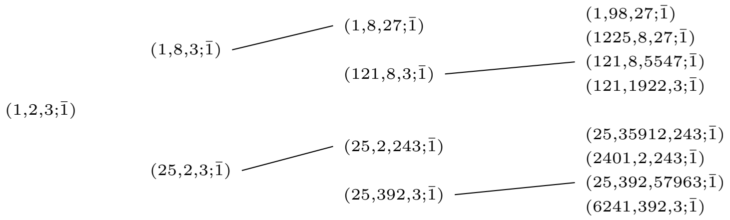
Similarly as before, we indicate the jumps between the involved series by the blue edges. We expect the pattern to continue.

Remark 2.6.14. The series (1-8-1) and (1-8-5) are neither connected to $T(1,8)$ via adjacencies nor to each other. We obtain the following pattern of adjacencies.

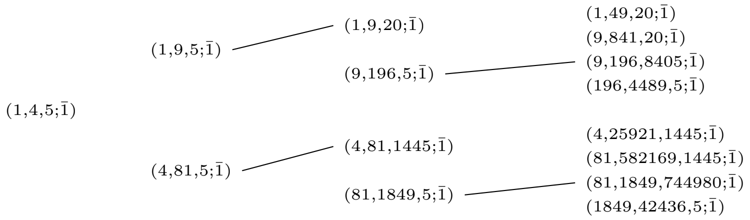




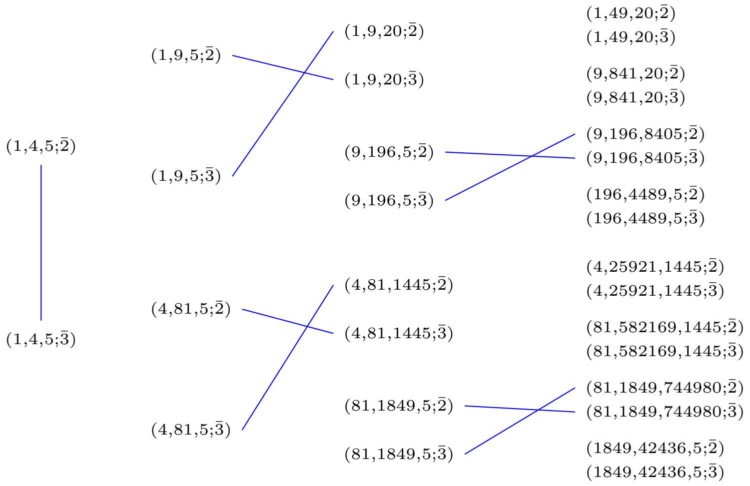
Remark 2.6.15. The series (1-6-1) is not connected to $T(1,6)$ by adjacency. Inside (1-6-1), the pattern of adjacencies equals that of (2,3,1):



Remark 2.6.16. The series (1-5-1) is not connected to any other series (1-5-∗) via adjacencies. Inside (1-5-1), we have the following pattern of adjacencies:



The series (1-5-2), (1-5-3) are not connected to $T(1,5)$ by adjancency. Inside these series, the adjacency pattern resembles the previous one, but only jumps occur:



Remark 2.6.17. Note that Construction 2.6.8 does not take “self-adjacency” into account, that means fake weighted projective planes Z of integral degree that are adjacent to themselves. This happens precisely in the following 16 cases:

$$
\begin{aligned}
u = (1,1,2): &\quad \text{(8-1-0), (4-2-1), (2-4-1), (2-4-3), (1-8-1), (1-8-3), (1-8-5),}\\
u = (1,2,3): &\quad \text{(6-1-0), (3-2-1), (2-3-1), (2-3-2), (1-6-1), (1-6-5),}\\
u = (1,4,5): &\quad \text{(5-1-0), (1-5-1), (1-5-4),}
\end{aligned}
$$

where u denotes the first row of the degree matrix. Among these cases, we have $Z \leftsquigarrow X \rightsquigarrow Z$ with a non-toric X from Construction 2.5.1 in precisely 6 cases, namely

$$\text{(2-4-3),} \quad \text{(1-8-3),} \quad \text{(1-8-5),} \quad \text{(1-6-5),} \quad \text{(1-5-1),} \quad \text{(1-5-4).}$$

2.7 A criterion for non-existence of Sasaki-Einstein metrics

The main result of this section, Theorem 2.7.1, shows the non-existence of Sasaki-Einstein metrics for a certain subclass of quasismooth $\mathbb{K}^*$-surfaces of Picard number one.

Let $f \in \mathbb{C}[T_1, \ldots, T_{n+1}]$ be a non-constant polynomial and let $x \in V(f)$ be a singular point. We call the intersection of $V(f)$ with a sphere of real dimension $2n+1$ of small radius centered at x the *link of the singular point x*. By the *anticanonical cone link S_X* of X we mean the link of the singularity that the anticanonical cone $\tilde{X}$ over X has in its apex. A *Sasaki-Einstein metric* on the link S_X is the restriction of a Ricci-flat Kähler metric on the anticanonical cone.

Theorem 2.7.1. *Let X be a non-toric, quasismooth, rational, projective $\mathbb{C}^*$-surface and let $1 < l_1 \le l_2$ denote the orders of its non-trivial finite $\mathbb{C}^*$-isotropy groups. If $2l_1 \le l_2$, then S_X admits no Sasaki-Einstein metric.*

We recall the necessary background from Section 1.4 and from [18, Chapter 6].

Setting 2.7.2. Let X be a projective, rational, quasismooth $\mathbb{C}^*$-surface of Picard number one arising from Construction 1.4.1, that means $X = X(P)$ with

$$
P \;:=\; [v_1, v_2, v_3, v_4] \;:=\; \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \end{bmatrix}, \qquad
\begin{array}{c}
1 \le d_1 \le l_1 \le l_2, \quad \gcd(l_i, d_i) = 1, \\
d_0 + \frac{d_1}{l_1} + \frac{d_2}{l_2} \;<\; 0 \;<\; \frac{d_1}{l_1} + \frac{d_2}{l_2},
\end{array}
$$

where Z is the fake weighted projective space with generator matrix P and where

$$X \;:=\; X(P) \;:=\; \overline{V(h)} \;\subseteq\; Z, \qquad h \;:=\; 1 + U_1 + U_2 \;\in \mathcal{O}(\mathbb{T}^3).$$

By Proposition 1.3.7, the anticanonical cone over X is the 3-dimensional affine variety $\tilde{X}$ with 2-torus action given by the defining matrix

$$
\tilde{P} \;:=\; [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4] \;:=\; \begin{bmatrix} -1 & -1 & l_1 & 0 \\ -1 & -1 & 0 & l_2 \\ 0 & d_0 & d_1 & d_2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.
$$

The columns $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ of $\tilde{P}$ are the primitive generators of a polyhedral cone $\tilde{\sigma}$ of full dimension in $\mathbb{Z}^4$ and we have

$$\tilde{X} \ = \ \overline{V(1+S_1+S_2)} \ \subseteq \ \tilde{Z},$$

where $\tilde{Z}$ is the affine toric 4-fold associated with $\tilde{\sigma}$. For $\kappa = 1,2$ we associate to $\tilde{P}$ the cones

$$\tilde{\tau}_\kappa \ := \ \eta_\kappa^{-1}(\tilde{\sigma}) \ \subseteq \ \mathbb{R}^3,$$

where the linear maps $\eta_\kappa \colon \mathbb{Z}^3 \to \mathbb{Z}^4$ are the antitropical coordinates, given by their matrices

$$\eta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Remark 2.7.3. Consider $X \subseteq Z$ as in Setting 2.7.2. According to Proposition 1.3.7 (iv) the fibers $\mathcal{X}_{\kappa,0}$ are normal for $\kappa = 1,2$, and the fiber $\mathcal{X}_{0,0}$ is normal if and only if $l_1 = 1$.

Construction 2.7.4 (See [18, Constr. 6.11])**.** Let $\tilde{\tau}_\kappa$ be as in Setting 2.7.2 and let $\kappa = 1,2$. Let $\tilde{b}_1, \ldots, \tilde{b}_k$ be the primitive generators of $\tilde{\tau}_\kappa$. Then we define

$$b_i' \ := \ \left(\tilde{b}_{i1}, 1, \tilde{b}_{i3}\right), \qquad \tau_\kappa' \ := \ \operatorname{cone}\left(b_1', b_2', b_3'\right).$$

We set $\omega_\kappa' := (\tau_\kappa')^\vee$ and with any vector $\xi = (\xi_1, 1, \xi_2) \in (\tau_\kappa')^\circ$ we associate the polytope

$$\mathcal{B}_\kappa'(\xi) \ := \ \{u \in \omega_\kappa';\ u(\xi) \le 1\} \ \subseteq \ \omega_\kappa' \ \subseteq \ \mathbb{R}^3.$$

Then, for each $\kappa = 1,2$ we define a *volume function* as

$$\operatorname{vol}_\kappa(x_1, x_2) \ := \ \operatorname{vol}(\mathcal{B}_\kappa'(x_1, 1, x_2)).$$

Theorem 2.7.5. *Consider $X \subseteq Z$ as in Setting 2.7.2 and let τ_κ' and $\operatorname{vol}_\kappa$ be as in Construction 2.7.4. Then the following statements hold:*

- (i) *The function $\operatorname{vol}_\kappa$ is rational on $\tau_\kappa' \cap \mathbb{R} \times \{1\} \times \mathbb{R}$.*
- (ii) *There exists a unique x_0 such that $\partial \operatorname{vol}_\kappa / \partial x\,(x_0, 1, 0) = 0$ and such that $(x_0, 1, 0) \in \tau_\kappa'$ for $\kappa = 1,2$.*
- (iii) *If the anticanonical cone link S_X of X admits a Sasaki-Einstein metric, then we obtain $\partial \operatorname{vol}_\kappa / \partial y\,(x_0, 1, 0) < 0$ for $\kappa = 1,2$.*

Proof. This is a direct consequence of [18, Thm. 6.12 and Rem. 6.13] and Remark 2.7.3. ☐

Lemma 2.7.6. *Consider the cone τ_1' as in Construction 2.7.4 Then τ_1' has the primitive generators*

$$v_1 := (d_2, 1, l_2), \qquad v_2 := (d_0 l_2 + d_2, 1, l_2), \qquad v_3 := (d_1, 1, -l_1).$$

Proof. By Proposition 2.5.2 the generator matrix of the toric degeneration $\mathcal{X}_{1,0}$ of X is given by

$$P_{1,0} \;=\; [u_1, u_2, u_3] \;=\; \begin{bmatrix} d_2 & d_0 l_2 + d_2 & d_1 \\ l_2 & l_2 & -l_1 \end{bmatrix}.$$

According to Proposition 1.3.12 the generators of the cone τ_1' are given as $\iota(u_1), \iota(u_2), \iota(u_3)$, where

$$\iota\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad (x,y) \mapsto (x,1,y).$$

This yields the desired claim. $\square$

Lemma 2.7.7. *Consider the cone* τ_1' *as in Construction 2.7.4. We set*

$$\begin{aligned} \mu_1 &:= \gcd\left(l_1 + l_2, -d_0 l_1 l_2 - d_1 l_2 - d_2 l_1, -d_0 l_2 + d_1 - d_2\right), \\ \mu_3 &:= \gcd\left(l_1 + l_2, d_1 l_2 + d_2 l_1, d_1 - d_2\right). \end{aligned}$$

Then the dual cone $\omega_1' := \tau_1'^{\vee}$ *has the primitive generators*

$$\begin{aligned} u_1 &:= (0, l_2, -1), \\ u_2 &:= \tfrac{1}{\mu_1}\left(l_1 + l_2, -d_0 l_1 l_2 - d_1 l_2 - d_2 l_1, -d_0 l_2 + d_1 - d_2\right), \\ u_3 &:= \tfrac{1}{\mu_3}\left(-l_1 - l_2, d_1 l_2 + d_2 l_1, -d_1 + d_2\right). \end{aligned}$$

Proof. We denote by P_1 the matrix which columns are the primitive generators of τ_1'. Due to Lemma 2.7.6 we have

$$P_1 \;=\; \begin{bmatrix} d_2 & d_0 l_2 + d_2 & d_1 \\ 1 & 1 & 1 \\ l_2 & l_2 & -l_1 \end{bmatrix}.$$

In order to see that the u_i are ray generators of ω_1' it suffices to show that each u_i cuts out a facet of τ_1' and evaluates positively on the complementary ray. We obtain

$$\begin{aligned} u_1 \cdot P_1 &= (0, 0, l_1 + l_2)^{\top}, \\ u_2 \cdot P_1 &= \tfrac{1}{\mu_1}\left(0, -d_0 l_2 (l_1 + l_2), 0\right)^{\top}, \\ u_3 \cdot P_1 &= \tfrac{1}{\mu_3}\left(-d_0 l_2 (l_1 + l_2), 0, 0\right)^{\top}. \end{aligned}$$

According to Setting 2.7.2 we have $d_0 < 0$ and $l_1, l_2 > 0$. Hence, the non-zero entries on the right hand side are positive. $\square$

Lemma 2.7.8. *Let* $\omega \subseteq \mathbb{R}^3$ *be a three-dimensional cone with primitive generators* u_1, u_2, u_3 *and let* $\xi \in (\omega^{\vee})^{\circ}$. *Then we obtain*

$$\{v \in \omega;\; v(\xi) \le 1\} \;=\; \operatorname{conv}\left((0,0,0), \tfrac{1}{\langle \xi, u_1\rangle} u_1, \tfrac{1}{\langle \xi, u_2\rangle} u_2, \tfrac{1}{\langle \xi, u_3\rangle} u_3\right).$$

Moreover, the volume of this convex set is given by

$$\operatorname{vol}\left(\{v \in \omega;\; v(\xi) \le 1\}\right) \;=\; \tfrac{1}{2}\left|\det\left(\tfrac{1}{\langle \xi, u_1\rangle} u_1, \tfrac{1}{\langle \xi, u_2\rangle} u_2, \tfrac{1}{\langle \xi, u_3\rangle} u_3\right)\right|.$$

Proof. The first statement is clear. For the second claim, we use that the cone is full dimensional in $\mathbb{R}^3$ and has three generators. Consequently, the volume of the parallel epiped spanned by u_1, u_2, u_3 is equal to the absolute value of the determinant of the matrix $[u_1, u_2, u_3]$. This yields the claim. ☐

Lemma 2.7.9. *Let $p_1, \ldots, p_k \in \mathbb{R}^n$ and $C \coloneqq \operatorname{conv}\{p_1, \ldots, p_k\}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be an affine-linear function. Then we have $g \geq 0$ on C if and only if $g(p_i) \geq 0$ for $i = 1, \ldots, k$.*

Proof. Since $g\colon \mathbb{R}^n \to \mathbb{R}$ is affine-linear there exists a vector $a \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that $g(x) = a \cdot x + b$. We obtain

$$g\left(\sum_{i=1}^{k} \lambda_i p_i\right) = a \cdot \left(\sum_{i=1}^{k} \lambda_i p_i\right) + b = \sum_{i=1}^{k} \lambda_i a \cdot p_i + \sum_{i=1}^{k} \lambda_i b = \sum_{i=1}^{k} \lambda_i (a \cdot p_i + b) = \sum_{i=1}^{k} \lambda_i g(p_i),$$

where we used $\sum_{i=1}^{k} \lambda_i = 1$ for the second equality. Then, the assertion follows from $\lambda_i \geq 0$. ☐

Proposition 2.7.10. *Let τ_1' and vol_1 be as in Construction 2.7.4 and let $(x, 1, y) \in (\tau_1')^{\circ}$. We set*

$$\begin{aligned} f_{1,1}(y) &\coloneqq l_2 - y, \\ f_{2,1}(x,y) &\coloneqq (l_1 + l_2)x + (-d_0 l_2 + d_1 - d_2)y - d_0 l_1 l_2 - d_1 l_2 - d_2 l_1, \\ f_{3,1}(x,y) &\coloneqq (l_1 + l_2)x + (d_1 - d_2)y - d_1 l_2 - d_2 l_1. \end{aligned}$$

Then the volume function is given by

$$\operatorname{vol}_1(x,y) = \tfrac{d_0 l_2 (l_1 + l_2)^2}{2 f_{1,1}(y) f_{2,1}(x,y) f_{3,1}(x,y)}$$

and its derivatives are given by

$$\begin{aligned} {}^{\partial \operatorname{vol}_1}/_{\partial x}\,(x,1,y) &= \tfrac{-d_0 l_2 (l_1+l_2)^3}{2 f_{1,1}(y) f_{2,1}(x,y)^2 f_{3,1}(x,y)^2} \cdot (f_{2,1}(x,y) + f_{3,1}(x,y)), \\ {}^{\partial \operatorname{vol}_1}/_{\partial y}\,(x,1,y) &= \tfrac{-d_0 l_2 (l_1+l_2)^2}{2 f_{1,1}(y)^2 f_{2,1}(x,y)^2 f_{3,1}(x,y)^2} \cdot (-f_{2,1}(x,y) f_{3,1}(x,y) \\ &\quad + f_{1,1}(y) f_{3,1}(x,y)(-d_0 l_2 + d_1 - d_2) + f_{1,1}(y) f_{2,1}(x,y)(d_1 - d_2)). \end{aligned}$$

Proof. We denote by P_1 the matrix which columns are the primitive generators of ω_1'. We infer from Lemma 2.7.7 that

$$P_1 = [u_1, u_2, u_3] = \begin{bmatrix} 0 & \frac{1}{\mu_1}(l_1 + l_2) & \frac{1}{\mu_3}(-l_1 - l_2) \\ l_2 & \frac{1}{\mu_1}(-d_0 l_1 l_2 - d_1 l_2 - d_2 l_1) & \frac{1}{\mu_3}(d_1 l_2 + d_2 l_1) \\ -1 & \frac{1}{\mu_1}(-d_0 l_2 + d_1 - d_2) & \frac{1}{\mu_3}(-d_1 + d_2) \end{bmatrix}.$$

We derive $d_0 < 0$ and $l_1, l_2 > 0$ from Setting 2.7.2. Hence, we obtain

$$\det(P_1) = \tfrac{1}{\mu_1 \mu_3} d_0 l_2 (l_1 + l_2)^2 < 0,$$

Moreover, we compute

$$\begin{aligned}
\langle u_1,(x,1,y)\rangle &= l_2-y \;=\; f_{1,1}(y),\\
\langle u_2,(x,1,y)\rangle &= \tfrac{1}{\mu_1}\left((l_1+l_2)x+(-d_0l_2+d_1-d_2)y-d_0l_1l_2-d_1l_2-d_2l_1\right)\\
&= \tfrac{1}{\mu_1}f_{2,1}(x,y),\\
\langle u_3,(x,1,y)\rangle &= -\tfrac{1}{\mu_3}\left((l_1+l_2)x+(d_1-d_2)y-d_1l_2-d_2l_1\right) \;=\; -\tfrac{1}{\mu_3}f_{3,1}(x,y).
\end{aligned}$$

We set $E := \mathbb{R}\times\{1\}\times\mathbb{R}$ and let $\mathrm{pr}\colon E\to\mathbb{R}^2, (x,1,y)\mapsto(x,y)$ be the projection. Then $\mathrm{pr}(\tau_1'\cap E)$ is a convex set. Using $l_1,l_2>0$ we see

$$f_{1,1}(d_2,l_2)=0,\quad f_{1,1}(d_0l_2+d_2,l_2)=0,\quad f_{1,1}(d_1,-l_1)=l_1+l_2>0.$$

Hence, Lemma 2.7.9 yields that $f_{1,1}$ is a positive function on $\mathrm{pr}\,(\tau_1'\cap E)$. Again by $l_1,l_2>0$, we get

$$\begin{aligned}
f_{2,1}(d_2,l_2) &= (l_1+l_2)d_2+(-d_0l_2+d_1-d_2)l_2-d_0l_1l_2-d_1l_2-d_2l_1\\
&= -d_0l_2(l_1+l_2)>0,\\
f_{2,1}(d_0l_2+d_2,l_2) &= (l_1+l_2)(d_0l_2+d_2)+(-d_0l_2+d_1-d_2)l_2-d_0l_1l_2-d_1l_2-d_2l_1\\
&= -d_0l_2(l_1+l_2)+(l_1+l_2)d_0l_2=0,\\
f_{2,1}(d_1,-l_1) &= (l_1+l_2)d_1-(-d_0l_2+d_1-d_2)l_1-d_0l_1l_2-d_1l_2-d_2l_1=0.
\end{aligned}$$

Now Lemma 2.7.9 implies that $f_{2,1}$ is a positive function on $\mathrm{pr}\,(\tau_1'\cap E)$. Moreover, by using $d_0<0$ and $l_1,l_2>0$, we have

$$\begin{aligned}
f_{3,1}(d_2,l_2) &= (l_1+l_2)d_2+(d_1-d_2)l_2-d_1l_2-d_2l_1=0,\\
f_{3,1}(d_0l_2+d_2,l_2) &= (l_1+l_2)(d_0l_2+d_2)+(d_1-d_2)l_2-d_1l_2-d_2l_1=d_0l_2(l_1+l_2)<0,\\
f_{3,1}(d_1,-l_1) &= (l_1+l_2)d_1-(d_1-d_2)l_1-d_1l_2-d_2l_1=0.
\end{aligned}$$

Therefore, we apply Lemma 2.7.9 to $-f_{3,1}$. Consequently, we obtain that $f_{3,1}$ is a negative function on $\mathrm{pr}\,(\tau_1'\cap E)$. Hence, using Lemma 2.7.8, the volume function vol_1 is given by

$$\mathrm{vol}_1(x,y) \;=\; \tfrac{1}{2}\cdot\tfrac{1}{|\langle(x,1,y),u_1\rangle|}\cdot\tfrac{1}{|\langle(x,1,y),u_2\rangle|}\cdot\tfrac{1}{|\langle(x,1,y),u_3\rangle|}\cdot|\det P_1| \;=\; \tfrac{-d_0l_2(l_1+l_2)^2}{2f_{1,1}(y)f_{2,1}(x,y)(-f_{3,1}(x,y))},$$

since $d_0<0$. Lastly, the formulas of the derivatives follow from direct calculations. ☐

Lemma 2.7.11. *Consider the cone τ_2' as in Setting 2.7.2. Then τ_2' has the primitive generators*

$$v_1 := (d_1,1,l_1)\,,\quad v_2 := (d_0l_1+d_1,1,l_1)\,,\quad v_3 := (d_2,1,-l_2)\,.$$

Proof. From Proposition 2.5.2 we obtain that the generator matrix of the toric degeneration $\mathcal{X}_{2,0}$ of X is given by

$$P_{2,0} \;=\; [u_1,u_2,u_3] \;=\; \begin{bmatrix} d_1 & d_0l_1+d_1 & d_2\\ l_1 & l_1 & -l_2\end{bmatrix}$$

Proposition 1.3.12 tells us that the generators of the cone τ_2' are given by $\iota(u_1), \iota(u_2), \iota(u_3)$, where

$$\iota\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad (x,y) \mapsto (x,1,y)$$

and the claim is proven. ☐

Lemma 2.7.12. *Consider the cone τ_2' as in Setting 2.7.2. We set*

$$\begin{aligned} \mu_2 &:= \gcd\left(l_1+l_2, -d_0l_1l_2 - d_1l_2 - d_2l_1, -d_0l_1 + d_2 - d_1\right), \\ \mu_3 &:= \gcd\left(l_1+l_2, d_1l_2 + d_2l_1, d_1 - d_2\right). \end{aligned}$$

Then the dual cone $\omega_2' := \tau_2'^{\vee}$ has the primitive generators

$$\begin{aligned} u_1 &:= (0, l_1, -1), \\ u_2 &:= \tfrac{1}{\mu_2}\left(l_1+l_2, -d_0l_1l_2 - d_1l_2 - d_2l_1, -d_0l_1 + d_2 - d_1\right), \\ u_3 &:= \tfrac{1}{\mu_3}\left(-l_1-l_2, d_1l_2 + d_2l_1, d_1 - d_2\right). \end{aligned}$$

Proof. We denote by P_2 the matrix which columns are the primitive generators of τ_2'. According Lemma 2.7.6 we obtain

$$P_2 := \begin{bmatrix} d_1 & d_0l_1 + d_1 & d_2 \\ 1 & 1 & 1 \\ l_1 & l_1 & -l_2 \end{bmatrix}.$$

To see that the u_i are ray generators of ω_2' we have to check that that each of them cuts out a facet of τ_2' and evaluates positively on the remaining ray. We obtain

$$\begin{aligned} u_1 \cdot P_2 &= (0,0,l_1+l_2)^{\top}, \\ u_2 \cdot P_2 &= \tfrac{1}{\mu_2}\left(0, -d_0l_1(l_1+l_2), 0\right)^{\top}, \\ u_3 \cdot P_2 &= \tfrac{1}{\mu_3}\left(-d_0l_1(l_1+l_2), 0, 0\right)^{\top}. \end{aligned}$$

Setting 2.7.2 yields $d_0 < 0$ and $l_1, l_2 > 0$. Thus, all non-zero entries on the right hand side are positive. ☐

Proposition 2.7.13. *Let τ_2' and vol_2 be as in Construction 2.7.4 and let $(x,1,y) \in (\tau_2')^{\circ}$. We set*

$$\begin{aligned} f_{1,2}(y) &:= l_1 - y, \\ f_{2,2}(x,y) &:= (l_1+l_2)x + (d_2 - d_1 - d_0l_1)y - d_0l_1l_2 - d_2l_1 - d_1l_2, \\ f_{3,2}(x,y) &:= (l_1+l_2)x + (d_2 - d_1)y - d_1l_2 - d_2l_1. \end{aligned}$$

Then, the volume function is given by

$$\mathrm{vol}_2(x,y) = \tfrac{-d_0l_1(l_1+l_2)^2}{2f_{1,2}(y)f_{2,2}(x,y)f_{3,2}(x,y)}$$

and its derivatives are given by

$$\begin{aligned}
{}^{\partial \mathrm{vol}_2}/_{\partial x}\,(x,1,y) &= \tfrac{d_0 l_1 (l_1+l_2)^3}{2 f_{1,2}(y) f_{2,2}(x,y)^2 f_{3,2}(x,y)^2} \cdot (f_{2,2}(x,y) + f_{3,2}(x,y)),\\
{}^{\partial \mathrm{vol}_2}/_{\partial y}\,(x,1,y) &= \tfrac{d_0 l_1 (l_1+l_2)^2}{2 f_{1,2}(y)^2 f_{2,2}(x,y)^2 f_{3,2}(x,y)^2} \cdot \big(-f_{2,2}(x,y) f_{3,2}(x,y)\\
&\quad + f_{1,2}(y) f_{3,2}(x,y)(d_2 - d_1 - d_0 l_1) + f_{1,2}(y) f_{2,2}(x,y)(d_2 - d_1)\big).
\end{aligned}$$

Proof. Let P_2 be the matrix whose columns are the primitive generators of ω_2'. The generators of ω_2' are determined in Lemma 2.7.11 and we obtain

$$P_2 \;=\; [u_1, u_2, u_3] \;=\; \begin{bmatrix} 0 & \frac{1}{\mu_2}(l_1 + l_2) & \frac{1}{\mu_3}(-l_1 - l_2) \\ l_1 & \frac{1}{\mu_2}(-d_0 l_1 l_2 - d_1 l_2 - d_2 l_1) & \frac{1}{\mu_3}(d_1 l_2 + d_2 l_1) \\ -1 & \frac{1}{\mu_2}(-d_0 l_1 + d_2 - d_1) & \frac{1}{\mu_3}(d_1 - d_2) \end{bmatrix}.$$

Due to Setting 2.7.2, we get $d_0 < 0$ and $l_1, l_2 > 0$. Therefore, its determinant is

$$\det(P_2) \;=\; \tfrac{1}{\mu_2 \mu_3} d_0 l_1 (l_1 + l_2)^2 < 0,$$

Further, we have

$$\begin{aligned}
\langle u_1, (x,1,y)\rangle &= l_1 - y = f_{1,2}(y),\\
\langle u_2, (x,1,y)\rangle &= \tfrac{1}{\mu_2}\left((l_1+l_2)x + (d_2 - d_1 - d_0 l_1)y - d_0 l_1 l_2 - d_2 l_1 - d_1 l_2\right) = \tfrac{1}{\mu_2} f_{2,2}(x,y),\\
\langle u_3, (x,1,y)\rangle &= -\tfrac{1}{\mu_3}\left((l_1+l_2)x + (d_2 - d_1)y - d_1 l_2 - d_2 l_1\right) = -\tfrac{1}{\mu_3} f_{3,2}(x,y).
\end{aligned}$$

Set $E \coloneqq \mathbb{R} \times \{1\} \times \mathbb{R}$. Applying $\mathrm{pr}\colon E \to \mathbb{R}^2, (x,1,y) \mapsto (x,y)$ we see that $\mathrm{pr}(\tau_2' \cap E)$ is a convex set in $\mathbb{R}^2$. In order to see that the functions $f_{i,2}$ do not change signs on τ_2' we treat them separately. Using $l_1, l_2 > 0$ we see

$$f_{1,2}(d_1, l_1) = 0, \quad f_{1,2}(d_0 l_1 + d_1, l_1) = 0, \quad f_{1,2}(d_2, -l_2) = l_1 + l_2 > 0.$$

By Lemma 2.7.9 we obtain that $f_{1,2}$ is a positive function on $\mathrm{pr}(\tau_2' \cap E)$. Moreover, since $l_1, l_2 > 0$ and $d_0 < 0$ we obtain

$$\begin{aligned}
f_{2,2}(d_1, l_1) &= 0,\\
f_{2,2}(d_0 l_1 + d_1, l_1) &= (l_1 + l_2) d_0 l_1 < 0,\\
f_{2,2}(d_2, -l_2) &= 0.
\end{aligned}$$

Applying Lemma 2.7.9 to $-f_{2,2}$, we conclude that $f_{2,2}$ is a negative function on $\mathrm{pr}(\tau_2' \cap E)$. Because of $l_1, l_2 > 0$ and $d_0 < 0$ we obtain

$$\begin{aligned}
f_{3,2}(d_1, l_1) &= (l_1 + l_2) d_1 + (d_2 - d_1) l_1 - d_1 l_2 - d_2 l_1 = 0,\\
f_{3,2}(d_0 l_1 + d_1, l_1) &= (l_1 + l_2) d_0 l_1 < 0,\\
f_{3,2}(d_2, -l_2) &= (l_1 + l_2) d_2 - (d_2 - d_1) l_2 - d_1 l_2 - d_2 l_1 = 0.
\end{aligned}$$

Thus, Lemma 2.7.9 yields that $f_{3,2}$ is a negative function on $\mathrm{pr}(\tau_2' \cap E)$. Hence, according to Lemma 2.7.8, the volume vol_2 is given as

$$\mathrm{vol}_2(x,y) = \frac{1}{2} \cdot \tfrac{1}{|\langle (x,1,y),u_1\rangle|} \cdot \tfrac{1}{|\langle (x,1,y),u_2\rangle|} \cdot \tfrac{1}{|\langle (x,1,y),u_3\rangle|} \cdot |\det P_2| = \tfrac{-d_0 l_2 (l_1+l_2)^2}{2 f_{1,2}(y) f_{2,2}(x,y) f_{3,2}(y)}.$$

Finally, the formulas of the derivatives follow from direct calculations. □

Proof of Theorem 2.7.1. We consider the point

$$x_0 := \tfrac{d_0 l_1 l_2 + 2 d_1 l_2 + 2 d_2 l_1}{2(l_1+l_2)}$$

and obtain

$$\begin{aligned}
f_{1,1}(x_0,0) &= l_2,\\
f_{2,1}(x_0,0) &= (\tfrac{1}{2} d_0 l_1 l_2 + d_1 l_2 + d_2 l_1) - d_0 l_1 l_2 - d_1 l_2 - d_2 l_1 = -\tfrac{1}{2} d_0 l_1 l_2,\\
f_{3,1}(x_0,0) &= (\tfrac{1}{2} d_0 l_1 l_2 + d_1 l_2 + d_2 l_1) - d_1 l_2 - d_2 l_1 = \tfrac{1}{2} d_0 l_1 l_2\\
f_{1,2}(x_0,0) &= l_1,\\
f_{2,2}(x_0,0) &= (\tfrac{1}{2} d_0 l_1 l_2 + d_1 l_2 + d_2 l_1) - d_0 l_1 l_2 - d_2 l_1 - d_1 l_2 = -\tfrac{1}{2} d_0 l_1 l_2,\\
f_{3,2}(x_0,0) &= (\tfrac{1}{2} d_0 l_1 l_2 + d_1 l_2 + d_2 l_1) - d_1 l_2 - d_2 l_1 = \tfrac{1}{2} d_0 l_1 l_2.
\end{aligned}$$

It follows from Proposition 2.7.10 and Proposition 2.7.13 that

$$\partial \mathrm{vol}_1/\partial x\,(x_0,1,0) \;=\; 0 \;=\; \partial \mathrm{vol}_2/\partial x\,(x_0,1,0).$$

Further, for the other derivatives $\partial \mathrm{vol}_\kappa/\partial y(x,1,0)$ we obtain

$$\begin{aligned}
\partial \mathrm{vol}_1/\partial y\,(x_0,1,0) &= \tfrac{2(l_1+l_2)^2(2l_2-l_1)}{d_0 l_1^3 l_2^3},\\
\partial \mathrm{vol}_2/\partial y\,(x_0,1,0) &= \tfrac{2(l_1+l_2)^2(2l_1-l_2)}{d_0 l_1^3 l_2^3}.
\end{aligned}$$

Using $1 \le l_1 \le l_2$ and $d_0 < 0$ we obtain $\partial \mathrm{vol}_1/\partial y\,(x_0,1,0) < 0$, and for $\kappa = 2$, that $\partial \mathrm{vol}_2/\partial y\,(x_0,1,0) < 0$ if and only if $2l_1 - l_2 > 0$. Altogether, Theorem 2.7.5 (iii) provides us with the assertion. □

CHAPTER

THREE

FULL INTRINSIC QUADRIC SURFACES

We study in detail full intrinsic quadric surfaces and their geometry in terms of local Gorenstein indices. For instance, we present upper and lower bounds for the degree, the log canonicity and the Picard index. Moreover, we determine all full intrinsic quadric surfaces admitting a Kähler-Einstein metric. The results of Section 3.1 to 3.5 have been published in the joint article [25] with Jürgen Hausen.

3.1 Full intrinsic quadric surfaces admit a $\mathbb{K}^*$-action

In this section we show that every full intrinsic quadric surface is rational, $\mathbb{Q}$-factorial, projective and admits a non-trivial $\mathbb{K}^*$-action. This will allow us to work with the approach to $\mathbb{K}^*$-surfaces provided by [20, 28]; see also [3, Sec. 5.4]. First let us give a precise definition of a full intrinsic quadric; see also [4, Sec. 9].

Definition 3.1.1. A *full intrinsic quadric* is a normal complete variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and Cox ring of the form

$$\mathcal{R}(X) \;=\; \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)) \;=\; \mathbb{K}[T_1, \ldots, T_n]/\langle g \rangle$$

with $\mathrm{Cl}(X)$-homogeneous generators $T_1, \ldots, T_n \in \mathcal{R}(X)$ and a $\mathrm{Cl}(X)$-homogeneous quadric $g \in \mathbb{K}[T_1, \ldots, T_n]$ of full rank.

Remark 3.1.2. In the setting of Definition 3.1.1, the divisor class group is generated by any $n-1$ of the degrees $w_i = \deg(T_i) \in \mathrm{Cl}(X)$, see [3, Def. 3.2.1.1 and Cor. 3.2.1.11]. Moreover, if X is $\mathbb{Q}$-factorial, then the Picard number $\rho(X)$ equals the dimension of the rational vector space $\mathrm{Cl}_{\mathbb{Q}}(X) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Cl}(X)$ and as well the dimension of the convex cone $\mathrm{Mov}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ generated by the movable divisor classes, see for instance [3, Lemma 4.3.3.2].

Theorem 3.1.3. *Let X be a full intrinsic quadric surface. Then X is $\mathbb{Q}$-factorial, rational, projective and admits an effective $\mathbb{K}^*$-action. Moreover, the Picard number of X satisfies $\rho(X) \le 3$ and its Cox ring allows a $\mathrm{Cl}(X)$-graded presentation as*

$$\mathcal{R}(X) \;\cong\; \begin{cases} \mathbb{K}[T_1,\dots,T_4]/\langle T_1T_2+T_3^2+T_4^2\rangle, & \rho(X)=1,\\ \mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^2\rangle, & \rho(X)=2,\\ \mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, & \rho(X)=3. \end{cases}$$

Proof. By definition, X is a normal complete surface with finitely generated Cox ring. From [3, Thm. 4.3.3.5] we infer that X is $\mathbb{Q}$-factorial and projective. Moreover, by [12, Prop. 2.1], we have a $\mathrm{Cl}(X)$-graded presentation

$$\mathcal{R}(X) \cong \mathbb{K}[T_1,\dots,T_{n+m}]/\langle g\rangle, \quad g = T_1T_2+\ldots+T_{n-1}T_n+T_{n+1}^2+\ldots+T_{n+m}^2.$$

We use this to show $\rho(X)\le 3$. Recall from [3, Cor. 1.6.2.7 and Constr. 1.6.3.1] that X is the geometric quotient of an open subset of the total coordinate space

$$\bar{X} \;=\; \mathrm{Spec}\,\mathcal{R}(X) \;=\; V(g) \;\subseteq\; \mathbb{K}^{n+m}$$

by the quasitorus $H = \mathrm{Spec}\,\mathbb{K}[\mathrm{Cl}(X)]$, which, due to $\mathbb{Q}$-factoriality of X, is of dimension $\rho(X)$. Consequently, we have

$$2 \;=\; \dim(X) \;=\; \dim(\bar{X})-\dim(H) \;=\; n+m-1-\rho(X).$$

The degrees $w_1,\dots,w_{n+m}$ of $T_1,\dots,T_{n+m}$ generate $\mathrm{Cl}(X)$. Moreover, the degree $\mu = \deg(g)\in \mathrm{Cl}(X)$ satisfies $\mu = w_i+w_{i+1}$ for $i=1,3,\dots,n-1$ and $\mu = 2w_{n+j}$ for $j=1,\dots,m$. Thus, $\mathrm{Cl}(X)$ is generated by $w_1,w_2,w_3,w_5,\dots,w_{n-1}$ and we see

$$n+m-3 \;=\; \rho(X) \;\le\; 2+\frac{n-2}{2}.$$

We conclude $n/2+m\le 4$ and thus $n\le 8$. Assume $n=8$. Then $m=0$ and $\rho(X)=5$ hold. Consequently, (w_1,w_2,w_3,w_5,w_7) is a basis for the rational vector space $\mathrm{Cl}_{\mathbb{Q}}(X)$. Let u be a linear form on $\mathrm{Cl}_{\mathbb{Q}}(X)$ such that

$$\langle u,w_1\rangle = \langle u,w_2\rangle = \langle u,w_3\rangle = \langle u,w_5\rangle = 0, \qquad \langle u,w_7\rangle < 0.$$

Then u annihilates as well $\mu = w_1+w_2$ and thus also $w_4=\mu-w_3$ and $w_6=\mu-w_5$. Moreover, u evaluates positively on $w_8=\mu-w_7$. Consequently, computing the cone of movable divisor classes according to [3, Prop. 3.3.2.3], we obtain

$$\mathrm{Mov}(X) \;=\; \bigcap_{i=1}^{8} \mathrm{cone}(w_j;\ j\ne i) \;\subseteq\; \ker(u) \;\subseteq\; \mathrm{Cl}_{\mathbb{Q}}(X).$$

This contradicts to Remark 3.1.2, telling us that $\mathrm{Mov}(X)$ is a cone of full dimension in $\mathrm{Cl}_{\mathbb{Q}}(X)$. We conclude, $n\le 6$.

Next, we treat the case $n = 6$. Because of $n/2 + m \leq 4$, we conclude $m = 0, 1$. First, we exclude $n = 6, m = 1$. There, we have $\varrho(X) = 4$ and (w_1, w_2, w_3, w_5) is a vector space basis for $\mathrm{Cl}_{\mathbb{Q}}(X)$. As above, we choose a linear form u on $\mathrm{Cl}_{\mathbb{Q}}(X)$ such that

$$\langle u, w_1\rangle = \langle u, w_2\rangle = \langle u, w_3\rangle = 0, \qquad \langle u, w_5\rangle < 0.$$

Since $\mu = w_1 + w_2$ lies inside of $\ker(u)$, we obtain $w_4 = \mu - w_3 \in \ker(u)$. Because of $\mu = 2w_7$, we conclude $w_7 \in \ker(u)$. Hence, by using [3, Prop. 3.3.2.3], we get

$$\mathrm{Mov}(X) \ = \ \bigcap_{i=1}^{7} \mathrm{cone}(w_j;\ j \neq i) \ \subseteq \ \ker(u) \ \subseteq \ \mathrm{Cl}_{\mathbb{Q}}(X).$$

Thus, the dimension of $\mathrm{Mov}(X)$ is at most $\dim(\ker(u)) = 3$, which is a contradiction to the fact that $\mathrm{Mov}(X)$ is of full dimension in $\mathrm{Cl}_{\mathbb{Q}}(X)$.

Thus, we have $\rho(X) \leq 3$. If $\rho(X) = 3$, then $n = 6$ and $m = 0$, which leads to third case in the assertion. For $\rho(X) = 2$, we are left with the choices $n = 4$ with $m = 1$ and $n = 0, 2$. The first one gives the second case of the assertion.

Next, we exclude the case $\varrho(X) = 2, n = 2$. Here we get $m = 3$ and (w_1, w_2) is a vector space basis for $\mathrm{Cl}_{\mathbb{Q}}(X)$. The relations

$$\mu = 2w_3 = 2w_4 = 2w_5$$

yield $\dim(\mathrm{lin}(w_3, w_4, w_5)) = 1$. Moreover, we derive $w_1, w_2 \notin \mathrm{lin}(w_3, w_4, w_5)$ from $w_1 + \ w_2 = 2w_i$ for all $i = 3, 4, 5$, and the fact that (w_1, w_2) is a basis. So we may choose a linear form u on $\mathrm{Cl}_{\mathbb{Q}}(X)$ such that

$$\langle u, w_3\rangle = \langle u, w_4\rangle = \langle u, w_5\rangle = 0, \qquad \langle u, w_1\rangle < 0.$$

Using [3, Prop. 3.3.2.3] to compute the moving cone, we get

$$\mathrm{Mov}(X) \ = \ \bigcap_{i=1}^{5} \mathrm{cone}(w_j;\ j \neq i) \ \subseteq \ \ker(u) \ \subseteq \ \mathrm{Cl}_{\mathbb{Q}}(X).$$

This contradicts $\dim(\ker(u)) = 1$ and $\dim(\mathrm{Mov}(X)) = 2$.

Now, we show that the case $\varrho(X) = 2, n = 0$ can't occur. Here we get $m = 5$ and $\mu = 2w_i$ for all $i = 1, \ldots, 5$. Consequently, one obtains

$$\dim(\mathrm{lin}(w_1, \ldots, w_5)) = 1.$$

This yields $\dim(\mathrm{Mov}(X)) \leq 1$, which is a contradiction to $\mathrm{Mov}(X)$ being of dimension $\varrho(X) = 2$.

For $\rho(X) = 1$, we find the possibility $n = m = 2$, which is the first case of the assertion. Also, $n = 0, 4$ might happen. We first exclude the case $n = 4$. There, the prospective total coordinate space $\bar{X} = \mathrm{Spec}\,\mathbb{K}[\mathcal{R}(X)]$ is explicitly given as

$$\bar{X} \ = \ V(T_1T_2 + T_3T_4) \ \subseteq \ \mathbb{K}^4.$$

In this setting, we find a diagonal action of a three-dimensional torus $\mathbb{T}$ on $\mathbb{K}^4$ turning $\bar{X}$ into a toric variety. Thus, X as a GIT-quotient of $\bar{X}$ by a one-dimensional subgroup of $\mathbb{T}$ is as well a toric variety and must have a polynomial ring as its Cox ring; a contradiction to $\bar{X}$ being singular.

Finally, we treat the case $\rho(X) = 1$ and $n = 0$. If any two of the degrees $w_i = \deg(T_i)$ coincide, say $w_1 = w_2$ then we may substitute $T_1' = T_1 + IT_2$ and $T_2' = T_1 - IT_2$ with $I = \sqrt{-1}$, which brings us into the setting $n > 0$ just discussed. Thus, we are left with discussing the situation

$$\mathcal{R}(X) \ = \ \mathbb{K}[T_1, T_2, T_3, T_4]/\langle T_1^2 + T_2^2 + T_3^2 + T_4^2 \rangle, \qquad w_i \neq w_j \text{ for } i \neq j.$$

Due to $\mathbb{Q}$-factoriality of X, the divisor class group $\mathrm{Cl}(X)$ is of rank one and hence of the form $\mathbb{Z} \oplus \Gamma$ with a finite abelian group Γ. We claim that up to renumbering the variables and an automorphism of $\mathrm{Cl}(X)$, we have

$$\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \qquad Q = [w_1, \ldots, w_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}.$$

Since $w_1, \ldots, w_4$ are non-torsion elements generating a pointed cone in $\mathrm{Cl}_{\mathbb{Q}}(X) = \mathbb{Q}$, we may assume $w_i = (s_i, \eta_i) \in \mathbb{Z} \oplus \Gamma$ with $s_i > 0$. By $\mathrm{Cl}(X)$-homogeneity of the relation, all s_i coincide. Hence, as the s_i generate $\mathbb{Z}$, they are all equal to one. Write Γ as a direct product of finite cyclic groups. Then, subtracting suitable multiples of the first row of Q from the last ones, we can achieve

$$Q = [w_1, \ldots, w_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \eta_2 & \eta_3 & \eta_4 \end{bmatrix}, \qquad \eta_2, \eta_3, \eta_4 \in \Gamma.$$

Note that this adjusting process is realized by an automorphism of $\mathrm{Cl}(X)$. By Remark 3.1.2, any two of η_2, η_3, η_4 generate Γ as a group. Thus, Γ is in fact either cyclic or a sum of two cyclic groups. Moreover, we have $2\eta_i = 0$ for all $i = 2, 3, 4$ by homogeneity of the relation. Hence, any element of Γ is of order two and we are left with the cases

$$\Gamma \ = \ \mathbb{Z}/2\mathbb{Z}, \qquad\qquad \Gamma \ = \ \mathbb{Z}/2 \oplus \mathbb{Z}/2\mathbb{Z}.$$

The first case can't occur as it will not allow a choice of pairwise different $w_1, \ldots, w_4$. Thus, suitable renumbering of the variables and applying a suitable automorphism of $\mathbb{Z} \oplus \Gamma$ to the w_i leads to $\mathrm{Cl}(X)$ and Q as claimed.

The task is to show that the above $\mathrm{Cl}(X)$-graded algebra $\mathcal{R}(X)$ can't be a Cox ring. Assume that $\mathcal{R}(X)$ is a Cox ring. Then, since the variables $T_1, \ldots, T_4$ define pairwisse non-associated primes in $\mathcal{R}(X)$, we are in the setting of [24, Constr. 3.2.1.3] and can apply the theory developed thereafter. In particular, as $\bar{X}$ is smooth apart from the origin, X would be quasismooth [12, Prop. 2.8], hence log terminal. Moreover, we can apply [3, Cor. 3.3.3.3] to see that X is a del Pezzo surface of Picard number one and

Gorenstein index one; we used the software package [23] for the computation. The Cox rings of all log del Pezzo surfaces of Picard number one and Gorenstein index one without torus action have been computed in [24, Thm. 4.1] and for those with torus action the Cox rings are listed in [3, 5.4.4.2]; none of these Cox rings is isomorphic to $\mathcal{R}(X)$ from above.

We verified, that the Cox ring of any full intrinsic quadric surface X is as in the assertion, in particular it is defined by trinomial relations. Consequently, the associated total coordinate space $\bar{X}$ allows a diagonal torus action of complexity one. This action induces a non-trivial $\mathbb{K}^*$-action on X. Since $\mathrm{Cl}(X)$ is finitely generated by assumption, this forces X to be rational. □

3.2 Picard number one

The main result of this section, Theorem 3.2.5, provides the description of all full intrinsic quadric surfaces of Picard number one in terms of the local Gorenstein indices of two of their possibly singular points.

Construction 3.2.1 (Full intrinsic quadric surfaces X of Picard number one as $\mathbb{K}^*$-surfaces)**.** Consider an integral matrix of the form

$$P \;:=\; [v_1, v_2, v_3, v_4] \;:=\; \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}, \qquad b \le -2,\; 0 \le a \le -b-2.$$

Let Z be the toric variety arising from the fan Σ in $\mathbb{Z}^3$ with *generator matrix P*, i.e. $v_1, \ldots, v_4$ are the primitive ray generators of Σ, and the maximal cones

$$\sigma^+ := \mathrm{cone}(v_1, v_3, v_4), \qquad \sigma^- := \mathrm{cone}(v_2, v_3, v_4), \qquad \tau_0 := \mathrm{cone}(v_1, v_2).$$

Denote by U_1, U_2, U_3 the coordinate functions on the standard 3-torus $\mathbb{T}^3 \subseteq Z$. Then we obtain a normal, non-toric, rational, projective surface

$$X \;:=\; X(P) \;:=\; \overline{V(h)} \;\subseteq\; Z, \qquad h \;:=\; 1 + U_1 + U_2 \;\in \mathcal{O}(\mathbb{T}^3).$$

Moreover, the $\mathbb{K}^*$-action on $\mathbb{T}^3$ given by $t \cdot x = (x_1, x_2, tx_3)$ extends to an action on Z, it leaves $V(h) \subseteq \mathbb{T}^3$ invariant and hence induces a $\mathbb{K}^*$-action on X.

Proposition 3.2.2. *Consider P and $X \subseteq Z$ as in Construction 3.2.1, let P^* be the transpose of P and set $K := \mathbb{Z}^4/\mathrm{im}(P^*)$. For the divisor class group of X, we have*

$$\mathrm{Cl}(X) \;\cong\; \mathrm{Cl}(Z) \;\cong\; K \;\cong\; \mathbb{Z} \oplus \mathbb{Z}\,/\,2\gcd(2a+2,\, a-b)\mathbb{Z}.$$

Moreover, denoting by $Q\colon \mathbb{Z}^4 \to K$ the projection, we obtain the following description of the Cox ring of X as a graded algebra:

$$\mathcal{R}(X) \;\cong\; \mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2 + T_3^2 + T_4^2 \rangle, \qquad \deg(T_i) = Q(e_i) = [D_i^X],$$

where $D_i^X \subseteq X$ is the prime divisor on X obtained by intersecting X with the toric prime divisor of Z given by the ray through v_i and $[D_i^X] \in \mathrm{Cl}(X)$ denotes its class.

Proof of Construction 3.2.1 and Proposition 3.2.2. According to their definition, the columns $v_1, \ldots, v_4$ of P are pairwise different primitive integral vectors. Moreover, they generate $\mathbb{Q}^3$ as a convex cone, as we have

$$2v_1 + v_3 + v_4 = [0, 0, 2a + 2],\ a \geq 0, \qquad 2v_2 + v_3 + v_4 = [0, 0, 2b + 2],\ b \leq -2.$$

Thus, P is a defining matrix of a normal, rational, projective $\mathbb{K}^*$-surface X' in the sense of [3, Constr. 5.4.1.3 and 5.4.1.6 (e-e)]. Both, X' and X from Construction 3.2.1 share the same ambient toric variety Z and are given in homogeneous coordinates of Z by

$$X' \ = \ V(T_1T_2 + T_3^2 + T_4^2) \ = \ X.$$

Now [3, Thm. 3.4.3.7] tells us that the divisor class group of $X = X'$ is given as $\mathrm{Cl}(X) = \mathrm{Cl}(Z) = K$, that the Cox ring $\mathcal{R}(X)$ of X is as claimed and that the generator degrees satisfy $\deg(T_i) = [D_i]$. Note that X is non-toric as its Cox ring is not a polynomial ring. □

The *local class group* $\mathrm{Cl}(X, x)$ of a point $x \in X$ is the group of Weil divisors of X modulo those being principal near x, and by $\mathrm{cl}(X, x)$ the order of $\mathrm{Cl}(X, x)$.

Proposition 3.2.3. *Let $X = X(P)$ arise from Construction 3.2.1. The fixed points of the $\mathbb{K}^*$-action on X are given in Cox coordinates by*

$$x^+ \ := \ [0, 1, 0, 0], \qquad x^- \ := \ [1, 0, 0, 0], \qquad x_0 \ := \ [0, 0, 1, I].$$

Moreover, for the orders of the local class groups of the fixed points of the $\mathbb{K}^*$-action we obtain

$$\mathrm{cl}(X, x^+) = 4a + 4, \qquad \mathrm{cl}(X, x^-) = -4b - 4, \qquad \mathrm{cl}(X, x_0) = a - b.$$

Finally, the ordered pair $(4a + 4,\ -4 - 4b)$ is an isomorphy invariant of the algebraic surface X.

Proof. For the first statement, we refer to [17, Rem. 5.6]. For the second one, we use the description [3, Prop. 3.3.1.5] of the local class groups and its Gale dual representation provided by [3, Lemma 2.1.4.1]. Concretely, for the fixed points x^+ and x^- this means

$$\begin{aligned} \mathrm{cl}(X, x^+) \ &= \ |K/Q(\mathrm{lin}_{\mathbb{Z}}(e_2))| \ = \ |\mathbb{Z}^3/\mathrm{lin}_{\mathbb{Z}}(v_1, v_3, v_4)| \ = \ \det[v_1, v_3, v_4], \\ \mathrm{cl}(X, x^-) \ &= \ |K/Q(\mathrm{lin}_{\mathbb{Z}}(e_1))| \ = \ |\mathbb{Z}^3/\mathrm{lin}_{\mathbb{Z}}(v_2, v_3, v_4)| \ = \ \det[v_2, v_3, v_4]. \end{aligned}$$

Similarly, we obtain that the local class group order $\mathrm{cl}(x_0)$ of the fixed point x_0 is given by

$$|K/Q(\mathrm{lin}_{\mathbb{Z}}(e_3, e_4))| \ = \ |\mathrm{lin}_{\mathbb{Z}}(-e_1 - e_2, e_3)/\mathrm{lin}_{\mathbb{Z}}(v_1, v_2)| \ = \ \det \begin{bmatrix} -1 & -1 \\ a & b \end{bmatrix}.$$

For the last statement recall from [3, Prop. 5.4.1.9] that x^+, x^- are the only $\mathbb{K}^*$-fixed points lying in the closure of infinitely many orbits. Thus, $\{\mathrm{cl}(X, x^+), \mathrm{cl}(X, x^-)\}$ and $\mathrm{cl}(X, x_0)$ are invariants of the $\mathbb{K}^*$-surface X. Since on a non-toric, rational, projective surface any two $\mathbb{K}^*$-actions are conjugate in the automorphism group, the assertion follows. □

Proposition 3.2.4. *Every full intrinsic quadric surface X of Picard number one is isomorphic to an $X(P)$ for precisely one matrix P from Construction 3.2.1.*

Proof. According to Theorem 3.1.3 and [22, Ex. 7.1], the defining matrix P is of the format 3×4 and the first two rows are as in the assertion:

$$P \ = \ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}.$$

Note that d_3 and d_4 are odd by primitivity of the columns. Thus, subtracting the $(d_3-1)/2$-fold of the first and the $(d_4-1)/2$-fold of the second row from the last one turns our matrix into a defining matrix

$$P \ = \ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}.$$

These are *admissible operations* in the sense of [17, Def. 6.3] which do not affect the resulting $\mathbb{K}^*$-surface due to [17, Prop. 6.7]. Moreover, swapping the first two columns if necessary, we achieve that P is slope-ordered, meaning

$$a > b.$$

Again this is an admissible operation. As for any defining matrix of a rational $\mathbb{K}^*$-surface with two elliptic fixed points, slope orderedness implies

$$a + \tfrac{1}{2} + \tfrac{1}{2} =: m^+ > 0, \qquad\qquad b + \tfrac{1}{2} + \tfrac{1}{2} =: m^- < 0,$$

see [17, Rem. 7.5]. Multiplying the last row by -1 is another admissible operation and turns $m^\pm$ into $m^\mp$. Doing so, if necessary, and re-arranging via the first two admissible operation steps yields

$$a + 1 \ \le \ -b - 1.$$

If $X(P) \cong X(P')$ holds with P, P' as in Construction 3.2.1, then we have $P = P'$, as due to Proposition 3.2.3, the entries a, b of P and a', b' of P' satisfy

$$(4a + 4, \ -4b - 4) \ = \ (4a' + 4, \ -4b' - 4).$$

□

Recall that the *Gorenstein index* of a $\mathbb{Q}$-factorial variety X is the smallest positive integer ι_X such that the ι_X-fold of a canonical divisor of X is Cartier. The *local Gorenstein index* ι_x of a point $x \in X$ is the smallest positive integer such that the ι_x-fold of a canonical divisor of X is Cartier near x.

Theorem 3.2.5. *For any* $\iota \in \mathbb{Z}_{\geq 1}$*, consider the set* M_ι *of pairs* $\eta = (\iota^+, \iota^-) \in \mathbb{Z}^2_{\geq 1}$ *with* $\operatorname{lcm}(\iota^+, \iota^-) = \iota$*. Define subsets*

$$\begin{array}{lcl} S_{11}(1,\iota) & := & \{\eta \in M_\iota;\ \iota^+ \textit{ odd},\ \iota^- \textit{ odd},\ \iota^+ \leq \iota^-\}, \\ S_{12}(1,\iota) & := & \{\eta \in M_\iota;\ \iota^+ \textit{ odd},\ \iota^- \textit{ even},\ 4 \mid \iota^-,\ 2\iota^+ \leq \iota^-\}, \\ S_{21}(1,\iota) & := & \{\eta \in M_\iota;\ \iota^+ \textit{ even},\ \iota^- \textit{ odd},\ 4 \mid \iota^+,\ \iota^+ \leq 2\iota^-\}, \\ S_{22}(1,\iota) & := & \{\eta \in M_\iota;\ \iota^+ \textit{ even},\ \iota^- \textit{ even},\ 4 \mid \iota^+,\ 4 \mid \iota^-,\ \iota^+ \leq \iota^-\}. \end{array}$$

Then each set $S_{ij}(1,\iota)$ *provides us with a series of defining matrices* P_η *of full intrinsic quadric surfaces:*

$$\eta = (\iota^+, \iota^-) \in S_{11}(1,\iota): \qquad\qquad \eta = (\iota^+, \iota^-) \in S_{12}(1,\iota):$$

$$P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota^+ - 1 & -\iota^- - 1 & 1 & 1 \end{bmatrix}, \qquad P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota^+ - 1 & -\frac{\iota^-}{2} - 1 & 1 & 1 \end{bmatrix},$$

$$\eta = (\iota^+, \iota^-) \in S_{21}(1,\iota): \qquad\qquad \eta = (\iota^+, \iota^-) \in S_{22}(1,\iota):$$

$$P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota^+}{2} - 1 & -\iota^- - 1 & 1 & 1 \end{bmatrix}, \qquad P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota^+}{2} - 1 & -\frac{\iota^-}{2} - 1 & 1 & 1 \end{bmatrix}.$$

Each surface $X(P_\eta)$ *is of Picard number one, Gorenstein index* $\iota = \operatorname{lcm}(\iota^+, \iota^-)$ *and* $\iota^\pm$ *are the local Gorenstein indices of the points*

$$x^+ \ = \ [0,1,0,0], \qquad\qquad x^- \ = \ [1,0,0,0].$$

Moreover, every full intrinsic quadric surface of Picard number one and Gorenstein index ι *is isomorphic to* $X(P_\eta)$ *for precisely one* P_η *from the above list.*

Proof. Let X be a full intrinsic quadric surface of Picard number one. We first show $X \cong X(P_\eta)$ with P_η from the above list and check the local Gorenstein indices. Proposition 3.2.4 allows us to assume $X = X(P)$ with a unique P of the form

$$P \ = \ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}, \qquad b \leq -2,\ 0 \leq a \leq -b-2.$$

According to Remark 1.2.10, we have the anticanonical divisor $-\mathcal{K}_X = D_3^X + D_4^X$ on X and [17, Prop. 8.9] tells us that the linear forms $u^\pm$ representing the $\iota^\pm$-fold of $-\mathcal{K}_X$ near $x^\pm$ are given by

$$u^+ \ = \ \left[\tfrac{a\iota^+}{2a+2}, \tfrac{a\iota^+}{2a+2}, \tfrac{\iota^+}{a+1},\right], \qquad u^- \ = \ \left[\tfrac{b\iota^-}{2b+2}, \tfrac{b\iota^-}{2b+2}, \tfrac{\iota^-}{b+1}\right].$$

By the definition of the local Gorenstein index, these are primitive integral vectors. Consequently, the local Gorenstein indices $\iota^{\pm}$ of $x^{\pm}$ are

$$\iota^{+} \ = \ \begin{cases} a+1, & a \text{ even}, \\ 2a+2, & a \text{ odd}, \end{cases} \qquad \iota^{-} \ = \ \begin{cases} -b-1, & b \text{ even}, \\ -2b-2, & b \text{ odd}. \end{cases}$$

In particular, ι^{+}, ι^{-} is even (odd) if and only if a, b is odd (even), respectively. Moreover, if $\iota^{\pm}$ is even, then it is divisible by four. Thus, P is one of the matrices P_{η} with $\eta = (\iota^{+}, \iota^{-})$ listed in the assertion and $\iota^{\pm}$ is the local Gorenstein index of $x^{\pm}$.

Conversely, all the matrices P_{η} listed in the assertion fit into the shape of Construction 3.2.1 and thus deliver full intrinsic quadric surfaces $X = X(P_{\eta})$. By [17, Prop. 8.8], the point $x_0 = [0, 0, 1, I] \in X$ has local Gorenstein index one, hence the resulting X is of Gorenstein index $\iota = \operatorname{lcm}(\iota^{+}, \iota^{-})$.

Finally, we ensure that the matrices P_{η} listed in the assertion define pairwise non-isomorphic $X(P_{\eta})$. By Proposition 3.2.4, this means to show that any two matrices arising from different $S_{ij}(1, \iota)$ differ from each other. This is done by comparing the parity vectors $(\bar{a}, \bar{b})$ in $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ of the first two entries a, b of the third row of P_{η} for the $\eta \in S_{ij}(1, \iota)$:

	$S_{11}(1, \iota)$	$S_{12}(1, \iota)$	$S_{21}(1, \iota)$	$S_{22}(1, \iota)$
$(\bar{a}, \bar{b})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$

□

Example 3.2.6. Consider the full intrinsic quadric surface $X = X(P)$ of Picard number one given by the defining matrix

$$P \ = \ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 0 & -2 & 1 & 1 \end{bmatrix}.$$

Then X stems from the series $S_{11}(1, \iota)$ and we have $\iota = \iota^{+} = \iota^{-} = 1$. Theorem 3.2.5 also says that X is the only Gorenstein full intrinsic quadric surface with $\rho(X) = 1$.

3.3 Picard number two

The main result of this section, Theorem 3.3.5, provides the description of all full intrinsic quadric surfaces of Picard number two in terms of the local Gorenstein indices of two of their possibly singular points and the local class group order of another possibly singular point.

Construction 3.3.1 (Full intrinsic quadric surfaces X of Picard number two as $\mathbb{K}^*$-surfaces)**.** Consider an integral matrix of the form

$$P \;:=\; [v_1, v_2, v_3, v_4, v_5] \;:=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \qquad \begin{array}{l} b < a,\ c < 0,\ a \geq 0, \\ b + c \leq -1,\ a - b \leq -c, \\ a \leq -b - c - 1. \end{array}$$

Let Z be the toric variety arising from the fan Σ in $\mathbb{Z}^3$ with generator matrix P and the maximal cones

$$\sigma^+ := \mathrm{cone}(v_1, v_3, v_5), \qquad \sigma^- := \mathrm{cone}(v_2, v_4, v_5),$$

$$\tau_0 := \mathrm{cone}(v_1, v_2), \qquad \tau_1 := \mathrm{cone}(v_3, v_4).$$

Denote by U_1, U_2, U_3 the coordinate functions on the standard 3-torus $\mathbb{T}^3 \subseteq Z$. Then we obtain a normal, non-toric, rational, projective surface

$$X \;:=\; X(P) \;:=\; \overline{V(h)} \;\subseteq\; Z, \qquad h \;:=\; 1 + U_1 + U_2 \;\in \mathcal{O}(\mathbb{T}^3).$$

Moreover, the $\mathbb{K}^*$-action on $\mathbb{T}^3$ given by $t \cdot x = (x_1, x_2, tx_3)$ extends to an action on Z, it leaves $V(h) \subseteq \mathbb{T}^3$ invariant and hence induces a $\mathbb{K}^*$-action on X.

Proposition 3.3.2. *Consider P and $X \subseteq Z$ as in Construction 3.3.1, let P^* be the transpose of P and set $K := \mathbb{Z}^5/\mathrm{im}(P^*)$. For the divisor class group of X, we have*

$$\mathrm{Cl}(X) \;\cong\; K \;\cong\; \mathrm{Cl}(Z) \;\cong\; \mathbb{Z}^2 \oplus \mathbb{Z} \,/\, \gcd(2a+1, a-b, -c)\mathbb{Z}.$$

Moreover, denoting by $Q\colon \mathbb{Z}^5 \to K$ the projection, we obtain the following description of Cox ring of X as graded algebra:

$$\mathcal{R}(X) \;\cong\; \mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2 \rangle, \qquad \deg(T_i) = Q(e_i) = [D_i],$$

where $D_i^X \subseteq X$ is the prime divisor on X obtained by intersecting X with the toric prime divisor of Z given by the ray through v_i and $[D_i^X] \in \mathrm{Cl}(X)$ denotes its class.

Proof of Construction 3.3.1 and Proposition 3.3.2. According to their definition, the columns $v_1, \ldots, v_5$ of P are pairwise different primitive integral vectors. Moreover, they generate $\mathbb{Q}^3$ as a convex cone, as we have

$$2v_1 + 2v_3 + v_5 = [0, 0, 2a+1],\ a \geq 0,$$

$$2v_2 + 2v_4 + v_5 = [0, 0, 2b + 2c + 1],\ b + c \leq -1.$$

Consequently, P is a defining matrix of a rational projective $\mathbb{K}^*$-surface X' in the sense of [3, Constr. 5.4.1.3 and 5.4.1.6 (e-e)]. One shows $X' = X$ exactly as for Picard number one and infers the desired statements on the divisor class group and the Cox ring from the same reference. ☐

Proposition 3.3.3. *Let $X = X(P)$ arise from Construction 3.3.1. The fixed points of the $\mathbb{K}^*$-action on X are given in Cox coordinates by*

$$x^+ \ := \ [0,1,0,1,0], \qquad x^- \ := \ [1,0,1,0,0],$$

$$x_0 \ := \ [0,0,1,1,I], \qquad x_1 \ := \ [1,1,0,0,I].$$

Moreover, the orders of the local class groups of the fixed points of the $\mathbb{K}^*$-action on X are given by

$$\begin{array}{llll} \mathrm{cl}(X,x^+) & = 1+2a, & \mathrm{cl}(X,x^-) & = -1-2b-2c, \\ \mathrm{cl}(X,x_0) & = a-b, & \mathrm{cl}(X,x_1) & = -c. \end{array}$$

Finally, the ordered pairs $(1+2a,\ -1-2b-2c)$ and $(a-b,\ -c)$ are isomorphy invariants of the algebraic surface X.

Proof. The same references and arguing as in the proof of Proposition 3.2.3, give us the fixed points and show that the local class group orders of x^+, x^-, x_0 and x_1 compute as

$$\det[v_1,v_3,v_5], \qquad \det[v_2,v_4,v_5], \qquad \det\begin{bmatrix} -1 & -1 \\ a & b \end{bmatrix}, \qquad -\det\begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}.$$

As mentioned in the proof of Proposition 3.2.3, the fixed points x^+, x^- are the only ones lying in the closure of infinitely many orbits. Moreover, each of the remaining two fixed points x_0, x_1 lies in the closure of precisely two non-trivial orbits. Thus, $\{\mathrm{cl}(X,x^+),\mathrm{cl}(X,x^-)\}$ as well as $\{\mathrm{cl}(X,x_0),\mathrm{cl}(X,x_1)\}$ are invariants of the $\mathbb{K}^*$-surface X. As before, the assertion follows from the fact that on a non-toric, rational, projective surface any two $\mathbb{K}^*$-actions are conjugate in the automorphism group. ☐

Proposition 3.3.4. *Every full intrinsic quadric surface X of Picard number two is isomorphic to an $X(P)$ for precisely one matrix P from Construction 3.3.1.*

Proof. Using again Theorem 3.1.3 and [22, Ex. 7.1], we see that the defining matrix P is of the format 3×5 and the first two rows look as in the assertion:

$$P \ = \ \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix}.$$

As in the proof of Proposition 3.2.4, we achieve the desired shape of P via admissible operations [17, Def. 6.3]. First, adding suitable multiples of the first two rows to the last one yields

$$P \ = \ \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}.$$

Second, swapping the columns v_1 and v_2 as well as v_3 and v_4 if neccesary and re-arranging via the first step, we achieve that P is slope-ordered, meaning

$$a > b, \qquad 0 > c.$$

Third, swapping the first two columns blocks, that means $[v_1, v_2]$ and $[v_3, v_4]$, if neccesary and re-adjusting the entries, we can ensure

$$a - b \leq -c.$$

As for any defining matrix of a rational $\mathbb{K}^*$-surface with two elliptic fixed points, slope orderedness implies

$$a + \tfrac{1}{2} =: m^+ > 0, \qquad\qquad b + c + \tfrac{1}{2} =: m^- < 0.$$

Multiplying the last row by -1 turns $m^{\pm}$ into $m^{\mp}$. Doing so, if necessary, and re-arranging via the first two steps yields

$$a \ \leq \ -b - c - 1.$$

We show that $X(P) \cong X(P')$ with matrices P and P' as in Construction 3.3.1 implies $P = P'$. Proposition 3.3.3 yields equality of the ordered tuples

$$(1 + 2a,\ -1 - 2b - 2c) = (1 + 2a',\ -1 - 2b' - 2c'), \quad (a - b,\ -c) = (a' - b', -c')$$

built from the entries of the third row of P and P' respectively. From this we directly derive $P = P'$. ☐

Theorem 3.3.5. *For any $\iota \in \mathbb{Z}_{\geq 1}$, consider the set M_ι of triples $\eta = (\iota^+, \iota^-, c)$, where $\iota^+, \iota^- \in \mathbb{Z}_{\geq 1}$ with $\operatorname{lcm}(\iota^+, \iota^-) = \iota$ and $c \in \mathbb{Z}_{\leq -1}$. Define subsets*

$$\begin{aligned}
S_{11}(2,\iota) &:= \{\eta \in M_\iota;\ 2 \nmid \iota^+, \iota^-,\ 3 \nmid \iota^+, \iota^-,\ \iota^+ \leq \iota^-,\ 1 - \tfrac{\iota^+ + \iota^-}{2} \leq c \leq -\tfrac{\iota^+ + \iota^-}{4}\},\\
S_{12}(2,\iota) &:= \{\eta \in M_\iota;\ 2 \nmid \iota^+, \iota^-,\ 3 \nmid \iota^+,\ \iota^+ \leq 3\iota^-,\ 1 - \tfrac{\iota^+ + 3\iota^-}{2} \leq c \leq -\tfrac{\iota^+ + 3\iota^-}{4}\},\\
S_{21}(2,\iota) &:= \{\eta \in M_\iota;\ 2 \nmid \iota^+, \iota^-,\ 3 \nmid \iota^-,\ 3\iota^+ \leq \iota^-,\ 1 - \tfrac{3\iota^+ + \iota^-}{2} \leq c \leq -\tfrac{3\iota^+ + \iota^-}{4}\},\\
S_{22}(2,\iota) &:= \{\eta \in M_\iota;\ 2 \nmid \iota^+, \iota^-,\ \iota^+ \leq \iota^-,\ 1 - \tfrac{3\iota^+ + 3\iota^-}{2} \leq c \leq -\tfrac{3\iota^+ + 3\iota^-}{4}\}.
\end{aligned}$$

Then each set $S_{ij}(2,\iota)$ provides us with a series of defining matrices P_η of full intrinsic quadric surfaces:

$\eta = (\iota^+, \iota^-, c) \in S_{11}(2,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{\iota^+ - 1}{2} & -\frac{\iota^- + 1}{2} - c & 0 & c & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c) \in S_{12}(2,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{\iota^+ - 1}{2} & -\frac{3\iota^- + 1}{2} - c & 0 & c & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c) \in S_{21}(2,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{3\iota^+ - 1}{2} & -\frac{\iota^- + 1}{2} - c & 0 & c & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c) \in S_{22}(2,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{3\iota^+ - 1}{2} & -\frac{3\iota^- + 1}{2} - c & 0 & c & 1 \end{bmatrix}.$$

Each surface $X(P_\eta)$ is of Picard number two, Gorenstein index $\iota = \operatorname{lcm}(\iota^+, \iota^-)$ and ι^+, ι^- resp. $-c$ are the local Gorenstein indices resp. local class group order of

$$x^+ \ = \ [0,1,0,1,0], \qquad x^- \ = \ [1,0,1,0,0], \qquad x_1 \ = \ [1,1,0,0,I].$$

Finally, every full intrinsic quadric surface of Picard number two and Gorenstein index ι is isomorphic to $X(P_\eta)$ for precisely one P_η from the above list.

Proof. Let X be a full intrinsic quadric surface of Picard number two. Then Proposition 3.3.4 allows us to assume $X = X(P)$ with

$$P \ = \ \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \qquad \begin{array}{l} b < a, \ c < 0, \ a \ge 0, \\ b + c \le -1, \ a - b \le -c, \\ a \le -b - c - 1. \end{array}$$

Consider the anticanonical divisor $-\mathcal{K}_X = D_3^X + D_4^X + D_5^X$ on $X(P)$; see Remark 1.2.10. The linear forms $u^\pm$ representing the $\iota^\pm$-fold of $-\mathcal{K}_X$ near $x^\pm$ are given by

$$u^+ = \left[\iota^+, \, \tfrac{(a-1)\iota^+}{1+2a}, \, \tfrac{3\iota^+}{1+2a}\right], \quad u^- = \left[\tfrac{(2b-c+1)\iota^-}{2b+2c+1}, \, \tfrac{(b+c-1)\iota^-}{2b+2c+1}, \, -\tfrac{3\iota^-}{2b+2c+1}\right].$$

By the definition of the local Gorenstein index, these are primitive integral vectors. Together with the fact that $\iota^\pm$ divides $\operatorname{cl}(X, x^\pm)$, we obtain

$$3\iota^+ = y^+(1+2a), \qquad 1 + 2a = z^+\iota^+,$$

$$3\iota^- = -y^-(2b+2c+1), \qquad -(2b+2c+1) = z^-\iota^-$$

with positive integers $y^\pm$ and $z^\pm$. We conclude $y^+z^+ = 3$ and $y^-z^- = 3$. This leaves us with the following four cases:

Case 1.1: $y^+ = 3$, $y^- = 3$. Then we have $\iota^+ = 1 + 2a$ and $\iota^- = -2b - 2c - 1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+, \, \tfrac{\iota^+-3}{2}, \, 3\right], \qquad u^- = \left[\iota^- + 3c, \, \tfrac{\iota^-+3}{2}, \, -3\right].$$

We conclude that ι^+ as well as ι^- is odd and none of them is divisible by three. Substituting also in P and the conditions on its entries leads to setting $S_{11}(2, \iota)$.

Case 1.2: $y^+ = 3$, $y^- = 1$. Then we have $\iota^+ = 1 + 2a$ and $3\iota^- = -2b - 2c - 1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+, \, \tfrac{\iota^+-3}{2}, \, 3\right], \qquad u^- = \left[\iota^- + c, \, \tfrac{\iota^-+1}{2}, \, -1\right].$$

We conclude that ι^+ as well as ι^- is odd and ι^+ is not divisible by three. Substituting also in P and the conditions on its entries leads to setting $S_{12}(2, \iota)$.

Case 2.1: $y^+ = 1$, $y^- = 3$. Then we have $3\iota^+ = 1 + 2a$ and $\iota^- = -2b - 2c - 1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+,\ \tfrac{\iota^+-1}{2}\ 1,\right], \qquad u^- = \left[\iota^- + 3c,\ \tfrac{\iota^-+3}{2},\ -3\right].$$

We conclude that ι^+ as well as ι^- is odd and ι^- is not divisible by three. Substituting also in P and the conditions on its entries leads to setting $S_{21}(2, \iota)$.

Case 2.2: $y^+ = 1$, $y^- = 1$. Then we have $3\iota^+ = 1 + 2a$ and $3\iota^- = -2b - 2c - 1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+,\ \tfrac{\iota^+-1}{2},\ 1\right], \qquad u^- = \left[\iota^- + c,\ \tfrac{\iota^-+1}{2},\ -1\right].$$

We conclude that ι^+ as well as ι^- is odd. Substituting also in P and the conditions on its entries leads to setting $S_{22}(2, \iota)$.

We showed that every full intrinsic quadric surface of Picard number two is isomorphic to some $X(P_\eta)$ with P_η as in the assertion. Moreover, x_0 and x_1 are of local Gorenstein index one, see [17, Prop. 8.8 (iii)], we obtain that $X(P_\eta)$ has Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$. Conversely, one directly checks that every matrix P from the assertion defines a full intrinsic quadric surface of Picard number two and Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$.

Finally, we want to see that the matrices P_η listed in the assertion define pairwise non-isomorphic $X(P_\eta)$. Due to Proposition 3.3.4, this means to show that the sets $S_{ij}(2, \iota)$ are pairwise disjoint. With the aid of Proposition 3.3.3, we compare the local Gorenstein indices $\iota^\pm$ and the local class group orders $\mathrm{cl}(X, x^\pm)$:

	$S_{11}(2,\iota)$	$S_{12}(2,\iota)$	$S_{21}(2,\iota)$	$S_{22}(2,\iota)$
$(\iota^+, \mathrm{cl}(X, x^+))$	(ι^+, ι^+)	(ι^+, ι^+)	$(\iota^+, 3\iota^+)$	$(\iota^+, 3\iota^+)$
$(\iota^-, \mathrm{cl}(X, x^-))$	(ι^-, ι^-)	$(\iota^-, 3\iota^-)$	(ι^-, ι^-)	$(\iota^-, 3\iota^-)$

The listed pairs are invariants of the surface $X(P_\eta)$ up to switching x^+ and x^-. Thus, we see that $S_{11}(2, \iota)$ as well as $S_{22}(2, \iota)$ has trivial intersection with any other $S_{ij}(2, \iota)$. For $S_{12}(2, \iota)$ observe $\iota^+ < 3\iota^-$, as we have $3 \nmid \iota^+$. Similarly, $3\iota^+ < \iota^-$ holds for $S_{21}(3, \iota)$. Thus, in both cases, $\mathrm{cl}(X, x^+)$ is the strictly smallest of $\mathrm{cl}(X, x^\pm)$. Consequently, $S_{12}(2, \iota)$ and $S_{21}(2, \iota)$ intersect trivially. ☐

Example 3.3.6. Consider the full intrinsic quadric surfaces X and X' of Picard number two given by the defining matrices

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix}, \qquad P' = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix}.$$

Then X stems from the series $S_{12}(2,\iota)$ and X' from $S_{22}(2,\iota)$. Theorem 3.3.5 yields that X and X' are the only Gorenstein full intrinsic quadric surfaces with $\rho(X)=2$.

We conclude the section by taking a look at the possible contractions of the two-dimensional full intrinsic quadrics. Recall that a *contraction* of a prime divisor D on a normal variety X is a proper birational morphism $\pi\colon X \to X'$ such that the image $\pi(D)$ is of codimension at least two in X' and $X \setminus D$ maps isomorphically onto $X' \setminus \pi(D)$.

Proposition 3.3.7. *Let $X = X(P)$ arise from Construction 3.3.1. At most the prime divisors $D_1^X, \ldots, D_4^X \subseteq X$ are contractible and all possible contractions are projective toric surfaces of Picard number one. More precisely,*

$$D_1^X:\ b \geq 0 \qquad D_2^X:\ a{+}c \leq -1 \qquad D_3^X:\ a{+}c \geq 0 \qquad D_4^X:\ b \leq -1$$

$$\begin{bmatrix} -1 & -1 & 2 \\ b & b{+}c & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 2 \\ a & a{+}c & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 2 \\ a{+}c & b{+}c & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 2 \\ a & b & 1 \end{bmatrix}$$

gives us for each D_i^X the characterizing property of contractibility in terms of the entries a, b, c of P and, for the case that D_i^X is contractible, also the generator matrix of the contracted surface.

Proof. [17, Rem. 10.4 (i)] tells us that a contractible divisor must be $\mathbb{K}^*$-invariant and [17, Prop. 10.8] tells us that the divisor is an orbit closure containing a hyperbolic fixed point. Hence, the contractible prime divisors are among the $D_i^X = V(T_i) \subseteq X$, where $i = 1, \ldots, 5$. The same references show that the divisor D_5^X is not contractible. Recall that the matrix P is given as

$$P \;=\; [v_1, v_2, v_3, v_4, v_5] \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \qquad \begin{array}{l} b < a,\ c < 0,\ a \geq 0, \\ b + c \leq -1,\ a - b \leq -c, \\ a \leq -b - c - 1. \end{array}$$

The task is to characterize contractibility for each of $D_1^X, \ldots, D_4^X$ and to determine the possible contraction in terms of the entries of P. We exemplarily perform this for the divisor D_1^X. Consider the matrix

$$P_1 \;:=\; [v_2, v_3, v_4, v_5] \;=\; \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 2 \\ b & 0 & c & 1 \end{bmatrix},$$

obtained from P by removing the colmun v_1, which corresponds the prime divisor $D_1^X \subseteq X$. Then D_1^X is contractible if and only if P_1 is a defining matrix of a $\mathbb{K}^*$-surface. The latter in turn holds if and only if

$$m^+ \;=\; b + \tfrac{1}{2} \;>\; 0,$$

as P_1 inherits all the other properties from P. Thus, D_1^X is contractible if and only if $b \geq 0$ holds. If so, then contracing D_1^X gives the $\mathbb{K}^*$-surface X_1 defined by P_1. Via admissible operations, we can turn P_1 into the shape

$$P_1 \;=\; \begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \\ b & b+c & 0 & 1 \end{bmatrix}.$$

Indeed, we swap the first two column blocks, re-arrange the shape and then subtract the b-fold of the first row from the last one. The makes the third column *erasable* [17, Def. 6.2] and we obtain a defining matrix

$$P_1' \;=\; \begin{bmatrix} -1 & -1 & 2 \\ b & b+c & 1 \end{bmatrix}$$

by *erasing* the third column [17, Def. 6.3, Prop. 6.7]. This process reflects removing the redundant Cox ring generator $T_3 = T_1T_2 - T_4^2$ in the first presentation of X_1. We conclude that X_1 is the toric surface defined by the generator matrix P_1'. □

Remark 3.3.8. Consider the two Gorenstein full intrinsic quadric surfaces X and X' of Picard number two from Example 3.3.6.

(i) In the surface X, the contractible divisors are D_2^X and D_4^X. In each case, the contracted surface is the projective plane $\mathbb{P}^2$.
(ii) In the surface X', the contractible divisors are $D_1^{X'}$ and $D_2^{X'}$. In each case, the contracted surface is the weighted projective plane $\mathbb{P}(1,2,3)$.

3.4 Picard number three

The main result of this section, Theorem 3.4.5, provides the description of all full intrinsic quadric surfaces of Picard number three in terms of the local Gorenstein indices of two of their possibly singular points and the local class group orders of two further possibly singular points.

Construction 3.4.1 (Full intrinsic quadric surfaces X of Picard number three as $\mathbb{K}^*$-surfaces)**.** Consider an integral matrix of the form

$$P := [v_1,v_2,v_3,v_4,v_5,v_6] := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}, \qquad \begin{array}{l} a > b,\ 0 > c,\ 0 > d, \\ a-b \geq -c \geq -d, \\ b+c+d < 0 < a, \\ a \leq -b-c-d. \end{array}$$

Let Z be the toric variety arising from the fan Σ in $\mathbb{Z}^3$ with generator matrix P and the maximal cones

$$\sigma^+ := \operatorname{cone}(v_1,v_3,v_5), \qquad \sigma^- := \operatorname{cone}(v_2,v_4,v_6),$$

$$\tau_0 := \mathrm{cone}(v_1, v_2), \quad \tau_1 := \mathrm{cone}(v_3, v_4), \quad \tau_2 := \mathrm{cone}(v_5, v_6).$$

Denote by U_1, U_2, U_3 the coordinate functions on the standard 3-torus $\mathbb{T}^3 \subseteq Z$. Then we obtain a normal, non-toric, rational, projective surface

$$X \ := \ X(P) \ := \ \overline{V(h)} \ \subseteq \ Z, \qquad h \ := \ 1 + U_1 + U_2 \ \in \mathcal{O}(\mathbb{T}^3).$$

Moreover, the $\mathbb{K}^*$-action on $\mathbb{T}^3$ given by $t \cdot x = (x_1, x_2, tx_3)$ extends to an action on Z, it leaves $V(h) \subseteq \mathbb{T}^3$ invariant and hence induces a $\mathbb{K}^*$-action on X.

Proposition 3.4.2. *Consider P and $X \subseteq Z$ as in Construction 3.4.1, let P^* be the transpose of P and set $K := \mathbb{Z}^6/\mathrm{im}(P^*)$. For the divisor class group of X, we have Then the divisor class group of X equals that of Z and is given by*

$$\mathrm{Cl}(X) \ \cong \ \mathrm{Cl}(Z) \ \cong \ K \ \cong \ \mathbb{Z}^3 \oplus \mathbb{Z} \,/\, \gcd(a, b, c, d)\mathbb{Z}.$$

Moreover, denoting by $Q\colon \mathbb{Z}^6 \to K$ the projection, we obtain the following description of the Cox ring of X as a graded algebra:

$$\mathcal{R}(X) \ \cong \ \mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5T_6 \rangle, \qquad \deg(T_i) = Q(e_i) = [D_i],$$

where $D_i^X \subseteq X$ is the prime divisor on X obtained by intersecting X with the toric prime divisor of Z given by the ray through v_i and $[D_i^X] \in \mathrm{Cl}(X)$ denotes its class.

Proof of Construction 3.2.1 and Proposition 3.2.2. According to their definition, the columns $v_1, \ldots, v_6$ of P are pairwise different primitive integral vectors. Moreover, they generate $\mathbb{Q}^3$ as a convex cone, as we have

$$v_1 + v_3 + v_5 = [0, 0, a], \ a > 0, \qquad v_2 + v_4 + v_6 = [0, 0, b + c + d], \ b + c + d < 0.$$

Consequently, P is a defining matrix of a rational projective $\mathbb{K}^*$-surface X' in the sense of [3, Constr. 5.4.1.3 and 5.4.1.6 (e-e)]. Again, one shows $X' = X$ exactly as in the case of Picard number one, and the same reference gives the desired statements on the divisor class group and the Cox ring. □

Proposition 3.4.3. *Let $X = X(P)$ arise from Construction 3.4.1. The fixed points of the $\mathbb{K}^*$-action on X are given in Cox coordinates by*

$$x^+ \ := \ [0, 1, 0, 1, 0, 1], \qquad x^- \ := \ [1, 0, 1, 0, 1, 0],$$

$$x_0 \ := \ [0, 0, 1, 1, 1, -1], \quad x_1 \ := \ [1, 1, 0, 0, 1, -1], \quad x_2 \ := \ [1, 1, 1, -1, 0, 0].$$

Moreover, the orders of the local class groups of the fixed points of the $\mathbb{K}^*$-action are given by

$$\mathrm{cl}(X, x^+) = a, \qquad \mathrm{cl}(X, x^-) = -b - c - d,$$

$$\mathrm{cl}(X, x_0) = a - b, \quad \mathrm{cl}(X, x_1) = -c, \quad \mathrm{cl}(X, x_2) = -d.$$

Finally, the ordered tuples $(a, -b - c - d)$ and $(a - b, -c, -d)$ are isomorphy invariants of the algebraic surface X.

Proof. As in the previous section, the references from the proof of Proposition 3.2.3 deliver the description of the fixed points and show that the local class group orders of x^+, x^-, x_0, x_1 and x_2 are

$$\det[v_1, v_3, v_5], \qquad \det[v_2, v_4, v_6],$$

$$\det\begin{bmatrix} -1 & -1 \\ a & b \end{bmatrix}, \qquad -\det\begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}, \qquad -\det\begin{bmatrix} 1 & 1 \\ 0 & d \end{bmatrix}.$$

Simlarly as in the corresponding earlier proofs, x^+, x^- are the only fixed points lying in the closure of infinitely many orbits and each of x_0, x_1, x_2 lies in the closure of precisely two non-trivial orbits. Thus, the sets $\{\mathrm{cl}(X, x^+), \mathrm{cl}(X, x^-)\}$ and $\{\mathrm{cl}(X, x_0), \mathrm{cl}(X, x_1), \mathrm{cl}(X, x_2)\}$ are invariants of the $\mathbb{K}^*$-surface X. Again, the assertion follows from the fact that on a non-toric, rational, projective, surface any two $\mathbb{K}^*$-actions are conjugate in the automorphism group. □

Proposition 3.4.4. *Every full intrinsic quadric surface X of Picard number three is isomorphic to an $X(P)$ for precisely one matrix P from Construction 3.4.1.*

Proof. Applying once more Theorem 3.1.3 and [22, Ex. 7.1] yields that the defining matrix P is of the format 3×6 and the first two rows look as wanted:

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \end{bmatrix}.$$

Again, suitable admissible operations [17, Def. 6.3] bring us to the setting of Construction 3.4.1. First, adding suitable multiples of the first two rows to the last one, we achieve

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}.$$

Second, swapping columns inside the pairs (v_1, v_2), (v_3, v_4) and (v_5, v_6) and re-arranging via the first step, we achieve that P is slope-ordered, meaning

$$a > b, \quad 0 > c, \quad 0 > d.$$

Third, suitable swapping the columns blocks $[v_1, v_2]$, $[v_3, v_4]$ and $[v_5, v_6]$ and re-adjusting the entries, we can ensure

$$a - b \geq -c \geq -d.$$

As for any defining matrix of a rational $\mathbb{K}^*$-surface with two elliptic fixed points, slope orderedness implies

$$a =: m^+ > 0, \qquad b + c + d =: m^- < 0.$$

Multiplying the last row by -1 turns $m^{\pm}$ into $m^{\mp}$. Doing so, if necessary, and re-arranging via the first two steps yields

$$a \ \leq \ -b-c-d.$$

We show that $X(P) \cong X(P')$ with matrices P and P' as in Construction 3.4.1 implies $P = P'$. Proposition 3.4.3 yields equality of the ordered tuples

$$(a,\ -b-c-d) \ = \ (a',\ -b'-c'-d'), \qquad (a-b,\ -c,\ -d) \ = \ (a'-b',\ -c',\ -d')$$

built from the entries of the third row of P and P' respectively. From this we directly derive $P = P'$. $\square$

Theorem 3.4.5. *For any $\iota \in \mathbb{Z}_{\geq 1}$, consider the set M_ι of 4-tuples $\eta = (\iota^+, \iota^-, c, d)$, where $\iota^+, \iota^- \in \mathbb{Z}_{\geq 1}$ with $\operatorname{lcm}(\iota^+, \iota^-) = \iota$ and $c, d \in \mathbb{Z}_{\leq -1}$. Define subsets*

$$\begin{aligned}
S_{11}(3,\iota) \ &:= \ \{\eta \in M_\iota;\ 2 \nmid \iota^+, \iota^-,\ \iota^+ \leq \iota^-,\ -\iota^+ - \iota^- \leq 2c+d,\ c \leq d \leq -1\},\\
S_{12}(3,\iota) \ &:= \ \{\eta \in M_\iota;\ 2 \nmid \iota^+,\ \iota^+ \leq 2\iota^-,\ -\iota^+ - 2\iota^- \leq 2c+d,\ c \leq d \leq -1\},\\
S_{21}(3,\iota) \ &:= \ \{\eta \in M_\iota;\ 2 \nmid \iota^-,\ 2\iota^+ \leq \iota^-,\ -2\iota^+ - \iota^- \leq 2c+d,\ c \leq d \leq -1\},\\
S_{22}(3,\iota) \ &:= \ \{\eta \in M_\iota;\ \iota^+ \leq \iota^-,\ -2\iota^+ - 2\iota^- \leq 2c+d,\ c \leq d \leq -1\}.
\end{aligned}$$

Then each set $S_{ij}(3,\iota)$ provides us with a series of defining matrices P_η of full intrinsic quadric surfaces:

$\eta = (\iota^+, \iota^-, c, d) \in S_{11}(3,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota^+ & -\iota^- - c - d & 0 & c & 0 & d \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c, d) \in S_{12}(3,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota^+ & -2\iota^- - c - d & 0 & c & 0 & d \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c, d) \in S_{21}(3,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota^+ & -\iota^- - c - d & 0 & c & 0 & d \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c, d) \in S_{22}(3,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota^+ & -2\iota^- - c - d & 0 & c & 0 & d \end{bmatrix}.$$

Each $X(P_\eta)$ is of Picard number three, Gorenstein index $\iota = \operatorname{lcm}(\iota^+, \iota^-)$ and ι^+, ι^-, resp. $-c$, $-d$ are the local Gorenstein indices resp. local class group orders of

$$x^+ \ = \ [0,1,0,1,0,1], \qquad x^- \ = \ [1,0,1,0,1,0],$$

$$x_1 \ = \ [1,1,0,0,1,-1], \qquad x_2 \ = \ [1,1,1,-1,0,0].$$

Finally, every full intrinsic quadric surface of Picard number three and Gorenstein index ι is isomorphic to $X(P_\eta)$ for precisely one P_η from the above list.

Proof. Let X be a full intrinsic quadric surface of Picard number three. Then Construction 3.4.1 and Proposition 3.4.4 allow us to assume $X = X(P)$ with

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}, \qquad \begin{array}{l} a > b,\ 0 > c,\ 0 > d, \\ a - b \ge -c \ge -d, \\ b + c + d < 0 < a, \\ a \le -b - c - d. \end{array}$$

According to Remark 1.2.10 the anticanonical divisor is $-\mathcal{K}_X = D_3^X + D_4^X + D_5^X + D_6^X$ on $X(P)$. The linear forms $u^\pm$ representing the $\iota^\pm$-fold of $-\mathcal{K}_X$ near $x^\pm$ are given as

$$u^+ = \left[\iota^+,\, \iota^+,\, \tfrac{2\iota^+}{a}\right], \qquad u^- = \left[\tfrac{(b-c+d)\iota^-}{b+c+d},\, \tfrac{(b+c-d)\iota^-}{b+c+d},\, -\tfrac{2\iota^-}{b+c+d}\right].$$

By the definition of the local Gorenstein index, these are primitive integral vectors. Together with the fact that $\iota^\pm$ divides $\mathrm{cl}(X, x^\pm)$, we obtain

$$2\iota^+ = y^+ a, \qquad a = z^+ \iota^+,$$

$$2\iota^- = -y^-(b + c + d), \qquad -(b + c + d) = z^- \iota^-$$

with positive integers $y^\pm$ and $z^\pm$. We conclude $y^+ z^+ = 2$ and $y^- z^- = 2$. The possible constellations of (y^+, y^-) yield the following four cases:

Case 1.1: $a = \iota^+$, $b = -\iota^- - c - d$. Inserting this, we see that P arises from $S_{11}(3, \iota)$ and its entries satisfy the required estimates. Moreover, $u^\pm$ become

$$u^+ = \left[\iota^+,\, \iota^+,\, 2\right], \qquad u^- = \left[2c + \iota^-,\, 2d + \iota^-,\, -2\right].$$

As these are integral primitive vectors, we see that ι^+ as well as ι^- are odd and that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

Case 1.2: $a = \iota^+$, $b = -2\iota^- - c - d$. As in the previous subcase, inserting shows that P stems from $S_{12}(3, \iota)$. Note that this time we have

$$u^+ = \left[\iota^+,\, \iota^+,\, 2\right], \qquad u^- = \left[c + \iota^-,\, d + \iota^-,\, -1\right].$$

Thus, ι^+ is odd and we have no divisibility condition on ι^-. As before, we obtain that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

Case 2.1: $a = 2\iota^+$, $b = -\iota^- - c - d$. Inserting shows that P is given by $S_{21}(3, \iota)$ and its entries satisfy the required estimates. Moreover, we have

$$u^+ = \left[\iota^+,\, \iota^+,\, 1\right], \qquad u^- = \left[2c + \iota^-,\, 2d + 2\iota^+ + \iota^-,\, -2\right].$$

These must be integral primitive vectors. Consequently, ι^- is odd and we obtain that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

Case 2.2: $a = 2\iota^+$, $b = -2\iota^- - c - d$. Inserting shows that the matrix P arises from $S_{22}(3,\iota)$. Moreover, the linear forms $u^\pm$ are given by

$$u^+ = \left[\iota^+,\, \iota^+,\, 1\right], \qquad\qquad u^- = \left[c + \iota^-,\, d + \iota^+ + \iota^-,\, -1\right].$$

Thus, there are no divisibility conditions on $\iota^\pm$ and we see that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

We showed that every full intrinsic quadric surface of Picard number three is isomorphic to some $X(P)$ with P as in the assertion. Moreover, as $x_0, x_1, x_2 \in X(P)$ are all of local Gorenstein index one, see [17, Prop. 8.9 (iii)], we obtain that $X(P)$ has Gorenstein index $\iota = \operatorname{lcm}(\iota^+, \iota^-)$. Conversely, one directly checks that every matrix P from the assertion defines a full intrinsic quadric surface of Picard number three and Gorenstein index $\iota = \operatorname{lcm}(\iota^+, \iota^-)$.

Finally, we want to see that the matrices P listed in the assertion define pairwise non-isomorphic $X(P)$. According to Proposition 3.4.4, this amounts to showing that the sets $S_{ij}(3,\iota)$ are pairwise disjoint. We use Proposition 3.4.3 to compare the local Gorenstein indices $\iota^\pm$ and the local class group orders $\operatorname{cl}(X, x^\pm)$:

	$S_{11}(3,\iota)$	$S_{12}(3,\iota)$	$S_{21}(3,\iota)$	$S_{22}(3,\iota)$
$(\iota^+, \operatorname{cl}(X, x^+))$	(ι^+, ι^+)	(ι^+, ι^+)	$(\iota^+, 2\iota^+)$	$(\iota^+, 2\iota^+)$
$(\iota^-, \operatorname{cl}(X, x^-))$	(ι^-, ι^-)	$(\iota^-, 2\iota^-)$	(ι^-, ι^-)	$(\iota^-, 2\iota^-)$

The listed pairs are invariants of the surface up to switching x^+ and x^-. Thus, we see that $S_{11}(3,\iota)$ as well as $S_{22}(3,\iota)$ has trivial intersection with any other $S_{ij}(3,\iota)$. For $S_{12}(3,\iota)$ observe $\iota^+ < 2\iota^-$ as ι^+ is odd. Similarly, for $S_{21}(3,\iota)$, we have $2\iota^+ < \iota^-$. Thus, in both cases, $\operatorname{cl}(X, x^+)$ is the strictly smallest of $\operatorname{cl}(X, x^\pm)$. It follows that $S_{12}(3,\iota)$ and $S_{21}(3,\iota)$ intersect trivially. ☐

Example 3.4.6. Consider the full intrinsic quadric surfaces X and X' of Picard number three given by the defining matrices

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}, \qquad P' = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}.$$

Then X stems from the series $S_{12}(3,\iota)$ and X' from $S_{22}(3,\iota)$. Theorem 3.4.5 yields that X and X' are the only Gorenstein full intrinsic quadric surfaces with $\rho(X) = 3$.

Proposition 3.4.7. *Let $X = X(P)$ arise from Construction 3.4.1. At most the prime divisors $D_1^X, \ldots, D_6^X \subseteq X$ are contractible and all possible contractions are projective toric surfaces of Picard number three. More precisely,*

$$D_1^X\colon\ b \ge 1 \qquad\qquad D_2^X\colon\ a+c+d \le -1 \qquad\qquad D_3^X\colon\ a+c \ge 1$$

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ b & b+c & 0 & d \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a & a+c & 0 & d \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a+c & b+c & 0 & d \end{bmatrix}$$

$D_4^X: \ b+d \leq -1$ $\qquad$ $D_5^X: \ a+d \geq 1$ $\qquad$ $D_6^X: \ b+c \leq -1$

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ a & b & 0 & d \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a+d & b+d & 0 & c \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a & b & 0 & c \end{bmatrix}$$

gives us for each D_i^X the characterizing property of contractibility in terms of the entries a, b, c, d of P and, for the case that D_i^X is contractible, also the generator matrix of the contracted surface.

Proof. One succeeds by the same arguments as in the proof of Proposition 3.3.7. ☐

A normal surface singularity is of *type A_n* if the exceptional divisor of its minimal resolution is a string of n smooth rational curves, each of self intersection -2.

Remark 3.4.8. Consider the two Gorenstein full intrinsic quadric surfaces X and X' of Picard number three from Example 3.4.6.

(i) On X, the contractible divisors are D_2^X, D_4^X and D_6^X. In each case, the contracted surface is $\mathbb{P}^1 \times \mathbb{P}^1$.

(ii) On X', the contractible divisors are $D_3^{X'}$, $D_4^{X'}$, $D_5^{X'}$ and $D_6^{X'}$. In each case, the contraction is the toric del Pezzo surface of Picard number 2 with two singularities, both of type A_1.

3.5 Geometry of full intrinsic quadric surfaces

We present direct applications of Theorem 3.2.5, 3.3.5 and 3.4.5, exploring the geometry of full intrinsic quadric surfaces. In Theorem 3.5.16 we determine the weighted resolution graphs for the canonical resolution of singularities. Moreover, Theorem 3.5.3, 3.5.23 and 3.5.29, 3.5.30, 3.5.31 give explicit upper and lower bounds on the degree, the log canonicity and the Picard index in terms of the Gorenstein index. Finally, Theorem 3.5.32, 3.5.43, 3.5.48 characterize the existence of Kähler-Einstein metrics in terms of the Gorenstein index.

We recall that a *del Pezzo surface* is a normal projective surface X admitting an ample anticanonical divisor $-\mathcal{K}_X$. Moreover, a del Pezzo surface X is *log terminal* if all the exceptional divisors of its minimal resolution of singularities have discrepancies strictly bigger than -1; if so then one refers to X also as a *log del Pezzo surface.*

Proposition 3.5.1. *Every full intrinsic quadric surface X is a log del Pezzo surface.*

Proof. We may assume $X = X(P)$. Then log terminality is a direct consequence of [17, Cor. 8.12]. According to the possible values of the Picard number $\rho = \rho(X)$, the degree $\mu \in \mathrm{Cl}(X)$ of the defining quadric of X is given as

$$\mu \;=\; \begin{cases} w_1 + w_2 = 2w_3 = 2w_4, & \rho = 1, \\ w_1 + w_2 = w_3 + w_4 = 2w_5, & \rho = 2, \\ w_1 + w_2 = w_3 + w_4 = w_5 + w_6, & \rho = 3, \end{cases}$$

where $w_i = \deg(T_i) \in \mathrm{Cl}(X)$. Due to Remark 1.2.10 the anticanonical class of X equals $w_1 + \ldots + w_{\rho+3} - \mu$ and thus is a positive multiple of μ. From [3, Prop. 3.3.2.9] we infer that the cone of movable divisor classes of X is given by

$$\mathrm{Mov}(X) \;=\; \bigcap_{i=1}^{\varrho+3} \tau_i, \qquad \tau_i \;\coloneqq\; \mathrm{cone}(w_j;\; j \neq i) \;\subseteq\; \mathrm{Cl}_{\mathbb{Q}}(X).$$

Observe that μ is an interior point of each τ_i. All involved cones are of full dimension; see Remark 1.1.13. Hence we obtain that μ is an interior point of $\mathrm{Mov}(X)$. Thus, [3, Prop. 3.3.2.9, Thm. 4.3.3.5] show that μ, and hence the anticanonical class of X, is ample. □

Remark 3.5.2. The surfaces from Example 3.2.6, 3.3.6, 3.4.6 are the only Gorenstein two-dimensional full intrinsic quadrics. By Proposition 3.5.1 they are all log del Pezzo and thus we can recover them as well as the only full intrinsic quadrics in the classification of all rational Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces [3, Thms. 5.4.4.2 to 5.4.4.5].

3.5.1. The anticanonical degree

• Recall that the *(anticanonical) degree* of a del Pezzo surface X is the self intersection number of an anticanonical divisor of X.

Let X be a $\mathbb{K}^*$-surface as in Construction 1.2.1. Then due to Proposition 1.2.7 X inherits $\mathbb{Q}$-factoriality from its ambient toric variety Z. Moreover, X has a complete intersection complete intersection Cox ring, and we can compute intersection numbers according to [3, Constr. 3.3.3.4]; see also [3, Sect. 5.4.2] and [17, Sum 7.7].

The results of this section are summarised in the following theorem relating the degree to the local Gorenstein indices; see Proposition 3.5.6, Proposition 3.5.10 and Proposition 3.5.14 for the proof.

Theorem 3.5.3. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.2.5, 3.3.5 or 3.4.5. Then the degree $\mathcal{K}_X^2$ of X is given as*

$$\begin{array}{lllll} \rho = 1: & \mathcal{K}_X^2 = \frac{1}{\iota^+} + \frac{1}{\iota^-}, & \eta \in S_{11}(1,\iota), & \mathcal{K}_X^2 = \frac{1}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{12}(1,\iota), \\ & \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{1}{\iota^-}, & \eta \in S_{21}(1,\iota), & \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{22}(1,\iota), \\ \rho = 2: & \mathcal{K}_X^2 = \frac{9}{2\iota^+} + \frac{9}{2\iota^-}, & \eta \in S_{11}(2,\iota), & \mathcal{K}_X^2 = \frac{9}{2\iota^+} + \frac{3}{2\iota^-}, & \eta \in S_{12}(2,\iota), \\ & \mathcal{K}_X^2 = \frac{3}{2\iota^+} + \frac{9}{2\iota^-}, & \eta \in S_{21}(2,\iota), & \mathcal{K}_X^2 = \frac{3}{2\iota^+} + \frac{3}{2\iota^-}, & \eta \in S_{22}(2,\iota), \\ \rho = 3: & \mathcal{K}_X^2 = \frac{4}{\iota^+} + \frac{4}{\iota^-}, & \eta \in S_{11}(3,\iota), & \mathcal{K}_X^2 = \frac{4}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{12}(3,\iota), \\ & \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{4}{\iota^-}, & \eta \in S_{21}(3,\iota), & \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{22}(3,\iota). \end{array}$$

Here, ρ is the Picard number, ι the Gorenstein index of X and $\iota^{\pm}$ the local Gorenstein index of $x^{\pm} \in X$. Moreover, we obtain the following upper and lower bounds:

$$\begin{aligned} \rho = 1: &\quad \tfrac{2}{\iota} \leq \mathcal{K}_X^2 \leq 1 + \tfrac{4}{\iota}, \\ \rho = 2: &\quad \tfrac{3}{\iota} \leq \mathcal{K}_X^2 \leq \tfrac{9}{2} + \tfrac{9}{2\iota}, \\ \rho = 3: &\quad \tfrac{4}{\iota} \leq \mathcal{K}_X^2 \leq 4 + \tfrac{4}{\iota}. \end{aligned}$$

We treat the cases of Picard number one, two and three separately. The combination of the respective lemmata and proposition then gives us Theorem 3.5.3.

Lemma 3.5.4. *Let $X = X(P)$ arise from Construction 3.2.1. Then the intersection numbers of the curves $D_i^X, i = 1, 2, 3, 4$ are given as*

	D_1^X	D_2^X	D_3^X	D_4^X
D_1^X	$\frac{1}{a+1} - \frac{1}{a-b}$	$\frac{1}{a-b}$	$\frac{1}{2(a+1)}$	$\frac{1}{2(a+1)}$
D_2^X	$\frac{1}{a-b}$	$-\frac{1}{b+1} - \frac{1}{a-b}$	$-\frac{1}{2(b+1)}$	$-\frac{1}{2(b+1)}$
D_3^X	$\frac{1}{2(a+1)}$	$-\frac{1}{2(b+1)}$	$\frac{1}{4}\left(\frac{1}{a+1} - \frac{1}{b+1}\right)$	$\frac{1}{4}\left(\frac{1}{a+1} - \frac{1}{b+1}\right)$
D_4^X	$\frac{1}{2(a+1)}$	$-\frac{1}{2(b+1)}$	$\frac{1}{4}\left(\frac{1}{a+1} - \frac{1}{b+1}\right)$	$\frac{1}{4}\left(\frac{1}{a+1} - \frac{1}{b+1}\right)$

Proof. This is a result of the formulas given by [3, Prop. 5.4.2.1, Cor. 5.4.2.2], cf. [17, Sum. 7.7], where we are simply plugging in the values

$$\begin{array}{llllllll} l_{01} & = 1, & l_{02} & = 1, & l_{11} & = 2, & l_{21} & = 2, \\ m_{01} & = a, & m_{02} & = b, & m_{11} & = \frac{1}{2}, & m_{21} & = \frac{1}{2}, \\ m^{+} & = a+1, & m^{-} & = b+1. & & & & \end{array}$$

□

Proposition 3.5.5. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ of Picard number one with P_η as in Theorem 3.2.5. Then the degree $\mathcal{K}_X^2$ of X is given by*

$$\begin{array}{llll} \mathcal{K}_X^2 = \frac{1}{\iota^+} + \frac{1}{\iota^-}, & \eta \in S_{11}(1,\iota), & \mathcal{K}_X^2 = \frac{1}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{12}(1,\iota), \\ \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{1}{\iota^-}, & \eta \in S_{21}(1,\iota), & \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{22}(1,\iota). \end{array}$$

Proof. According to Remark 1.2.10 we have the following anticanonical divisor

$$-\mathcal{K}_X = D_3^X + D_4^X.$$

By using the self intersection numbers from Lemma 3.5.4, we get

$$\mathcal{K}_X^2 = D_3^X \cdot D_3^X + 2D_3^X \cdot D_4^X + D_4^X \cdot D_4^X = \tfrac{1}{a+1} - \tfrac{1}{b+1};$$

compare Proposition 1.4.4. According to Theorem 3.2.5 we can express the variables a, b of the generator matrix P from Construction 3.2.1 in terms of the local Gorenstein indices of $X(P)$

$$(a,b) \;=\; \begin{cases} (\iota^+ - 1, -\iota^- - 1) & \text{, if } \eta \in S_{11}(1,\iota), \\ \left(\iota^+ - 1, -\frac{\iota^- + 2}{2}\right) & \text{, if } \eta \in S_{12}(1,\iota), \\ \left(\frac{\iota^+ - 2}{2}, -\iota^- - 1\right) & \text{, if } \eta \in S_{21}(1,\iota), \\ \left(\frac{\iota^+ - 2}{2}, -\frac{\iota^- + 2}{2}\right) & \text{, if } \eta \in S_{22}(1,\iota). \end{cases}$$

This yields the desired presentation of the anticanonical self intersection. $\square$

Proposition 3.5.6. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ of Picard number one with P_η as in Theorem 3.2.5. Then the degree $\mathcal{K}_X^2$ of X is bounded by*

$$\begin{array}{llll} \mathcal{K}_X^2 \;\le\; 1 + \frac{1}{\iota}, & \eta \;\in\; S_{11}(1,\iota), & \mathcal{K}_X^2 \;\le\; 1 + \frac{2}{\iota}, & \eta \;\in\; S_{12}(1,\iota), \\ \mathcal{K}_X^2 \;\le\; \frac{1}{2} + \frac{4}{\iota}, & \eta \;\in\; S_{21}(1,\iota), & \mathcal{K}_X^2 \;\le\; \frac{1}{2} + \frac{2}{\iota}, & \eta \;\in\; S_{22}(1,\iota). \end{array}$$

The bounds are attained precisely in the following cases

$$\begin{array}{ll} \iota^+ \;=\; 1 \textit{ for } \eta \;\in\; S_{11}(1,\iota), & \iota^+ \;=\; 1 \textit{ for } \eta \;\in\; S_{12}(1,\iota), \\ \iota^+ \;=\; 4 \textit{ for } \eta \;\in\; S_{21}(1,\iota), & \iota^+ \;=\; 4 \textit{ for } \eta \;\in\; S_{22}(1,\iota). \end{array}$$

Proof. First, we treat the case $(\iota^+, \iota^-) \in S_{11}(1,\iota)$. Then one has $1 \le \iota^+ \le \iota^-$. Hence we get $\iota^- - 1 \le \iota^+(\iota^- - 1)$. We add $\iota^+ + 1$ on both sides of the inequality to get

$$\iota^+ + \iota^- \;\le\; \iota^+\iota^- + 1. \tag{3.5.6.1}$$

Further, we note $\iota = \operatorname{lcm}(\iota^+, \iota^-)$. This yields $\iota \le \iota^+\iota$ and $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$. Together with the description of $\mathcal{K}_X^2$ from Proposition 3.5.5 and the inequality (3.5.6.1), we obtain

$$\mathcal{K}_X^2 \;=\; \frac{1}{\iota^+} + \frac{1}{\iota^-} \;=\; \frac{\iota^+ + \iota^-}{\iota^+\iota^-} \;\le\; \frac{\iota^+\iota^- + 1}{\iota^+\iota^-} \;\le\; 1 + \frac{1}{\iota}.$$

Equality in (3.5.6.1) holds if and only if $\iota^+ + \iota^- = \iota^+\iota^- + 1$. Equality in the second estimate holds if and only if $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Then, both conditions hold if and only if $\iota^+ = 1$.

Next, we treat the case $(\iota^+, \iota^-) \in S_{12}(1,\iota)$. We obtain $1 \le \iota^+$ and $4 \le \iota^-$. This yields $\iota^- - 2 \le \iota^+(\iota^- - 2)$. Adding $2\iota^+ + 2$ to the inequality leads us to

$$2\iota^+ + \iota^- \;\le\; \iota^+\iota^- + 2.$$

As above, we note $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$. By using Proposition 3.5.5 and the above inequality, we conclude

$$\mathcal{K}_X^2 \;=\; \frac{1}{\iota^+} + \frac{2}{\iota^-} \;=\; \frac{2\iota^+ + \iota^-}{\iota^+\iota^-} \;\le\; \frac{\iota^+\iota^- + 2}{\iota^+\iota^-} \;\le\; 1 + \frac{2}{\iota}.$$

The first inequality is an equality if and only if $2\iota^+ + \iota^- = \iota^+\iota^- + 2$. This is equivalent to $\iota^- - 2 = \iota^+(\iota^- - 2)$. Equality in the second inequality holds if and only if $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Both conditions hold if and only if $\iota^+ = 1$.

Next, we consider the case $(\iota^+, \iota^-) \in S_{21}(1, \iota)$. Then we get $4 \le \iota^+$ and $3 \le \iota^-$. Consequently, one obtains $4(\iota^- - 2) \le \iota^+(\iota^- - 2)$. We add $2\iota^+ + 8$ on both sides and this yields

$$2\iota^+ + 4\iota^- \ \le\ \iota^+\iota^- + 8.$$

By combining Proposition 3.5.5, the above inequality and $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$, we see

$$\mathcal{K}_X^2 \ =\ \tfrac{2}{\iota^+} + \tfrac{1}{\iota^-} \ =\ \tfrac{4\iota^- + 2\iota^+}{2\iota^+\iota^-} \ \le\ \tfrac{\iota^+\iota^- + 8}{2\iota^+\iota^-} \ \le\ \tfrac{1}{2} + \tfrac{4}{\iota}.$$

Equality holds if and only if $4(\iota^- - 2) = \iota^+(\iota^- - 2)$ and $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Then both statements hold if and only if $\iota^+ = 4$.

Lastly, we treat the case $(\iota^+, \iota^-) \in S_{22}(1, \iota)$. So one has $4 \mid \iota^+$ and $4 \mid \iota^-$. We treat the cases $8 \le \iota^+ \le \iota^-$ and $\iota^+ = 4$ separately. First, let $8 \le \iota^+ \le \iota^-$. This implies $\frac{4\iota^- - 4}{\iota^- - 4} \le 8$ and $8 \le \iota^+$. Hence, we get $4(\iota^- - 1) \le 8(\iota^- - 4)$ and $4(\iota^- - 1) \le \iota^+(\iota^- - 4)$. We add $4\iota^+ + 4$ on both sides of the last inequality to obtain

$$4\iota^+ + 4\iota^- \le \iota^+\iota^- + 4.$$

From this inequality, Proposition 3.5.5 and from $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$, we infer that

$$\mathcal{K}_X^2 \ =\ \tfrac{2}{\iota^+} + \tfrac{2}{\iota^-} \ =\ \tfrac{2\iota^+ + 2\iota^-}{\iota^+\iota^-} \ \le\ \tfrac{\iota^+\iota^- + 4}{2\iota^+\iota^-} \ \le\ \tfrac{1}{2} + \tfrac{2}{\iota}.$$

The first inequality is attained if and only if $4(\iota^- - 1) = \iota^+(\iota^- - 4)$. The second if and only if $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Thus, we have equality for both inequalities if and if $\iota^+ = 4$. This would contradict to the assumption $8 \le \iota^+ \le \iota^-$. Now, let $(\iota^+, \iota^-) \in S_{22}(1, \iota)$ and $\iota^+ = 4$. Then we have $\iota = \iota^-$ and

$$\mathcal{K}_X^2 \ =\ \tfrac{2}{\iota^+} + \tfrac{2}{\iota^-} \ =\ \tfrac{1}{2} + \tfrac{2}{\iota^-} \ =\ \tfrac{1}{2} + \tfrac{2}{\iota}.$$

□

For convenience, we explicitly list a series of maximizers.

Example 3.5.7. Let ι_1 be an odd and let ι_2 be an even integer. Consider the generator matrices

$$P_{11}(\iota_1) \ :=\ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 0 & -\iota_1 - 1 & 1 & 1 \end{bmatrix}, \qquad P_{12}(\iota_2) \ :=\ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 0 & -\frac{\iota_2}{2} - 1 & 1 & 1 \end{bmatrix},$$

$$P_{21}(\iota_1) \ :=\ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 1 & -\iota_1 - 1 & 1 & 1 \end{bmatrix}, \qquad P_{22}(\iota_2) \ :=\ \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 1 & -\frac{\iota_2}{2} - 1 & 1 & 1 \end{bmatrix}.$$

Then we have $(1, \iota_1) \in S_{i1}(1, \iota)$ and $(4, \iota_2) \in S_{i2}(1, \iota)$ and the generator matrices corresponding to these tuples give us full intrinsic quadric surfaces $X_{ij}(\iota_j) := X(P_{ij}(\iota_j))$ of Picard number one. Further, the surface $X_{ij}(\iota_j)$ has Gorenstein index $\iota = \iota_j$ and is of the following degree

$$\begin{array}{rclcrcl} \mathcal{K}^2_{X_{11}(\iota_1)} & = & 1 + \frac{1}{\iota}, & \qquad & \mathcal{K}^2_{X_{12}(\iota_2)} & = & 1 + \frac{2}{\iota}, \\ \mathcal{K}^2_{X_{21}(\iota_1)} & = & \frac{1}{2} + \frac{4}{\iota}, & \qquad & \mathcal{K}^2_{X_{22}(\iota_2)} & = & \frac{1}{2} + \frac{2}{\iota}. \end{array}$$

Lemma 3.5.8. *Let $X = X(P)$ arise from Construction 3.3.1. Then the intersection numbers of the curves $D_i^X, i = 1, \ldots, 5$ are given by*

	D_1^X	D_2^X	D_3^X	D_4^X	D_5^X
D_1^X	$\frac{2}{2a+1} - \frac{1}{a-b}$	$\frac{1}{a-b}$	$\frac{2}{2a+1}$	0	$\frac{1}{2a+1}$
D_2^X	$\frac{1}{a-b}$	$\frac{1}{b-a} - \frac{2}{1+2b+2c}$	0	$-\frac{2}{1+2b+2c}$	$-\frac{1}{1+2b+2c}$
D_3^X	$\frac{2}{2a+1}$	0	$\frac{1}{c} + \frac{2}{2a+1}$	$-\frac{1}{c}$	$\frac{1}{2a+1}$
D_4^X	0	$-\frac{2}{1+2b+2c}$	$-\frac{1}{c}$	$\frac{1}{c} - \frac{2}{1+2b+2c}$	$\frac{1}{2b+2c+1}$
D_5^X	$\frac{1}{2a+1}$	$-\frac{1}{1+2b+2c}$	$\frac{1}{2a+1}$	$\frac{1}{2b+2c+1}$	$\frac{1}{4a+2} - \frac{1}{2+4b+4c}$

Proof. To obtain the intersection numbers we enter the formulas from [3, Prop. 5.4.2.1, Cor. 5.4.2.2] or [17, Sum. 7.7] with

$$\begin{array}{rclrclrclrclrcl} l_{01} & = & 1, & l_{02} & = & 1, & l_{11} & = & 1, & l_{12} & = & 1, & l_{21} & = & 2, \\ m_{01} & = & a, & m_{02} & = & b, & m_{11} & = & 0, & m_{12} & = & c, & m_{21} & = & \frac{1}{2}, \\ m^+ & = & a+1, & m^- & = & b+c+1. & & & & & & & & & \end{array}$$

□

Proposition 3.5.9. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ of Picard number two with P_η as in Theorem 3.3.5. Then the degree $\mathcal{K}_X^2$ of X is given by*

$$\begin{array}{rclrcl} \mathcal{K}_X^2 & = & \frac{9}{2\iota^+} + \frac{9}{2\iota^-}, \quad \eta \in S_{11}(2, \iota), & \qquad \mathcal{K}_X^2 & = & \frac{9}{2\iota^+} + \frac{3}{2\iota^-}, \quad \eta \in S_{12}(2, \iota), \\ \mathcal{K}_X^2 & = & \frac{3}{2\iota^+} + \frac{9}{2\iota^-}, \quad \eta \in S_{21}(2, \iota), & \qquad \mathcal{K}_X^2 & = & \frac{3}{2\iota^+} + \frac{3}{2\iota^-}, \quad \eta \in S_{22}(2, \iota). \end{array}$$

Proof. Remark 1.2.10 yields the anticanonical divisor

$$-\mathcal{K}_X = D_3^X + D_4^X + D_5^X.$$

Then, by using Lemma 3.5.8 we obtain

$$\begin{array}{rcl} \mathcal{K}_X^2 & = & D_3 \cdot D_3 + 2D_3 \cdot D_4 + 2D_3 \cdot D_5 + D_4 \cdot D_4 + 2D_4 \cdot D_5 + D_5 \cdot D_5 \\ & = & \frac{9}{4a+2} - \frac{9}{2+4b+4c}. \end{array}$$

Theorem 3.2.5 tells us that we can express the variables a, b, c of the generator matrix P from Construction 3.3.1 by the local Gorenstein indices of $X(P)$

$$(a, b+c) \;=\; \begin{cases} \left(\frac{\iota^+-1}{2}, \frac{-\iota^--1}{2}\right) & \text{, if } \eta \in S_{11}(2,\iota), \\ \left(\frac{\iota^+-1}{2}, \frac{-3\iota^--1}{2}\right) & \text{, if } \eta \in S_{12}(2,\iota), \\ \left(\frac{3\iota^+-1}{2}, \frac{-\iota^--1}{2}\right) & \text{, if } \eta \in S_{21}(2,\iota), \\ \left(\frac{3\iota^+-1}{2}, \frac{-3\iota^--1}{2}\right) & \text{, if } \eta \in S_{22}(2,\iota). \end{cases}$$

By inserting these values, we get the desired presentations. □

Proposition 3.5.10. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ of Picard number two with P_η as in Theorem 3.3.5. Then the degree $\mathcal{K}_X^2$ of X is bounded by*

$$\begin{array}{llll} \mathcal{K}_X^2 \;\le\; \frac{9}{2}+\frac{9}{2\iota}, & \eta \;\in\; S_{11}(2,\iota), & \mathcal{K}_X^2 \;\le\; \frac{9}{2}+\frac{3}{2\iota}, & \eta \;\in\; S_{12}(2,\iota), \\ \mathcal{K}_X^2 \;\le\; \frac{3}{2}+\frac{9}{2\iota}, & \eta \;\in\; S_{21}(2,\iota), & \mathcal{K}_X^2 \;\le\; \frac{3}{2}+\frac{3}{2\iota}, & \eta \;\in\; S_{22}(2,\iota). \end{array}$$

The bounds are attained in each case if and only if $\iota^+ = 1$.

Proof. First, we treat the case $(\iota^+, \iota^-, c) \in S_{11}(2,\iota)$. Then we have $1 \le \iota^+ \le \iota^-$. Hence we get $\iota^- - 1 \le \iota^+(\iota^- - 1)$. We add $\iota^+ + 1$ on both sides of the inequality to get

$$\iota^+ + \iota^- \;\le\; \iota^+\iota^- + 1. \tag{3.5.10.1}$$

We derive $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ from $\iota \le \iota^+\iota$. Together with the representation of $\mathcal{K}_X^2$ from Proposition 3.5.9 we get

$$\mathcal{K}_X^2 \;=\; \frac{9}{2\iota^+} + \frac{9}{2\iota^-} \;=\; \frac{9\iota^+ + 9\iota^-}{2\iota^+\iota^-} \;\le\; \frac{9\iota^+\iota^- + 9}{2\iota^+\iota^-} \;\le\; \frac{9}{2} + \frac{9}{2\iota}.$$

Equality in the first estimate holds if and only if $\iota^- - 1 = \iota^+(\iota^- - 1)$. The second inequality is an equality if and only if $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Both conditions hold true if and only if $\iota^+ = 1$.

Now we consider $(\iota^+, \iota^-, c) \in S_{12}(2,\iota)$. So we get $1 \le \iota^+ \le 3\iota^-$. This implies $\iota^+ - 1 \le 3\iota^-(\iota^+ - 1)$. Adding $3\iota^- + 1$, we obtain

$$\iota^+ + 3\iota^- \le 3\iota^+\iota^- + 1.$$

We infer from Proposition 3.5.9, the above inequality and from $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ that

$$\mathcal{K}_X^2 \;=\; \frac{9}{2\iota^+} + \frac{3}{2\iota^-} \;=\; \frac{9\iota^- + 3\iota^+}{2\iota^+\iota^-} \;\le\; \frac{9\iota^+\iota^- + 3}{2\iota^+\iota^-} \;\le\; \frac{9}{2} + \frac{3}{2\iota}.$$

Equality holds if and only if $\iota^+ - 1 = 3\iota^-(\iota^+ - 1)$ and $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. This is equivalent to $\iota^+ = 1$.

Now, we consider $(\iota^+, \iota^-, c) \in S_{21}(2, \iota)$. Then we obtain $3 \le 3\iota^+ \le \iota^-$. Therefore $3(\iota^+ - 1) \le \iota^-(\iota^+ - 1)$. We add $\iota^- + 3$ and get

$$3\iota^+ + \iota^- \le \iota^+\iota^- + 3$$

By using Proposition 3.5.9, the above inequality and $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ we conclude

$$\mathcal{K}_X^2 \;=\; \tfrac{3}{2\iota^+} + \tfrac{9}{2\iota^-} \;=\; \tfrac{3\iota^- + 9\iota^+}{2\iota^+\iota^-} \;\le\; \tfrac{3\iota^+\iota^- + 9}{2\iota^+\iota^-} \;\le\; \tfrac{3}{2} + \tfrac{9}{2\iota}.$$

Equality holds if and only if $3(\iota^+ - 1) = \iota^-(\iota^+ - 1)$ and $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Hence equality holds if and only if $\iota^+ \;=\; 1$.

Lastly, we treat the case $(\iota^+, \iota^-, c) \in S_{22}(2, \iota)$. So we have $1 \le \iota^+ \le \iota^-$. (3.5.10.1) tells us $3\iota^- + 3\iota^+ \le 3\iota^+\iota^- + 3$. Thus, together with Proposition 3.5.9 and $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ we get

$$\mathcal{K}_X^2 \;=\; \tfrac{3}{2\iota^+} + \tfrac{3}{2\iota^-} \;=\; \tfrac{3\iota^- + 3\iota^+}{2\iota^+\iota^-} \;\le\; \tfrac{3\iota^+\iota^- + 3}{2\iota^+\iota^-} \;\le\; \tfrac{3}{2} + \tfrac{3}{2\iota}$$

Equality for the first inequality holds if and only if $\iota^- - 1 = \iota^+(\iota^- - 1)$. Whence, both equalities hold if and only if $\iota^+ = 1$. ☐

We conclude the case of Picard number two with a series of exlicit maximizers.

Example 3.5.11. Let i be an odd integer. Consider the generator matrices

$$\begin{aligned}
c_{11}(i) &:= -\tfrac{i-1}{2}, & c_{12}(i) &:= -\tfrac{3i-1}{2},\\
P_{11}(i) &:= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -\frac{i-1}{2} & 1 \end{bmatrix}, & P_{12}(i) &:= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -\frac{3i-1}{2} & 1 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
c_{21}(i) &:= -\tfrac{i+1}{2}, & c_{22}(i) &:= -\tfrac{3i+1}{2},\\
P_{21}(i) &:= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -\frac{i+1}{2} & 1 \end{bmatrix}, & P_{22}(i) &:= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -\frac{3i+1}{2} & 1 \end{bmatrix}.
\end{aligned}$$

Then we have triples corresponding to $P_{ij}(i)$ such that $(1, i, c_{ij}(i)) \in S_{ij}(2, \iota)$. Now set $X_{ij}(i) := X(P_{ij}(i))$ for the corresponding full intrinsic quadric surfaces of Picard number two. The Gorenstein index of the surface $X_{ij}(i)$ is $\iota = i$ and the surfaces are of the following degree

$$\begin{aligned}
\mathcal{K}_{X_{11}(i)}^2 &= \tfrac{9}{2} + \tfrac{9}{2\iota}, & \mathcal{K}_{X_{12}(i)}^2 &= \tfrac{9}{2} + \tfrac{3}{2\iota},\\
\mathcal{K}_{X_{21}(i)}^2 &= \tfrac{3}{2} + \tfrac{9}{2\iota}, & \mathcal{K}_{X_{22}(i)}^2 &= \tfrac{3}{2} + \tfrac{3}{2\iota}.
\end{aligned}$$

Lemma 3.5.12. *Let $X = X(P)$ arise from Construction 3.4.1. Then the intersection numbers of the curves $D_i^X, i = 1, \ldots, 6$ are given by*

	D_1^X	D_2^X	D_3^X	D_4^X	D_5^X	D_6^X
D_1^X	$\frac{1}{a} + \frac{1}{b-a}$	$\frac{1}{a-b}$	$\frac{1}{a}$	0	$\frac{1}{a}$	0
D_2^X	$\frac{1}{a-b}$	$\frac{1}{b-a} - \frac{1}{b+c+d}$	0	$-\frac{1}{b+c+d}$	0	$-\frac{1}{b+c+d}$
D_3^X	$\frac{1}{a}$	0	$\frac{1}{a} + \frac{1}{c}$	$-\frac{1}{c}$	$\frac{1}{a}$	0
D_4^X	0	$-\frac{1}{b+c+d}$	$-\frac{1}{c}$	$\frac{1}{c} - \frac{1}{b+c+d}$	0	$-\frac{1}{b+c+d}$
D_5^X	$\frac{1}{a}$	0	$\frac{1}{a}$	0	$\frac{1}{a} + \frac{1}{d}$	$-\frac{1}{d}$
D_6^X	0	$-\frac{1}{b+c+d}$	0	$-\frac{1}{b+c+d}$	$-\frac{1}{d}$	$\frac{1}{d} - \frac{1}{b+c+d}$

Proof. To obtain our statement, we insert $l_{ij} = 1$, for all $i = 0, 1, 2, j = 1, 2$, and the following values into the formulas from [3, Prop. 5.4.2.1], cf. [17, Sum. 7.7]

$$\begin{array}{rclrclrcl} m_{01} & = & a, & m_{02} & = & b, & m_{11} & = & 0, \\ m_{12} & = & c, & m_{21} & = & 0, & m_{22} & = & d, \\ m^{+} & = & a+1, & m^{-} & = & b+c+d. & & & \end{array}$$

□

Proposition 3.5.13. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ of Picard number three with P_η as in Theorem 3.4.5. Then the anticanonical self intersection of X is given by*

$$\begin{array}{rclrclrclrcl} \mathcal{K}_X^2 & = & \frac{4}{\iota^+} + \frac{4}{\iota^-}, & \eta & \in & S_{11}(3,\iota), & \mathcal{K}_X^2 & = & \frac{4}{\iota^+} + \frac{2}{\iota^-}, & \eta & \in & S_{12}(3,\iota), \\ \mathcal{K}_X^2 & = & \frac{2}{\iota^+} + \frac{4}{\iota^-}, & \eta & \in & S_{21}(3,\iota), & \mathcal{K}_X^2 & = & \frac{2}{\iota^+} + \frac{2}{\iota^-}, & \eta & \in & S_{22}(3,\iota). \end{array}$$

Proof. Remark 1.2.10 gives us the anticanonical divisor as

$$-\mathcal{K}_X = D_3^X + D_4^X + D_5^X + D_6^X.$$

We use the intersection numbers from Lemma 3.5.12 and this yields

$$\mathcal{K}_X^2 \;=\; \left(D_3^X + D_4^X + D_5^X + D_6^X\right)^2 \;=\; \frac{4}{a} - \frac{4}{b+c+d}.$$

We derive a description of a, b, c, d in terms of the local Gorenstein indices of $X(P)$ from Theorem 3.4.5, i.e.

$$(a, b+c+d) \;=\; \begin{cases} (\iota^+, -\iota^-) & \text{, if } \eta \in S_{11}(3,\iota), \\ (\iota^+, -2\iota^-) & \text{, if } \eta \in S_{12}(3,\iota), \\ (2\iota^+, -\iota^-) & \text{, if } \eta \in S_{21}(3,\iota), \\ (2\iota^+, -2\iota^-) & \text{, if } \eta \in S_{22}(3,\iota). \end{cases}$$

This yields the claim. □

Proposition 3.5.14. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ of Picard number three with P_η as in Theorem 3.4.5. Then the anticanonical self intersection of X is bounded by*

$$\begin{array}{llll} \mathcal{K}_X^2 \ \le \ 4+\frac{4}{\iota}, & \eta \ \in \ S_{11}(3,\iota), & \mathcal{K}_X^2 \ \le \ 4+\frac{2}{\iota}, & \eta \ \in \ S_{12}(3,\iota), \\ \mathcal{K}_X^2 \ \le \ 2+\frac{4}{\iota}, & \eta \ \in \ S_{21}(3,\iota), & \mathcal{K}_X^2 \ \le \ 2+\frac{2}{\iota}, & \eta \ \in \ S_{22}(3,\iota), \end{array}$$

The bounds are attained in each case if and only if $\iota^+ = 1$.

Proof. We begin with treating the case $(\iota^+, \iota^-, c, d) \in S_{11}(3,\iota)$. Then we have $1 \le \iota^+ \le \iota^-$. Therefore $\iota^- - 1 \le \iota^+(\iota^- - 1)$. We add $\iota^+ + 1$ to get

$$\iota^+ + \iota^- \ \le \ \iota^+\iota^- + 1. \qquad (3.5.14.1)$$

Together with the represenation of $\mathcal{K}_X^2$ from Proposition 3.5.13 and with and the inequality $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ we get

$$\mathcal{K}_X^2 \ = \ \frac{4}{\iota^+} + \frac{4}{\iota^-} \ = \ \frac{4\iota^+ + 4\iota^-}{\iota^+\iota^-} \ \le \ \frac{4\iota^+\iota^- + 4}{\iota^+\iota^-} \ \le \ 4 + \frac{4}{\iota}.$$

We obtain equality in the first inequality if and only if $\iota^- - 1 = \iota^+(\iota^- - 1)$. Equality for the second inequality holds if and only if $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Consequently both inequalities are attained if and only if $\iota^+ = 1$.

Next, we treat the case $(\iota^+, \iota^-, c, d) \in S_{12}(3,\iota)$. We obtain $1 \le \iota^+ \le 2\iota^-$. Hence $4\iota^- - 2 \le \iota^+(4\iota^- - 2)$. We add $2\iota^+ + 2$ to obtain $2\iota^+ + 4\iota^- \le 4\iota^+\iota^- + 2$. By combining this with Proposition 3.5.13 and with and the inequality $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ we conclude

$$\mathcal{K}_X^2 \ = \ \frac{4}{\iota^+} + \frac{2}{\iota^-} \ = \ \frac{4\iota^- + 2\iota^+}{\iota^+\iota^-} \ \le \ \frac{4\iota^+\iota^- + 2}{\iota^+\iota^-} \ \le \ 4 + \frac{2}{\iota}.$$

Equality in the first inequality holds if and only if $4\iota^- - 2 = \iota^+(4\iota^- - 2)$. Further, equality for the second inequality holds if and only if $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Then both conditions hold if and only if $\iota^+ = 1$.

The case $(\iota^+, \iota^-, c, d) \in S_{21}(3,\iota)$ works analogously. We have $2 \le 2\iota^+ \le \iota^-$ and thus $4(\iota^+ - 1) \le 2\iota^-(\iota^+ - 1)$. Therefore $4\iota^+ + 2\iota^- \le 2\iota^+\iota^- + 4$. Again by using Proposition 3.5.13 and the inequality $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ we obtain

$$\mathcal{K}_X^2 \ = \ \frac{2}{\iota^+} + \frac{4}{\iota^-} \ = \ \frac{2\iota^- + 4\iota^+}{\iota^+\iota^-} \ \le \ \frac{2\iota^+\iota^- + 4}{\iota^+\iota^-} \ \le \ 2 + \frac{4}{\iota}.$$

Equality for both inequalities holds if and only if $4(\iota^+ - 1) = 2\iota^-(\iota^+ - 1)$ and $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. This is equivalent to $\iota^+ = 1$.

Lastly, let $(\iota^+, \iota^-, c, d) \in S_{22}(3,\iota)$. Then we get $1 \le \iota^+ \le \iota^-$. We combine Proposition 3.5.13, the inequality $\frac{1}{\iota^+\iota^-} \le \frac{1}{\iota}$ and (3.5.14.1) to obtain

$$\mathcal{K}_X^2 \ = \ \frac{2}{\iota^+} + \frac{2}{\iota^-} \ = \ \frac{2\iota^- + 2\iota^+}{\iota^+\iota^-} \ \le \ \frac{2\iota^+\iota^- + 2}{\iota^+\iota^-} \ \le \ 2 + \frac{2}{\iota}.$$

Equality holds if and only if $\iota^- - 1 = \iota^+(\iota^- - 1)$ and $\operatorname{lcm}(\iota^+, \iota^-) = \iota^+\iota^-$. Hence this holds if and only if $\iota^+ = 1$. ☐

For convenience we also provide a series of exlicit maximizers in Picard number three.

Example 3.5.15. Let i be an odd integer. Consider the generator matrices

$$c_{11}(i) := -\tfrac{i-1}{2}, \qquad c_{12}(i) := -i,$$

$$P_{11}(i) := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -\frac{i-1}{2} & 0 & -\frac{i-1}{2} & 0 & -1 \end{bmatrix}, \qquad P_{12}(i) := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -i+1 & 0 & -i & 0 & -1 \end{bmatrix},$$

$$c_{21}(i) := -\tfrac{i+1}{2}, \qquad c_{22}(i) := -i,$$

$$P_{21}(i) := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2 & -\frac{i}{2}+\frac{3}{2} & 0 & -\frac{i+1}{2} & 0 & -1 \end{bmatrix}, \qquad P_{22}(i) := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2 & -i+1 & 0 & -i & 0 & -1 \end{bmatrix}.$$

Then we have $(1, i, c_{ij}(i), -1) \in S_{ij}(3, \iota)$. Set $X_{ij}(i) := X(P_{ij}(i))$ for the corresponding full intrinsic quadric surfaces of Picard number three. Then the Gorenstein index of the surface $X_{ij}(i)$ is $\iota = i$ and the surfaces are of the following degree

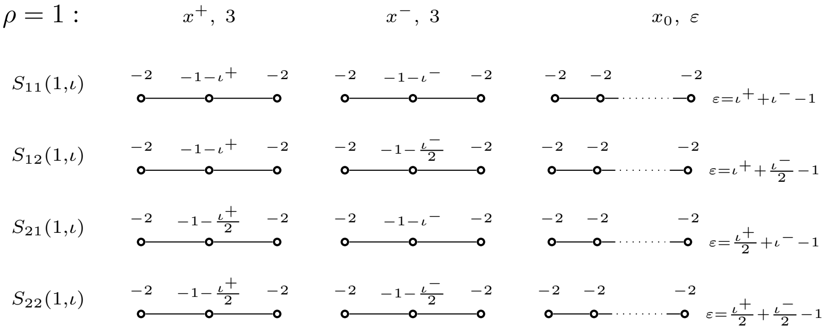
$$\begin{aligned} \mathcal{K}^2_{X_{11}(i)} &= 4 + \tfrac{4}{\iota}, & \mathcal{K}^2_{X_{12}(i)} &= 4 + \tfrac{2}{\iota}, \\ \mathcal{K}^2_{X_{21}(i)} &= 2 + \tfrac{4}{\iota}, & \mathcal{K}^2_{X_{22}(i)} &= 2 + \tfrac{2}{\iota}. \end{aligned}$$

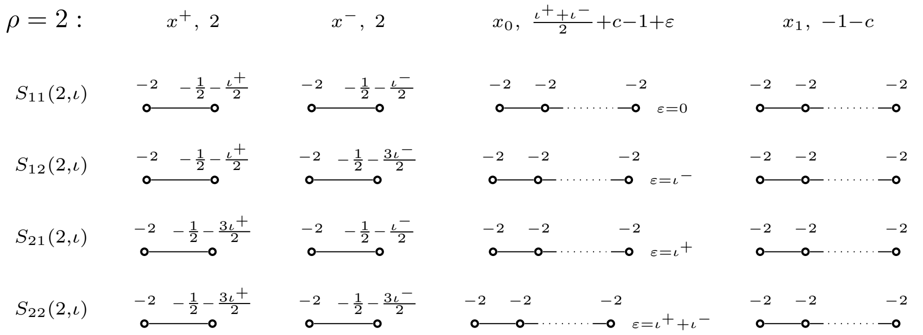
3.5.2. Singularities and resolution •

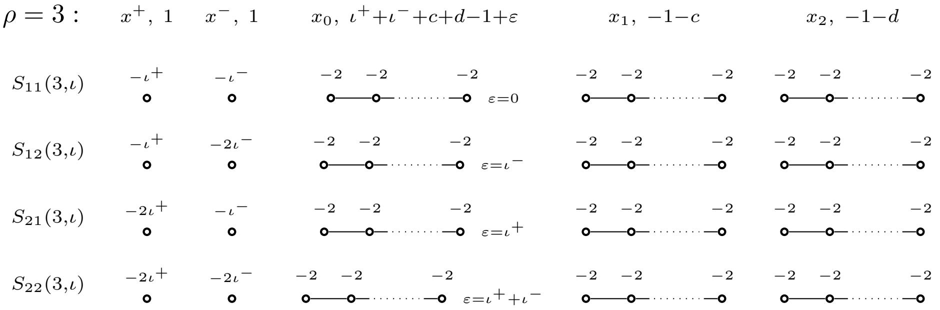
We turn to the singularities of full intrinsic quadric surfaces and determine their canonical resolution of singularities in the sense of [36, Sec. 3.2]; see also [3, Sec. 5.4.3].

Recall that the nodes of the resolution graph represent the irreducible components of the exceptional divisor over the corresponding singularity. For two curves we join the corresponding nodes with an edge if the curves intersect.

Theorem 3.5.16. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.2.5, Theorem 3.3.5 or Theorem 3.4.5 and its canonical resolution of singularities. Then the possible singularities $x^+, x^-, x_0, x_1, x_2 \in X$ have the following resolution graphs:*







Next to $x^{\pm}, x_i$ we place the number of exceptional curves, and the weights of the vertices are the self intersection numbers of the corresponding exceptional curves. The canonical resolution is minimal unless $x^+ \in X$ is smooth, where the latter happens if and only if

$$\iota^+ = 1, \quad \eta \in S_{11}(2,\iota) \cup S_{12}(2,\iota) \cup S_{11}(3,\iota) \cup S_{12}(3,\iota).$$

We recall the necessary background for the proof. The tropical variety of a full intrinsic quadric surface is given as

$$\mathrm{trop}(X) \;=\; \tau_0 \cup \tau_1 \cup \tau_2,$$

where its leaves are given by

$$\tau_0 \;:=\; \mathrm{cone}(e_0) + \mathrm{lin}(e_3), \qquad \tau_1 \;:=\; \mathrm{cone}(e_1) + \mathrm{lin}(e_3), \qquad \tau_2 \;:=\; \mathrm{cone}(e_2) + \mathrm{lin}(e_3).$$

Construction 3.5.17 (Canonical resolution of singularities; see also [3, Constr. 3.4.4.3, Constr. 5.4.3.2])**.** Let X be a $\mathbb{K}^*$-surface with generator matrix P as in Construction 1.2.1.

(i) Tropical step: We add $v^+ := e_3$ and $v^- := -e_3$ as new columns to the generator matrix P to obtain a new generator matrix P'. Let Σ' be the fan in $\mathbb{Z}^3$ having P' as generator matrix and $\mathrm{trop}(X)$ as its support. Then we have a canonical morphism $X' \to X$.

(ii) Toric step: Let Σ'' be the coarsest regular subdivision of Σ' with generator matrix P''. Then the corresponding surface X'' is smooth and we have a canonical morphism $X'' \to X'$.

(iii) The composition $X'' \to X$ of the tropical and the toric step yields the *canonical resolution of singularities* of X.

Remark 3.5.18 ([3, Rem. 5.4.3.3])**.** The tropical step replaces the elliptic fixed points with parabolic fixed point curves. According to [3, Prop. 3.4.4.6], the resulting surface X' is locally toric and hence admits at most toric singularities, each resolved by a chain of rational curves.

In the following we denote the prime divisors on X obtained by cutting down the toric divisors of the ambient toric variety Z by $D_{ij}^{X}, D_{ij}^{X'}, D_{ij}^{X''}$ in order to match the notation of the references [3,17]. Similarily, we denote by $D_{\pm}^{X'}, D_{\pm}^{X''}$ the prime divisors corresponding to the columns $v^{\pm}$ of the generator matrices P', P''.

Proposition 3.5.19. *Consider a full intrinsic quadric surface X as in Construction 3.2.1. Let $X'' \to X$ be its canonical resolution of singularities and let $X'' = X(P'')$ with the generator matrix P''. Then the following statements hold:*

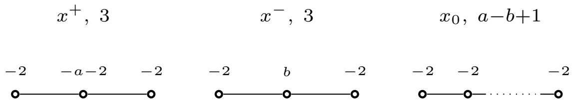
(i) *The matrix P'' is given by*

$$P'' = \begin{bmatrix} -1 & -1 & \dots & -1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ a & a-1 & \dots & b+1 & b & 1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

(ii) *We have the following self intersection numbers*

$$\begin{aligned} D_{ij}^{X''} \cdot D_{ij}^{X''} &= -2, \qquad \text{for all } i = 0,1,2,\ j = 2,\dots,n_i - 1, \\ D_{+}^{X''} \cdot D_{+}^{X''} &= -a-2, \\ D_{-}^{X''} \cdot D_{-}^{X''} &= b. \end{aligned}$$

(iii) *The resolution graph of the possible singularities $x^+, x^-, x_0 \in X$ looks as follows*



Proof. First, we show (i). To this end, we construct the new fan Σ' supported on the tropical variety from Construction 3.5.17 (i). We intersect the maximal cones $\sigma^+, \sigma^-, \tau_0$ of Σ from Construction 3.3.1 with the tropical variety and obtain the maximal cones of Σ' given by

$$\operatorname{cone}(v_1, e_3), \quad \operatorname{cone}(v_2, -e_3), \quad \operatorname{cone}(v_3, \pm e_3), \quad \operatorname{cone}(v_4, \pm e_3), \quad \operatorname{cone}(v_1, v_2).$$

As corresponding new generator matrix we get

$$P' \;:=\; \left[v'_{01}, v'_{02}, v'_{11}, v'_{21}, v^{+}, v^{-}\right] \;:=\; \begin{bmatrix} -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ a & b & 1 & 1 & 1 & -1 \end{bmatrix}.$$

Set $X' := X(P')$. Then we have no elliptic fixed points in X' and instead we have two fixed point curves $D^{X'}_{\pm}$. The only possible singular parabolic fixed points are the following:

$$\begin{array}{llll} x^{+}_{01} \in D^{X'}_{+} \cap D^{X'}_{01}, & x^{-}_{02} \in D^{X'}_{-} \cap D^{X'}_{02}, \\ x^{+}_{11} \in D^{X'}_{+} \cap D^{X'}_{11}, & x^{-}_{11} \in D^{X'}_{-} \cap D^{X'}_{11}, \\ x^{+}_{21} \in D^{X'}_{+} \cap D^{X'}_{21}, & x^{-}_{21} \in D^{X'}_{-} \cap D^{X'}_{21}. \end{array}$$

Moreover, the hyperbolic fixed point x_0 can be singular. According to [3, Prop. 3.4.4.6] and [17, Sum. 7.1] a point $x^{+}_{01}, x^{-}_{02}, x^{+}_{11}, x^{-}_{11}, x^{+}_{21}, x^{-}_{21}, x_0$ is singular if and only if its associated determinant

$$\begin{vmatrix} 0 & 1 \\ 1 & a \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ b & -1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}$$

differs from ±1. The new columns of the matrix P'', that means those not occurring in P', arise from resolving the singular points by regular subdivision of the associated cone:

$$\begin{array}{rlcl} x^{+}_{11}: & \operatorname{cone}((2,0,1),(0,0,1)) & = & \operatorname{cone}((2,0,1),(1,0,1)) \cup \operatorname{cone}((1,0,1),(0,0,1)), \\ x^{-}_{11}: & \operatorname{cone}((2,0,1),(0,0,-1)) & = & \operatorname{cone}((2,0,1),(1,0,0)) \cup \operatorname{cone}((1,0,0),(0,0,-1)), \\ x^{+}_{21}: & \operatorname{cone}((0,2,1),(0,0,1)) & = & \operatorname{cone}((0,2,1),(0,1,1)) \cup \operatorname{cone}((0,1,1),(0,0,1)), \\ x^{-}_{21}: & \operatorname{cone}((0,2,1),(0,0,-1)) & = & \operatorname{cone}((0,2,1),(0,1,0)), \cup \operatorname{cone}((0,1,0),(0,0,-1)), \\ x_0: & \operatorname{cone}((-1,-1,a),(-1,-1,b)) & = & \bigcup_{k=1}^{a-b} ((-1,-1,a+1-k),(-1,-1,a-k)). \end{array}$$

Next, we prove (ii). The self intersection numbers of the exceptional curves $D^{X''}_{ij}$, $i = 0,1,2, j = 2,\ldots,n_i - 1$ and $D^{X''}_{\pm}$ are obtained by inserting the following values into the formulas from [3, Prop. 5.4.2.1] resp. from [17, Sum 7.7]

$$\begin{array}{llllll} l_{0j} = 1, & j = 1,\ldots,a-b+1, \\ m_{0j} = a+1-j, & j = 1,\ldots,a-b+1, \\ l_{11} = 1, & l_{12} = 2, & l_{13} = 1, \\ l_{21} = 1, & l_{22} = 2, & l_{23} = 1, \\ m_{11} = 1, & m_{12} = \frac{1}{2}, & m_{13} = 0, \\ m_{21} = 1, & m_{22} = \frac{1}{2}, & m_{23} = 0. \end{array}$$

Lastly, the statement (iii) follows from combining (i) and (ii). □

Proposition 3.5.20. *Consider a full intrinsic quadric surface X as in Construction 3.3.1. Let $X'' \to X$ be its canonical resolution of singularities and let $X'' = X(P'')$ with the generator matrix P''. Then the following statements hold:*

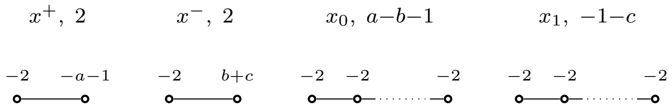
(i) *The matrix* P'' *is given by*

$$P'' \;:=\; \begin{bmatrix} -1 & -1 & \dots & -1 & -1 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 & 0 & \dots & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ a & a-1 & \dots & b+1 & b & 0 & -1 & \dots & c+1 & c & 1 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

(ii) *We have the following self intersection numbers*

$$\begin{aligned} D_{ij}^{X''} \cdot D_{ij}^{X''} &= -2, && \textit{for all } i = 0,1,2,\ j = 2,\dots,n_i - 1, \\ D_{+}^{X''} \cdot D_{+}^{X''} &= -a-1, \\ D_{-}^{X''} \cdot D_{-}^{X''} &= b+c. \end{aligned}$$

(iii) *The resolution graph of the possible singularities* $x^+, x^-, x_0, x_1 \in X$ *looks as follows*



Proof. We begin by proving (i). The fan Σ' from Construction 3.5.17 (i) is constructed by intersecting the maximal cones $\sigma^+, \sigma^-, \tau_0, \tau_1$ of Σ from Construction 3.3.1 with the tropical variety. This yields the maximal cones of Σ' given by

$$\operatorname{cone}(v_1, e_3), \quad \operatorname{cone}(v_3, e_3), \quad \operatorname{cone}(v_2, -e_3), \quad \operatorname{cone}(v_4, -e_3), \quad \operatorname{cone}(v_5, \pm e_3), \\ \operatorname{cone}(v_1, v_2), \quad \operatorname{cone}(v_3, v_4).$$

Then we obtain the generator matrix corresponding to Σ' given as

$$P' \;:=\; \left[v_{01}, v_{02}, v_{11}, v_{12}, v_{21}, v^+, v^-\right] \;:=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ a & b & 0 & c & 1 & 1 & -1 \end{bmatrix}.$$

Set $X' := X(P')$. Then we have two fixed point curves $D_{\pm}^{X'}$ in X' and six possibly singular fixed points:

$$\begin{array}{llll} x_{01}^+ \in D_+^{X'} \cap D_{01}^{X'}, & x_{02}^- \in D_-^{X'} \cap D_{02}^{X'}, \\ x_{11}^+ \in D_+^{X'} \cap D_{11}^{X'}, & x_{12}^- \in D_-^{X'} \cap D_{12}^{X'}, \\ x_{21}^+ \in D_+^{X'} \cap D_{21}^{X'}, & x_{21}^- \in D_-^{X'} \cap D_{21}^{X'}. \end{array}$$

Further, the hyperbolic fixed points x_0, x_1 are possibly singular. [3, Prop. 3.4.4.6] and [17, Sum. 7.1] tell us that $x_{01}^+, x_{02}^-, x_{11}^+, x_{12}^-, x_{21}^+, x_{21}^-, x_0, x_1$ are smooth if and only if the following determinants

$$\begin{vmatrix} 0 & 1 \\ 1 & a \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 \\ b & -1 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 \\ c & -1 \end{vmatrix}, \quad \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 \\ 0 & c \end{vmatrix}$$

equal ±1. We obtain P'' by adding new columns to P' which arise from regular subdivision of the cones corresponding to the singularities:

$$\begin{array}{rlcl}
x_{21}^{+}: & \operatorname{cone}((0,2,1),(0,0,1)) & = & \operatorname{cone}((0,2,1),(0,1,1)) \;\cup\; \operatorname{cone}((0,1,1),(0,0,1)),\\
x_{21}^{-}: & \operatorname{cone}((0,2,1),(0,0,-1)) & = & \operatorname{cone}((0,2,1),(0,1,0)), \;\cup \operatorname{cone}((0,1,0),(0,0,-1)),\\
x_0: & \operatorname{cone}((-1,-1,a),(-1,-1,b)) & = & \bigcup_{k=1}^{a-b}\left((-1,-1,a+1-k),(-1,-1,a-k)\right),\\
x_1: & \operatorname{cone}((1,0,0),(1,0,c)) & = & \bigcup_{k=0}^{-c}\left((1,0,-k),(1,0,-k+1)\right).
\end{array}$$

Next, we show (ii). We apply the formulas from [3, Prop. 5.4.2.1], see also [17, Sum 7.7], and get the desired self intersection numbers by plugging in the values

$$\begin{array}{llll}
l_{0j} = l_{1k} = 1, & j = 1,\dots,a-b+1, & k = 1,\dots,-c+1, & \\
m_{0j} = a+1-j, & j = 1,\dots,a-b+1, & & \\
m_{1j} = -c+1-j, & j = 1,\dots,-c, & & \\
l_{21} \;=\; 1, & l_{22} \;=\; 2, & l_{23} \;=\; 1, & \\
m_{21} \;=\; 1, & m_{22} \;=\; \frac{1}{2}, & m_{23} \;=\; 0. &
\end{array}$$

Furthermore, (iii) is a consequence of (i) and (ii). □

Proposition 3.5.21. *Consider a full intrinsic quadric surface X as in Construction 3.4.1. Let $X'' \to X$ be its canonical resolution of singularities and let $X'' = X(P'')$ with the generator matrix P''. Then the following statements hold:*

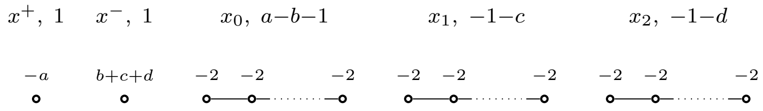
(i) *The matrix P'' is given by*

$$P'' := \begin{bmatrix} -1 & -1 & \dots & -1 & -1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 1 & 0 & 0 \\ a & a-1 & \dots & b+1 & b & 0 & \dots & c+1 & c & 0 & \dots & d+1 & d & 1 & -1 \end{bmatrix}.$$

(ii) *We have the following self intersection numbers*

$$\begin{array}{rcll}
D_{ij}^{X''} \cdot D_{ij}^{X''} & = & -2, & \text{for all } i = 0,1,2,\ j = 2,\dots,n_i-1,\\
D_{+}^{X''} \cdot D_{+}^{X''} & = & -a, & \\
D_{-}^{X''} \cdot D_{-}^{X''} & = & b+c+d. &
\end{array}$$

(iii) *The resolution graph of the possible singularities $x^+, x^-, x_0, x_1x_2 \in X$ looks as follows*



Proof. First, we prove (i). The maximal cones $\sigma^+, \sigma^-, \tau_0, \tau_1, \tau_2$ of the fan Σ are given by Construction 3.4.1. We intersect them with the tropical variety to obtain the fan Σ' from the tropical step of the resolution. The maximal cones of Σ' are given by

$$\begin{array}{lll}
\operatorname{cone}(v_1, e_3), & \operatorname{cone}(v_3, e_3), & \operatorname{cone}(v_5, e_3),\\
\operatorname{cone}(v_2, -e_3), & \operatorname{cone}(v_4, -e_3), & \operatorname{cone}(v_6, -e_3),\\
\operatorname{cone}(v_1, v_2), & \operatorname{cone}(v_3, v_4), & \operatorname{cone}(v_5, v_6).
\end{array}$$

Further, we denote the resulting generator matrix of Σ' by

$$P' \;:=\; \left[v_{01}, v_{02}, v_{11}, v_{12}, v_{21}, v_{22}, v^{+}, v^{-}\right] \;:=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ a & b & 0 & c & 0 & d & 1 & -1 \end{bmatrix}.$$

and by $X' := X(P')$ the corresponding $\mathbb{K}^*$-surface. This subdivision of the cones replaces the two elliptic fixed points of X with two parabolic fixed point curves where the following six points are possibly singular

$$x_{i1}^{+} \in D_{+}^{X'} \cap D_{i1}^{X'}, \qquad\qquad x_{i2}^{-} \in D_{-}^{X'} \cap D_{i2}^{X'}, \qquad i = 0, 1, 2.$$

[3, Prop. 3.4.4.6] and [17, Sum. 7.1] yield smoothness of the points $x_{01}^{+}, x_{02}^{-}, x_{11}^{+}, x_{12}^{-}, x_{21}^{+}, x_{22}^{-}$ and the hyperbolic fixed points x_0, x_1, x_2 if and only if the corresponding determinants

$$\begin{vmatrix} 0 & 1 \\ 1 & a \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ b & -1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ c & -1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ d & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & c \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & d \end{vmatrix}$$

differ from ±1. We obtain the new columns of the matrix P'', i.e. those that do not appear in P', by regular subdivision of the cones associated to the singular points :

$$\begin{array}{lll} x_0: & \operatorname{cone}((-1,-1,a),(-1,-1,b)) & = \bigcup_{k=1}^{a-b} ((-1,-1,a+1-k),(-1,-1,a-k)). \\ x_1: & \operatorname{cone}((1,0,0),(1,0,c)) & = \bigcup_{k=0}^{-c} ((1,0,-k),(1,0,-k+1)), \\ x_2: & \operatorname{cone}((1,0,0),(1,0,d)) & = \bigcup_{k=0}^{-d} ((1,0,-k),(1,0,-k+1)). \end{array}$$

For the statement (ii) we use the formulas from [3, Prop. 5.4.2.1], cf. [17, Sum 7.7], and insert

$$\begin{array}{ll} l_{ij} = 1, & i = 0,1,2,\ j = 1,\dots,n_i, \\ m_{0j} = a+1-j, & j = 1,\dots,a-b+1, \\ m_{1j} = -c+1-j, & j = 1,\dots,-c, \\ m_{2j} = -d+1-j, & j = 1,\dots,-d. \end{array}$$

Lastly, (i) and (ii) imply (iii). ☐

Proof of Theorem 3.5.16. The resolution graphs are given by Proposition 3.5.19, Proposition 3.5.20 and Proposition 3.5.21.

The values a, b, c, d are given in terms of the respective local Gorenstein indices by Theorem 3.2.5, Theorem 3.3.5 and Theorem 3.4.5. We use this representation to obtain the statement. ☐

Remark 3.5.22. Theorem 3.5.16 tells us in particular the following about the singularities of the Gorenstein full intrinsic quadric surfaces from Example 3.2.6, 3.3.6, and 3.4.6.

(i) On X from Example 3.2.6, the points x^{+}, x^{-} are singularities of type A_3 and x_0 is a singularity of type A_1.

(ii) On X from Example 3.3.6, the points x^+, x_0, x_1 are smooth and x^- is a singularity of type A_2.

(iii) On X' from Example 3.3.6, the points x^+, x^- are singularities of type A_2, the point x_1 is a singularity of type A_1 and x_0 is smooth.

(iv) On X from Example 3.4.6, the points x^- is a singularity of type A_1 and x^+, x_0, x_1, x_2 are smooth.

(v) On X' from Example 3.4.6, the points x^+, x^-, x_0 are singularities of type A_1 and x_1, x_2 are smooth.

The singularity types of the Gorenstein log del Pezzo surfaces are well known and we find those of our examples just discussed also in classification results, as for instance [3, Thms. 5.4.4.2 to 5.4.4.5].

3.5.3. The anticanonical complex • Recall that for any normal variety X with a $\mathbb{Q}$-Cartier canonical divisor $\mathcal{K}_X$ and any resolution of singularities $\pi\colon X' \to X$, the associated *ramification formula* is given by

$$\mathcal{K}_{X'} \ = \ \pi^*\mathcal{K}_X + \sum_{i=1}^{n} a_{E_i} E_i.$$

Here, $E_i, i = 1, \ldots, n$ are the exceptional prime divisors and the $a_{E_i} \in \mathbb{Q}$ are the so-called *discrepancies* of $\pi\colon X' \to X$. One speaks of *log terminal (log canonical)* singularities if $a_{E_i} > -1$ for each $i = 1, \ldots, n$ ($a_{E_i} \geq -1$ for each $i = 1, \ldots, n$). By the *log canonicity* of a log terminal projective surface X, we mean the number $\varepsilon_X \ := \ a_E + 1$, where a_E is the minimal possible discrepancy appearing among the exceptional divisors E of its minimal resolution of singularities. Note that $1/\iota_X$ is bounded by the log canonicity. Alexeev's results [1] show in particular that bounding the log canonicity gives finiteness for log del Pezzo surfaces.

Theorem 3.5.23. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.2.5, 3.3.5 or 3.4.5. Then the log canonicity ε_X of X is given by*

$$\begin{aligned}
\rho = 1: &\quad \varepsilon_X = \tfrac{1}{\iota^-},\ \eta \in S_{11}(1,\iota) \cup S_{21}(1,\iota), &\quad \varepsilon_X = \tfrac{2}{\iota^-},\ \eta \in S_{12}(1,\iota) \cup S_{22}(1,\iota),\\
\rho = 2: &\quad \varepsilon_X = \tfrac{3}{\iota^-},\ \eta \in S_{11}(2,\iota) \cup S_{21}(2,\iota), &\quad \varepsilon_X = \tfrac{1}{\iota^-},\ \eta \in S_{12}(2,\iota) \cup S_{22}(2,\iota),\\
\rho = 3: &\quad \varepsilon_X = \tfrac{2}{\iota^-},\ \eta \in S_{11}(3,\iota) \cup S_{21}(3,\iota), &\quad \varepsilon_X = \tfrac{1}{\iota^-},\ \eta \in S_{12}(3,\iota) \cup S_{22}(3,\iota).
\end{aligned}$$

In particular, we obtain the following upper and lower bounds for the log canonicity ε_X of X:

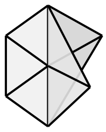
$$\rho = 1:\ \tfrac{1}{\iota} \leq \varepsilon_X \leq \tfrac{2}{\sqrt{\iota}}, \qquad \rho = 2:\ \tfrac{1}{\iota} \leq \varepsilon_X \leq \tfrac{3}{\sqrt{\iota}}, \qquad \rho = 3:\ \tfrac{1}{\iota} \leq \varepsilon_X \leq \tfrac{2}{\sqrt{\iota}}.$$

In the proof, we make use of the *anticanonical complex* $\mathcal{A}_X$ introduced in [5] for varieties X with a torus action of complexity one. The anticanonical complex can be constructed in higher dimension as well in higher complexity. We refer to [31, Sec. 3] and

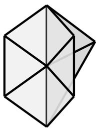
[17, Sec. 9] for more background. It is a polyhedral complex supported on the tropical variety. The anticanonical complex $\mathcal{A}_X$ is bounded if and only if X is log terminal and in this case, the discrepancy of a divisor E on the minimal resolution of X can be stated explicitly.

Proposition 3.5.24. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.2.5, 3.3.5 or 3.4.5. Then the anticanonical complex and its cells are given as*

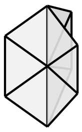
$$\begin{aligned}
\rho = 1: \quad & \tilde{v}^+ = (a+1)e_3, \ \tilde{v}^- = (b+1)e_3, \\
& \operatorname{conv}(0, \tilde{v}^+, v_1), \ \operatorname{conv}(0, v_1, v_2), \ \operatorname{conv}(0, v_2, \tilde{v}^-), \\
& \operatorname{conv}(0, \tilde{v}^+, v_3), \ \operatorname{conv}(0, v_3, \tilde{v}^-), \\
& \operatorname{conv}(0, \tilde{v}^+, v_4), \ \operatorname{conv}(0, v_4, \tilde{v}^-),
\end{aligned}$$



$$\begin{aligned}
\rho = 2: \quad & \tilde{v}^+ = \tfrac{2a+1}{3} e_3, \ \tilde{v}^- = \tfrac{2b+2c+1}{3} e_3, \\
& \operatorname{conv}(0, \tilde{v}^+, v_1), \ \operatorname{conv}(0, v_1, v_2), \ \operatorname{conv}(0, v_2, \tilde{v}^-), \\
& \operatorname{conv}(0, \tilde{v}^+, v_3), \ \operatorname{conv}(0, v_3, v_4), \ \operatorname{conv}(0, v_4, \tilde{v}^-), \\
& \operatorname{conv}(0, \tilde{v}^+, v_5), \ \operatorname{conv}(0, v_5, \tilde{v}^-),
\end{aligned}$$



$$\begin{aligned}
\rho = 3: \quad & \tilde{v}^+ = \tfrac{a}{2} e_3, \ \tilde{v}^- = \tfrac{b+c+d}{2} e_3, \\
& \operatorname{conv}(0, \tilde{v}^+, v_1), \ \operatorname{conv}(0, v_1, v_2), \ \operatorname{conv}(0, v_2, \tilde{v}^-), \\
& \operatorname{conv}(0, \tilde{v}^+, v_3), \ \operatorname{conv}(0, v_3, v_4), \ \operatorname{conv}(0, v_4, \tilde{v}^-), \\
& \operatorname{conv}(0, \tilde{v}^+, v_5), \ \operatorname{conv}(0, v_5, v_6), \ \operatorname{conv}(0, v_6, \tilde{v}^-).
\end{aligned}$$



Proof. [17, Thm. 9.17 (i) and (ii)] yield the maximal cells of $\mathcal{A}_X$. ☐

Lemma 3.5.25 ([17, Prop. 9.17 (iv)])**.** *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.2.5, 3.3.5 or 3.4.5 and its anticanonical complex $\mathcal{A}_X$. Then the discrepancy along an exceptional divisor E_ϱ of its minimal resolution of singularities is given by*

$$a_E = \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1,$$

where $\varrho \subseteq \mathbb{Q}^3$ is the ray corresponding to E, the vector v_ϱ is the primitive lattice vector in ϱ and v'_ϱ is the intersection point of ϱ and the boundary of $\mathcal{A}_X$.

Proof of Theorem 3.5.23. Proposition 3.5.24 gives us the respective cells of $\mathcal{A}_X$. Now Lemma 3.5.25 tells us that the discrepancies of the exceptional divisors E^+, E^- given by the rays through e_3, $-e_3$ are given for $\rho = 1, 2, 3$ by

$$\tfrac{1}{a+1} - 1, \ \tfrac{1}{-b-1} - 1, \qquad \tfrac{3}{2a+1} - 1, \ -\tfrac{3}{2b+2c+1} - 1, \qquad \tfrac{2}{a} - 1, \ -\tfrac{2}{b+c+d} - 1.$$

Proposition 3.5.24 tells us that v_ϱ and v'_ϱ coincide for the other exceptional divisors $E_\varrho \neq E^+, E^-$. Hence, due to Lemma 3.5.25, the other discrepancies are zero.

This implies that the discrepancies of $E^\pm$ are the minimal discrepancies of the canonical resolution and equal ε_X. Inserting the values of a, b, c, d from Theorem 3.2.5, Theorem 3.3.5 and Theorem 3.4.5, we arrive at the assertion. ☐

Remark 3.5.26. By Theorem 3.5.23, the surfaces X from Example 3.2.6, 3.3.6 and 3.4.6 are all of log canonicity $\varepsilon_X = 1$ in accordance with the fact that they are Gorenstein.

3.5.4. The Picard index • The *Picard index* $\mathfrak{p}_X$ of a normal variety X is the index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ of its Picard group in its divisor class group. Note that the Gorenstein index always divides the Picard index. Bounding the Picard index yields finiteness for del Pezzo surfaces of Picard number one with torus action [37]; see also [21] for a higher dimensional analogue in the special case of divisor class group $\mathbb{Z}$.

For the proof of the following theorem see Proposition 3.5.29, Proposition 3.5.30 and Proposition 3.5.31.

Theorem 3.5.27. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.2.5, 3.3.5 or 3.4.5. Then, according to the Picard number $\rho = \rho(X)$, the Picard index $\mathfrak{p} = \mathfrak{p}_X$ of X is given by*

$$\rho = 1:$$

$$\begin{aligned}
\mathfrak{p} &= \tfrac{8\iota^+\iota^-(\iota^++\iota^-)}{\gcd(2\iota^+,\iota^++\iota^-)}, \quad \eta \in S_{11}(1,\iota), & \mathfrak{p} &= \tfrac{4\iota^+\iota^-(2\iota^++\iota^-)}{\gcd(4\iota^+,2\iota^++\iota^-)}, \quad \eta \in S_{12}(1,\iota),\\
\mathfrak{p} &= \tfrac{4\iota^+\iota^-(\iota^++2\iota^-)}{\gcd(2\iota^+,\iota^++2\iota^-)}, \quad \eta \in S_{21}(1,\iota), & \mathfrak{p} &= \tfrac{2\iota^+\iota^-(\iota^++\iota^-)}{\gcd(2\iota^+,\iota^++\iota^-)}, \quad \eta \in S_{22}(1,\iota),
\end{aligned}$$

$$\rho = 2:$$

$$\begin{aligned}
\mathfrak{p} &= -\tfrac{c\iota^+\iota^-(\iota^++\iota^-+2c)}{\gcd(2\iota^+,\iota^++\iota^-,2c)}, \quad \eta \in S_{11}(2,\iota), & \mathfrak{p} &= -\tfrac{3c\iota^+\iota^-(\iota^++3\iota^-+2c)}{\gcd(2\iota^+,\iota^++3\iota^-,2c)}, \quad \eta \in S_{12}(2,\iota),\\
\mathfrak{p} &= -\tfrac{3c\iota^+\iota^-(3\iota^++\iota^-+2c)}{\gcd(6\iota^+,3\iota^++\iota^-,2c)}, \quad \eta \in S_{21}(2,\iota), & \mathfrak{p} &= -\tfrac{9c\iota^+\iota^-(3\iota^++3\iota^-+2c)}{\gcd(6\iota^+,3\iota^++3\iota^-,2c)}, \quad \eta \in S_{22}(2,\iota),
\end{aligned}$$

$$\rho = 3:$$

$$\begin{aligned}
\mathfrak{p} &= \tfrac{cd\iota^+\iota^-(\iota^++\iota^-+c+d)}{\gcd(\iota^+,\iota^-,c,d)}, \quad \eta \in S_{11}(3,\iota), & \mathfrak{p} &= \tfrac{2cd\iota^+\iota^-(\iota^++2\iota^-+c+d)}{\gcd(\iota^+,2\iota^-,c,d)}, \quad \eta \in S_{12}(3,\iota),\\
\mathfrak{p} &= \tfrac{2cd\iota^+\iota^-(2\iota^++\iota^-+c+d)}{\gcd(2\iota^+,\iota^-,c,d)}, \quad \eta \in S_{21}(3,\iota), & \mathfrak{p} &= \tfrac{4cd\iota^+\iota^-(2\iota^++2\iota^-+c+d)}{\gcd(2\iota^+,2\iota^-,c,d)}, \quad \eta \in S_{22}(3,\iota),
\end{aligned}$$

where ι is the Gorenstein index of X, $\iota^\pm$ the local Gorenstein index of $x^\pm \in X$, $-c$ the local class group order of $x_1 \in X$, and $-d$ the local class group order of $x_2 \in X$. In particular, we obtain the following upper and lower bounds:

$$\begin{aligned}
\rho = 1: &\quad \iota \le \mathfrak{p} \le 8\iota^2,\\
\rho = 2: &\quad \iota \le \mathfrak{p} \le \tfrac{27}{2}\iota^3(3\iota - 1),\\
\rho = 3: &\quad \iota \le \mathfrak{p} \le \tfrac{32}{3}\iota^3(2\iota - 1)^2.
\end{aligned}$$

Proposition 3.5.28 ([37, Thm. 1.1])**.** *The Picard index of a normal rational projective $\mathbb{K}^*$-surface X is given by*

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \;=\; \tfrac{1}{|\mathrm{Cl}(X)^{\mathrm{tors}}|} \prod_{x \in X^{\mathrm{sing}}} |\mathrm{Cl}(X,x)|\,.$$

Proposition 3.5.29. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.2.5. Then the following statements hold for the Picard index $\mathfrak{p} = \mathfrak{p}_X$*

(i) *We have*

$$\mathfrak{p} = \tfrac{8\iota^+\iota^-(\iota^++\iota^-)}{\gcd(2\iota^+,\,\iota^++\iota^-)}, \quad \eta \in S_{11}(1,\iota), \qquad \mathfrak{p} = \tfrac{4\iota^+\iota^-(2\iota^++\iota^-)}{\gcd(4\iota^+,\,2\iota^++\iota^-)}, \quad \eta \in S_{12}(1,\iota),$$

$$\mathfrak{p} = \tfrac{4\iota^+\iota^-(\iota^++2\iota^-)}{\gcd(2\iota^+,\,\iota^++2\iota^-)}, \quad \eta \in S_{21}(1,\iota), \qquad \mathfrak{p} = \tfrac{2\iota^+\iota^-(\iota^++\iota^-)}{\gcd(2\iota^+,\,\iota^++\iota^-)}, \quad \eta \in S_{22}(1,\iota).$$

(ii) *The following upper and lower bounds hold*

$$\iota \le \mathfrak{p} \le 8\iota^2, \quad \eta \in S_{11}(1,\iota), \qquad \iota \le \mathfrak{p} \le 2\iota^2, \quad \eta \in S_{12}(1,\iota),$$

$$\iota \le \mathfrak{p} \le 4\iota^2, \quad \eta \in S_{21}(1,\iota), \qquad \iota \le \mathfrak{p} \le 4\iota^2, \quad \eta \in S_{22}(1,\iota).$$

Proof. We begin by showing (i). Proposition 3.2.3 yields $\mathrm{Cl}(X)^{\mathrm{tors}} = 2\gcd(2a+2,\, a-b)$ and the local class group order of the fixed points are given in Proposition 3.2.3 as

$$\mathrm{cl}(X, x^+) \;=\; 4a+4, \qquad \mathrm{cl}(X, x^-) \;=\; -4b-4, \qquad \mathrm{cl}(X, x_0) \;=\; a-b.$$

Then Proposition 3.5.28 implies

$$\mathfrak{p} \;=\; -\tfrac{8(a+1)(b+1)(a-b)}{\gcd(2a+2,a-b)}.$$

By using the representation of the entries a, b in terms of local Gorenstein indices from Theorem 3.2.5 we obtain the assertion.

To show the lower bounds of (ii) we use $\iota = \mathrm{lcm}(\iota^+, \iota^-) \le \iota^+\iota^-$. For the upper bounds we note

$$\gcd(v, w) \;=\; \tfrac{vw}{\mathrm{lcm}(v,w)} \tag{3.5.29.1}$$

for all $v, w \in \mathbb{Z}_{\ge 1}$. Let $(\iota^+, \iota^-) \in S_{11}(1, \iota)$. We use (i) and (3.5.29.1) to get

$$\mathfrak{p} \;=\; \tfrac{8\iota^+\iota^-(\iota^++\iota^-)}{\gcd(2\iota^+,\iota^++\iota^-)} \;=\; \tfrac{8\iota^+\iota^-(\iota^++\iota^-)\mathrm{lcm}(2\iota^+,\iota^++\iota^-)}{2\iota^+(\iota^++\iota^-)} \;=\; 4\iota^-\mathrm{lcm}(2\iota^+, \iota^+ + \iota^-).$$

Note that $\iota^+ + \iota^-$ is even for $(\iota^+, \iota^-) \in S_{11}(1, \iota)$. Together with $\iota^+ \le \iota^-$ we obtain

$$\mathfrak{p} \;=\; 8\iota^-\mathrm{lcm}\left(\iota^+, \tfrac{\iota^++\iota^-}{2}\right) \;\le\; 8\iota^-\mathrm{lcm}\left(\iota^+, \iota^-\right) \;\le\; 8\iota^2.$$

Next, we treat the case $(\iota^+, \iota^-) \in S_{12}(1, \iota)$. Then (i) and (3.5.29.1) imply

$$\mathfrak{p} \;=\; \tfrac{4\iota^+\iota^-(2\iota^++\iota^-)}{\gcd(4\iota^+,2\iota^++\iota^-)} \;=\; \tfrac{4\iota^+\iota^-(2\iota^++\iota^-)\mathrm{lcm}(4\iota^+,2\iota^++\iota^-)}{4\iota^+(2\iota^++\iota^-)} \;=\; \iota^-\mathrm{lcm}(4\iota^+, 2\iota^+ + \iota^-).$$

Since ι^+ is odd and since $4 \mid \iota^-$ we have that $\iota^+ + \frac{\iota^-}{2}$ is odd. Hence

$$\mathfrak{p} \;=\; 2\iota^-\mathrm{lcm}(2\iota^+, \iota^+ + \tfrac{\iota^-}{2}) \;=\; 2\iota^-\mathrm{lcm}(\iota^+, \iota^+ + \tfrac{\iota^-}{2}).$$

Using $2\iota^+ \leq \iota^-$ we conclude $\operatorname{lcm}(\iota^+, \iota^+ + \frac{\iota^-}{2}) \leq \operatorname{lcm}(\iota^+, \iota^-)$. We incorporate this inequality into our considerations about the Picard index:

$$\mathfrak{p} \;=\; 2\iota^-\operatorname{lcm}(\iota^+, \iota^+ + \tfrac{\iota^-}{2}) \;\leq\; 2\iota^-\operatorname{lcm}(\iota^+, \iota^-) \;\leq\; 2\iota^2.$$

Let $(\iota^+, \iota^-) \in S_{21}(1, \iota)$. Due to (i) and (3.5.29.1) we get

$$\mathfrak{p} \;=\; \tfrac{4\iota^+\iota^-(\iota^+ + 2\iota^-)}{\gcd(2\iota^+, \iota^+ + 2\iota^-)} \;=\; \tfrac{4\iota^+\iota^-(\iota^+ + 2\iota^-)\operatorname{lcm}(2\iota^+, \iota^+ + 2\iota^-)}{2\iota^+(\iota^+ + 2\iota^-)} \;=\; 2\iota^-\operatorname{lcm}(2\iota^+, \iota^+ + 2\iota^-).$$

Note that ι^+ is even, ι^- is odd and $\iota^+ \leq 2\iota^-$. This implies

$$\mathfrak{p} \;=\; 4\iota^-\operatorname{lcm}(\iota^+, \tfrac{\iota^+}{2} + \iota^-) \;\leq\; 4\iota^-\operatorname{lcm}(\iota^+, 2\iota^-) \;\leq\; 4\iota^2.$$

Lastly, we treat the case $(\iota^+, \iota^-) \in S_{22}(1, \iota)$. The statement (i) and (3.5.29.1) yield

$$\mathfrak{p} \;=\; \tfrac{2\iota^+\iota^-(\iota^+ + \iota^-)}{\gcd(2\iota^+, \iota^+ + \iota^-)} \;=\; \tfrac{2\iota^+\iota^-(\iota^+ + \iota^-)\operatorname{lcm}(2\iota^+, \iota^+ + \iota^-)}{2\iota^+(\iota^+ + \iota^-)} \;=\; \iota^-\operatorname{lcm}(2\iota^+, \iota^+ + \iota^-).$$

Since ι^+ and ι^- are both even and since $\iota^+ \leq \iota^-$, we have

$$\mathfrak{p} \;=\; 4\iota^-\operatorname{lcm}\left(\tfrac{\iota^+}{2}, \tfrac{\iota^+ + \iota^-}{2}\right) \;\leq\; 4\iota^-\operatorname{lcm}\left(\tfrac{\iota^+}{2}, \iota^-\right) \;\leq\; 4\iota^2.$$

□

Proposition 3.5.30. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.3.5. Then the following statements hold for the Picard index $\mathfrak{p} = \mathfrak{p}_X$*

(i) *We have*

$$\mathfrak{p} = -\tfrac{c\iota^+\iota^-(\iota^+ + \iota^- + 2c)}{\gcd(2\iota^+, \iota^+ + \iota^-, 2c)}, \quad \eta \in S_{11}(2, \iota), \qquad \mathfrak{p} = -\tfrac{3c\iota^+\iota^-(\iota^+ + 3\iota^- + 2c)}{\gcd(2\iota^+, \iota^+ + 3\iota^-, 2c)}, \quad \eta \in S_{12}(2, \iota),$$

$$\mathfrak{p} = -\tfrac{3c\iota^+\iota^-(3\iota^+ + \iota^- + 2c)}{\gcd(6\iota^+, 3\iota^+ + \iota^-, 2c)}, \quad \eta \in S_{21}(2, \iota), \qquad \mathfrak{p} = -\tfrac{9c\iota^+\iota^-(3\iota^+ + 3\iota^- + 2c)}{\gcd(6\iota^+, 3\iota^+ + 3\iota^-, 2c)}, \quad \eta \in S_{22}(2, \iota).$$

(ii) *The following upper and lower bounds hold*

$$\iota \leq \mathfrak{p} \leq \tfrac{\iota^3(\iota - 1)}{2}, \qquad \eta \;\in\; S_{11}(2, \iota), \quad \iota \leq \mathfrak{p} \leq \tfrac{9}{2}\iota^3(3\iota - 1), \quad \eta \;\in\; S_{12}(2, \iota),$$

$$\iota \leq \mathfrak{p} \leq \tfrac{3}{2}\iota^3(\iota - 1), \quad \eta \;\in\; S_{21}(2, \iota), \quad \iota \leq \mathfrak{p} \leq \tfrac{27}{2}\iota^3(3\iota - 1), \quad \eta \;\in\; S_{22}(2, \iota).$$

Proof. For (i), we insert $\operatorname{Cl}(X)^{\mathrm{tors}} = \gcd(2a + 1, a - b, -c)$ from Construction 3.3.1 and the local class group orders

$$\operatorname{cl}(X, x^+) = 1 + 2a, \quad \operatorname{cl}(X, x^-) = -1 - 2b - 2c, \quad \operatorname{cl}(X, x_0) = a - b, \quad \operatorname{cl}(X, x_1) = -c.$$

from Proposition 3.3.3 into the formula of Proposition 3.5.28. This yields

$$\tfrac{c(1 + 2a)(1 + 2b + 2c)(a - b)}{\gcd(1 + 2a, a - b, c)}.$$

Theorem 3.3.5 gives us a representation of the entries a, b in terms of local Gorenstein indices and c, which yields the claim.

Next, we show the second assertion. Let $(\iota^+, \iota^-, c) \in S_{11}(2, \iota)$. Then we have $c \leq -\frac{\iota^+ + \iota^-}{4}$ and $1 \leq \iota^+ \leq \iota^-$. Consequently, we get

$$\iota^+ + \iota^- + 2c \leq \tfrac{\iota^+ + \iota^-}{2} \leq \iota^-.$$

Moreover, $(\iota^+, \iota^-, c) \in S_{11}(2, \iota)$ implies $1 - \frac{\iota^+ + \iota^-}{2} \leq c$. Hence

$$-c \leq \tfrac{\iota^+ + \iota^-}{2} - 1 \leq \iota^- - 1.$$

We use (i), the two inequalities above and that ι^+ and ι^- are odd to obtain

$$\mathfrak{p} \ = \ -\frac{c\iota^+ \iota^- (\iota^+ + \iota^- + 2c)}{\gcd(2\iota^+, \iota^+ + \iota^-, 2c)} \ \leq \ \frac{\iota^+ \iota^- \iota^- (\iota^- - 1)}{2 \gcd\left(\iota^+, \ \frac{\iota^+ + \iota^-}{2}, \ c\right)} \ \leq \ \frac{\iota^3 (\iota - 1)}{2}.$$

Next, we treat the case $(\iota^+, \iota^-, c) \in S_{12}(2, \iota)$. We get $c \leq -\frac{\iota^+ + 3\iota^-}{4}$ and $\iota^+ \leq 3\iota^-$. Hence, we obtain

$$\iota^+ + 3\iota^- + 2c \ \leq \ \tfrac{\iota^+ + 3\iota^-}{2} \ \leq \ 3\iota^-.$$

Further, $(\iota^+, \iota^-, c) \in S_{12}(2, \iota)$ implies $1 - \frac{\iota^+ + 3\iota^-}{2} \leq c$. So we get

$$-c \ \leq \ \tfrac{\iota^+ + 3\iota^-}{2} - 1 \ \leq \ 3\iota^- - 1.$$

Because ι^+ and ι^- are both odd, one has $\frac{\iota^+ + 3\iota^-}{2} \in \mathbb{Z}$. Together with (i) and the two inequalities above we conclude

$$\mathfrak{p} \ = \ -\frac{3c\iota^+ \iota^- (\iota^+ + 3\iota^- + 2c)}{\gcd(2\iota^+, \iota^+ + 3\iota^-, -2c)} \ \leq \ \frac{3\iota^+ \iota^- 3\iota^- (3\iota^- - 1)}{2 \gcd\left(\iota^+, \ \frac{\iota^+ + 3\iota^-}{2}, \ -c\right)} \ \leq \ \tfrac{9}{2}\iota^3 (3\iota - 1).$$

We proceed with $(\iota^+, \iota^-, c) \in S_{21}(2, \iota)$. Then we obtain $c \leq -\frac{3\iota^+ + \iota^-}{4}$ and $3\iota^+ \leq \iota^-$. We put this together as follows

$$3\iota^+ + \iota^- + 2c \leq \tfrac{3\iota^+ + \iota^-}{2} \leq \iota^-.$$

In addition, $(\iota^+, \iota^-, c) \in S_{21}(2, \iota)$ yields $1 - \frac{3\iota^+ + \iota^-}{2} \leq c$. Thus, we get

$$-c \leq \tfrac{\iota^+ + 3\iota^-}{2} - 1 \leq \iota^- - 1$$

Since ι^+, ι^- are both odd, we have $\frac{3\iota^+ + \iota^-}{2} \in \mathbb{Z}$. Thus, together with the inequalities from above and with (i), we obtain

$$\mathfrak{p} \ = \ -\frac{3c\iota^+ \iota^- (3\iota^+ + \iota^- + 2c)}{\gcd(6\iota^+, 3\iota^+ + \iota^-, 2c)} \ \leq \ \frac{3\iota^+ \iota^- \iota^- (\iota^- - 1)}{2} \ \leq \ \tfrac{3}{2}\iota^3 (\iota - 1).$$

Lastly, let $(\iota^+, \iota^-, c) \in S_{22}(2, \iota)$. Then we have $c \leq -\frac{3\iota^+ + 3\iota^-}{4}$ and $\iota^+ \leq \iota^-$ and we conclude

$$3\iota^+ + 3\iota^- + 2c \ \leq \ \tfrac{3\iota^+ + 3\iota^-}{2} \ \leq \ 3\iota^-.$$

Note that $(\iota^+, \iota^-, c) \in S_{22}(2, \iota)$ yields $1 - \frac{3\iota^+ + 3\iota^-}{2} \leq c$. Consequently, we get

$$-c \ \leq \ \tfrac{3\iota^+ + 3\iota^-}{2} - 1 \ \leq \ 3\iota^- - 1.$$

We have that ι^+, ι^- are both odd, hence $\frac{3\iota^+ + 3\iota^-}{2} \in \mathbb{Z}$. Thus, by combining this and (i) one obtains

$$\mathfrak{p} \ = \ -\tfrac{9c\iota^+\iota^-(3\iota^+ + 3\iota^- + 2c)}{\gcd(6\iota^+, 3\iota^+ + 3\iota^-, 2c)} \ \leq \ \tfrac{9\iota^+\iota^- 3\iota^-(3\iota^- - 1)}{2} \ \leq \ \tfrac{27}{2}\iota^3(3\iota - 1).$$

□

Proposition 3.5.31. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.4.5. Then the following statements hold for the Picard index $\mathfrak{p} = \mathfrak{p}_X$*

(i) *We have*

$$\mathfrak{p} = \tfrac{cd\iota^+\iota^-(\iota^+ + \iota^- + c + d)}{\gcd(\iota^+, \iota^-, c, d)}, \quad \eta \in S_{11}(3, \iota), \quad \mathfrak{p} = \tfrac{2cd\iota^+\iota^-(\iota^+ + 2\iota^- + c + d)}{\gcd(\iota^+, 2\iota^-, c, d)}, \quad \eta \in S_{12}(3, \iota),$$

$$\mathfrak{p} = \tfrac{2cd\iota^+\iota^-(2\iota^+ + \iota^- + c + d)}{\gcd(2\iota^+, \iota^-, c, d)}, \quad \eta \in S_{21}(3, \iota), \quad \mathfrak{p} = \tfrac{4cd\iota^+\iota^-(2\iota^+ + 2\iota^- + c + d)}{\gcd(2\iota^+, 2\iota^-, c, d)}, \quad \eta \in S_{22}(3, \iota),$$

(ii) *The following upper and lower bounds hold*

$$\iota \leq \mathfrak{p} \leq 3\iota^3(\iota - 1)^2, \quad \eta \in S_{11}(3, \iota), \quad \iota \leq \mathfrak{p} \leq \iota^3(3\iota - 1)(3\iota - 2), \quad \eta \in S_{12}(3, \iota),$$

$$\iota \leq \mathfrak{p} \leq 2\iota^3(\iota - 1)(3\iota - 1), \quad \eta \in S_{21}(3, \iota), \quad \iota \leq \mathfrak{p} \leq \tfrac{32}{3}\iota^3(2\iota - 1)^2, \quad \eta \in S_{22}(3, \iota).$$

Proof. We show (i). Construction 3.4.1 yields $\mathrm{Cl}(X)^{\mathrm{tors}} = \gcd(a, b, c, d)$ and the local class group orders are given by Proposition 3.4.3 as

$$\mathrm{cl}(X, x^+) = a, \qquad \mathrm{cl}(X, x^-) = -b - c - d,$$

$$\mathrm{cl}(X, x_0) = a - b, \quad \mathrm{cl}(X, x_1) = -c, \quad \mathrm{cl}(X, x_2) = -d.$$

Then using the formula from Proposition 3.5.28 we obtain

$$-\tfrac{acd(b+c+d)(a-b)}{\gcd(a,b,c,d)}.$$

Theorem 3.4.5 gives us a description of a, b in terms of the local Gorenstein indices and in terms of c, d. This yields the assertion.

Next, we prove (ii). We begin with $(\iota^+, \iota^-, c, d) \in S_{11}(3, \iota)$. Then we get $c \leq d \leq -1$, $-\iota^+ - \iota^- \leq 2c + d$ and $\iota^+ \leq \iota^-$. This yields

$$\iota^+ + \iota^- + c + d \ \leq \ \iota^+ + \iota^- - 2 \ \leq \ 2\iota^- - 2 \ \leq \ 2\iota - 2.$$

Further, we observe

$$-2\iota \ \le\ -2\iota^- \ \le\ -\iota^+ - \iota^- \ \le\ 2c + d \ \le\ 2c - 1.$$

Thus, we conclude $-c \le \iota - \frac{1}{2}$. Since c must be integer, we infer $-c \le \iota - 1$. Analogously, we conclude $-2\iota \le 3d$ and hence $-d \le \frac{2}{3}\iota$. Thus, together with (i) we obtain

$$\mathfrak{p} \ =\ \frac{cd\iota^+\iota^-(\iota^+ + \iota^- + c + d)}{\gcd(\iota^+, \iota^-, c, d)} \ \le\ 3\iota^3(\iota - 1)^2.$$

Now let $(\iota^+, \iota^-, c, d) \in S_{12}(3, \iota)$. So we obtain $c \le d \le -1$, $-\iota^+ - 2\iota^- \le 2c + d$ and $\iota^+ \le 2\iota^-$. Consequently, one gets

$$\iota^+ + 2\iota^- + c + d \ \le\ \iota^+ + 2\iota^- - 2 \ \le\ 3\iota - 2 \ \le\ 3\iota - 2.$$

Moreover, one has

$$-3\iota \ \le\ -\iota^+ - 2\iota^- \ \le\ 2c + d \ \le\ 2c - 1.$$

This implies $-c \le \frac{3}{2}\iota - \frac{1}{2}$. Similarily, we get $-3\iota^- \le 3d$ and thus $-d \le \iota$. Then the description of the Picard index from (i) and the inequalities above yield

$$\mathfrak{p} \ =\ \frac{2cd\iota^+\iota^-(\iota^+ + 2\iota^- + c + d)}{\gcd(\iota^+, 2\iota^-, c, d)} \ \le\ \iota^3(3\iota - 1)(3\iota - 2).$$

Next, we treat the case $(\iota^+, \iota^-, c, d) \in S_{21}(3, \iota)$. We have $c \le d \le -1$, $-2\iota^+ - \iota^- \le 2c + d$ and $2\iota^+ \le \iota^-$. Hence we get

$$2\iota^+ + \iota^- + c + d \ \le\ 2\iota^- - 2 \ \le\ 2\iota - 2$$

and

$$-3\iota \ \le\ -2\iota^+ - \iota^- \ \le\ 2c + d \ \le\ 2c - 1.$$

We obtain $-c \le \frac{3}{2}\iota - \frac{1}{2}$ and $-3\iota^- \le 3d$. Hence one gets $-d \le \iota$. Then, we use the statement from (i) and the inequalities above to get

$$\mathfrak{p} \ =\ \frac{2cd\iota^+\iota^-(2\iota^+ + \iota^- + c + d)}{\gcd(2\iota^+, \iota^-, c, d)} \ \le\ 2\iota^3(\iota - 1)(3\iota - 1).$$

Lastly, let $(\iota^+, \iota^-, c, d) \in S_{22}(3, \iota)$. Then one has $c \le d \le -1$, $-2\iota^+ - 2\iota^- \le 2c + d$ and $\iota^+ \le \iota^-$. From this we infer

$$2\iota^+ + 2\iota^- + c + d \ \le\ 2\iota^+ + 2\iota^- - 2 \ \le\ 4\iota^- - 2 \ \le\ 4\iota - 2.$$

Moreover, we have

$$-4\iota \ \le\ -4\iota^- \ \le\ -2\iota^+ - 2\iota^- \ \le\ 2c + d \ \le\ 2c - 1.$$

Thus, we conclude $-c \le 2\iota - \frac{1}{2}$. Since c must be integer, we infer $-c \le 2\iota - 1$. Similarily, we get $-4\iota^- \le 3d$ and hence $-d \le \frac{4}{3}\iota$. Together with (i) we obtain

$$\mathfrak{p} \ =\ \frac{4cd\iota^+\iota^-(2\iota^+ + 2\iota^- + c + d)}{\gcd(2\iota^+, 2\iota^-, c, d)} \ \le\ \frac{32}{3}\iota^3(2\iota - 1)^2.$$

□

3.5.5. Kähler-Einstein metrics • A *Kähler-Einstein metric* on a rational projective del Pezzo surface is a Kähler orbifold metric g such that the associated Kähler form ω_g equals its Ricci form $\mathrm{Ric}(\omega_g)$. The smooth del Pezzo surfaces with a Kähler-Einstein metric are $\mathbb{P}_2$, its blowing up in $k = 3, \ldots, 8$ points in general position and $\mathbb{P}_1 \times \mathbb{P}_1$; see [38, 39]. The case of quasismooth del Pezzo surfaces coming anticanonically embedded into a three-dimensional weighted projective space is understood as well; see [2, 7, 8, 33]. We settle the case of full intrinsic quadric surfaces.

Theorem 3.5.32. *Let X be a complex full intrinsic quadric surface admitting a Kähler-Einstein metric. Then $X \cong X(P)$ for precisely one P from the following:*

$\rho = 1,\ 2 \nmid \iota:$

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota - 1 & -\iota - 1 & 1 & 1 \end{bmatrix},$$

$\rho = 1,\ 4 \mid \iota:$

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota}{2} - 1 & -\frac{\iota}{2} - 1 & 1 & 1 \end{bmatrix},$$

$\rho = 3, \quad 2 \nmid \iota, \quad -2\iota \le 2c + d,$
$c \le d \le -1,\ c + d \le -\iota - 1:$

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota & -\iota - c - d & 0 & c & 0 & d \end{bmatrix},$$

$\rho = 3, \quad -4\iota \le 2c + d,\ c \le d \le -1,$
$c + d \le -2\iota - 1:$

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota & -2\iota - c - d & 0 & c & 0 & d \end{bmatrix},$$

where ρ denotes the Picard number and ι the Gorenstein index of $X(P)$. Conversely, each $X(P)$ with P from the above list admits a Kähler-Einstein metric.

We will verify existence or non-existence of Kähler-Einstein metrics via K-stability. Let us give an idea of the approach; we refer to Section 1.3 and [18] for more background. For a full intrinsic quadric $X = X(P) \subseteq Z$ arising from Construction 3.2.1, 3.3.1 or 3.4.1, consider the varieties $\mathcal{X}_\kappa \subseteq Z \times \mathbb{C}$, $\kappa = 0, 1, 2$, given by the equations

$$\begin{array}{cccc} & \kappa = 0 & \kappa = 1 & \kappa = 2 \\ \rho = 1: & ST_1T_2 + T_3^2 + T_4^2 & T_1T_2 + ST_3^2 + T_4^2 & T_1T_2 + T_3^2 + ST_4^2 \\ \rho = 2: & ST_1T_2 + T_3T_4 + T_5^2 & T_1T_2 + ST_3T_4 + T_5^2 & T_1T_2 + T_3T_4 + ST_5^2 \\ \rho = 3: & ST_1T_2 + T_3T_4 + T_5T_6 & T_1T_2 + ST_3T_4 + T_5T_6 & T_1T_2 + T_3T_4 + ST_5T_6 \end{array}$$

where $T_1, \ldots, T_6$ are the homogeneous coordinates on Z and S is the standard coordinate on $\mathbb{C}$. The projection $Z \times \mathbb{C} \to \mathbb{C}$ induces a flat family $\mathcal{X}_\kappa \to \mathbb{C}$. The fiber $\mathcal{X}_{\kappa,1}$ over $1 \in \mathbb{C}$ is our full intrinsic quadric surface and the fiber $\mathcal{X}_{\kappa,0}$ over $0 \in \mathbb{C}$, given by a binomial equation, is a toric surface. Recall that κ is called *special* if the corresponding fiber $\mathcal{X}_{\kappa,0}$ is a normal variety. Consider the sublattices

$$N_\kappa \ := \ \mathbb{Z} \cdot e_\kappa + \mathbb{Z} \cdot e_3, \qquad \kappa = 0, 1, 2, \quad e_0 := -e_1 - e_2.$$

The fan Δ_κ associated with the toric degeneration $\mathcal{X}_{\kappa,0}$ of X is obtained by restricting the fan of $Z \times \mathbb{C}$ to N_κ. It turns out that $\mathcal{X}_{\kappa,0}$ is normal in all cases except $(\rho, \kappa) = (1, 0)$, where Δ_0 describes the normalization. The families $\mathcal{X}_\kappa \to \mathbb{C}$ are so-called *equivariant test configurations* and we are provided with the following combinatorial K-stability criterion, characterizing existence of Kähler-Einstein metrics in terms of the barycenters b_κ of the moment polytopes $\mathcal{B}_\kappa$ associated with the toric degenerations $\mathcal{X}_{\kappa,0}$.

Proposition 3.5.33 ([18, Thm. 6.2])**.** *Consider a complex full intrinsic quadric surface X as in Construction 3.2.1, 3.3.1 or 3.4.1. Then X admits a Kähler-Einstein metric if and only if the coordinates of the barycenter of the moment polytope $b_\kappa \in \mathcal{B}_\kappa \subseteq \mathbb{Q}^2$ satisfy $b_{\kappa,1} = 0$ for all $\kappa = 0, 1, 2$ and $b_{\kappa,2} > 0$ for all special $\kappa = 0, 1, 2$.*

Proposition 3.5.34. *Consider a complex full intrinsic quadric surface X as in Construction 3.2.1 admitting a Kähler-Einstein metric. Then $X \cong X(P)$ for precisely one P from the following*

$$2 \nmid \iota : \quad \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota - 1 & -\iota - 1 & 1 & 1 \end{bmatrix}, \qquad 4 \mid \iota : \quad \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota}{2} - 1 & -\frac{\iota}{2} - 1 & 1 & 1 \end{bmatrix},$$

where ι denotes the Gorenstein index of X. Conversely, each X with generator matrix P from the above list admits a Kähler-Einstein metric.

Lemma 3.5.35. *Consider a full intrinsic quadric surface X as in Construction 3.2.1. Then the fibers $\mathcal{X}_{1,0}$ and $\mathcal{X}_{2,0}$ are normal whereas the fiber $\mathcal{X}_{0,0}$ is not.*

Proof. This follows directly from $l_{01} = l_{02} = 1, l_{11} = l_{21} = 2$ and Proposition 1.3.7 (iv). □

Remark 3.5.36. Consider a full intrinsic quadric surface X as in Construction 3.2.1. The generator matrix of the anticanonical cone over X is given by Proposition 1.2.11 as

$$\tilde{P} := [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4] := \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Moreover, we denote by $\tilde{\sigma}$ the cone over all columns of $\tilde{P}$. Since $\det(\tilde{P}) = 4a - 4b > 0$, the cone $\tilde{\sigma}$ is a simplicial cone of full dimension in $\mathbb{Z}^4$.

Lemma 3.5.37. *Consider a full intrinsic quadric surface X as in Construction 3.2.1, let $\kappa = 1, 2$ and let $\tilde{\tau}_\kappa$ be as in Construction 1.3.11. Then the primitive generators of $\tilde{\tau}_\kappa$ are given by*

$$u_1 := (2a + 1, 1, 2), \quad u_2 := (2b + 1, 1, 2), \quad u_3 := (1, 1, -2).$$

Proof. We follow Construction 1.3.11 to determine the generators of $\tilde{\tau}_\kappa$. Let $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ and $\tilde{\sigma}$ be as in Remark 3.5.36. Set $e_0 \coloneqq (-1,-1,0,0)$. Next, we prove

$$\sigma \ \coloneqq \ \operatorname{cone}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_4) \cap \operatorname{lin}(e_1, e_3, e_4) \ = \ \operatorname{cone}((-2,0,2a+1,1),(-2,0,2b+1,1)).$$

For the inclusion '$\subseteq$' let $w \coloneqq (w_1, w_2, w_3, w_4) \in \sigma$. Then we can present w as

$$w \ = \ \lambda_1\tilde{v}_1 + \lambda_2\tilde{v}_2 + \lambda_4\tilde{v}_4$$

with some $\lambda_i \in \mathbb{Q}_{\geq 1}$. We derive $w_2 = 0$ from $w \in \operatorname{lin}(e_1, e_3, e_4)$. Hence $\lambda_1 + \lambda_2 = 2\lambda_4$. This implies $\lambda_1 \leq 2\lambda_4$ and $\lambda_1 \leq 2\lambda_4$. We substitute λ_2 and obtain

$$w \ = \ \lambda_1\tilde{v}_1 + (2\lambda_4\tilde{v}_2 - \lambda_1\tilde{v}_2) + \lambda_4\tilde{v}_4.$$

After arranging the summands suitably, we get

$$\begin{aligned} w \ &= \ \lambda_1\tilde{v}_1 + (2\lambda_4\tilde{v}_2 - \lambda_1\tilde{v}_2) + \lambda_4\tilde{v}_4 + \tfrac{\lambda_1}{2}\tilde{v}_1 - \tfrac{\lambda_1}{2}\tilde{v}_1 \\ &= \ \tfrac{\lambda_1}{2}\left(2\tilde{v}_1 + \tilde{v}_4\right) + \left(\lambda_4 - \tfrac{\lambda_1}{2}\right)\left(2\tilde{v}_2 + \tilde{v}_4\right). \end{aligned}$$

This represents a positive linear combination of $2\tilde{v}_1 + \tilde{v}_4$ and $2\tilde{v}_2 + \tilde{v}_4$, where

$$2\tilde{v}_1 + \tilde{v}_4 \ = \ (-2,0,2a+1,1), \qquad 2\tilde{v}_2 + \tilde{v}_4 \ = \ (-2,0,2b+1,1).$$

The inclusion '$\supseteq$' is clear. Moreover, we have $\tilde{v}_3 \in \operatorname{lin}(e_1, e_3, e_4)$. This implies

$$\tilde{\sigma} \cap \operatorname{lin}(e_1, e_3, e_4) \ = \ \operatorname{cone}\left((-2,0,2a+1,1),(-2,0,2b+1,1),(2,0,1,1)\right).$$

Analogously, we obtain

$$\tilde{\sigma} \cap \operatorname{lin}(e_2, e_3, e_4) \ = \ \operatorname{cone}\left((0,-2,2a+1,1),(0,-2,2b+1,1),(0,2,1,1)\right).$$

Since $b < a$ both cones have three generators. Let $\eta_1 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$ and $\eta_2 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$ be the maps given by

$$\eta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \eta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then the two cones $\tilde{\tau}_\kappa, \kappa = 1,2$ are given by

$$\begin{aligned} \tilde{\tau}_1 \ &\coloneqq \ \eta_1^{-1}\left(\tilde{\sigma} \cap \operatorname{lin}(e_1, e_3, e_4)\right) \\ &= \ \operatorname{cone}\left(\eta_1^{-1}(-2,0,2a+1,1), \eta_1^{-1}(-2,0,2b+1,1), \eta_1^{-1}(2,0,1,1)\right) \\ &= \ \operatorname{cone}\left((2a+1,1,2),(2b+1,1,2),(1,1,-2)\right), \\ \tilde{\tau}_2 \ &\coloneqq \ \eta_2^{-1}\left(\tilde{\sigma} \cap \operatorname{lin}(e_2, e_3, e_4)\right) \\ &= \ \operatorname{cone}\left(\eta_2^{-1}(0,-2,2a+1,1), \eta_2^{-1}(0,-2,2b+1,1), \eta_2^{-1}(0,2,1,1)\right) \\ &= \ \operatorname{cone}\left((2a+1,1,2),(2b+1,1,2),(1,1,-2)\right). \end{aligned}$$

□

Lemma 3.5.38. *Consider a full intrinsic quadric surface X as in Construction 3.2.1, let $\kappa = 1, 2$ and let $\tilde{\tau}_\kappa$ be as in Lemma 3.5.37. Then the primitive generators of $\tilde{\omega}_\kappa := (\tilde{\tau}_\kappa)^\vee$ are given by*

$$(0, 2, -1), \qquad \tfrac{1}{\gcd(2,b)}(2, -2b-2, -b), \qquad \tfrac{1}{\gcd(2,a)}(-2, 2+2a, a).$$

Proof. The generators u_1, u_2, u_3 of $\tilde{\tau}_\kappa$ are given by Lemma 3.5.37. Then we have

$$\begin{aligned} (0, 2, -1) \cdot [u_1, u_2, u_3] &= (0, 0, 4)^\top, \\ \tfrac{1}{\gcd(2,b)}(2, -2b-2, -b) \cdot [u_1, u_2, u_3] &= \tfrac{1}{\gcd(2,b)}(4a-4b, 0, 0)^\top, \\ \tfrac{1}{\gcd(2,a)}(-2, 2+2a, a) \cdot [u_1, u_2, u_3] &= \tfrac{1}{\gcd(2,a)}(0, 4a-4b, 0)^\top. \end{aligned}$$

According to Construction 3.2.1 we have $b < a$. Hence, the non zero entries on the right hand side are all positive. Thus, the vectors listed in the assertion are ray generators of $\tilde{\omega}_\kappa$, since each of them evaluates to zero on two of the rays of $\tilde{\tau}_\kappa$ and evaluates positively on the third. □

Lemma 3.5.39. *Consider a full intrinsic quadric surface X as in Construction 3.2.1. Then for $\kappa = 1, 2$ the moment polytopes are given by*

$$\mathcal{B}_\kappa \;=\; \operatorname{conv}\left(\left(0, -\tfrac{1}{2}\right),\ \left(-\tfrac{1}{b+1}, \tfrac{b}{2b+2}\right),\ \left(-\tfrac{1}{a+1}, \tfrac{a}{2a+2}\right)\right).$$

Proof. The generators of $\tilde{\omega}_\kappa$ are given by Lemma 3.5.38. Moreover, let

$$\iota\colon \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3, \quad u \mapsto (u_1, 1, u_2).$$

Then the polytope $\mathcal{C}_\kappa$ is given as

$$\begin{aligned} \mathcal{C}_\kappa \;&:=\; \iota^{-1}\left(\operatorname{conv}(\tilde{\omega}_\kappa)\right) \\ &=\; \iota^{-1}\left(\operatorname{conv}\left((0, 2, -1), \tfrac{1}{\gcd(2,b)}(2, -2b-2, -b), \tfrac{1}{\gcd(2,a)}(-2, 2+2a, a).\right)\right) \\ &=\; \operatorname{conv}\left(\iota^{-1}\left(0, 1, -\tfrac{1}{2}\right),\ \iota^{-1}\left(-\tfrac{2}{2b+2}, 1, \tfrac{b}{2b+2}\right),\ \iota^{-1}\left(-\tfrac{2}{2a+2}, 1, \tfrac{a}{2a+2}\right)\right) \\ &=\; \operatorname{conv}\left(\left(0, -\tfrac{1}{2}\right),\ \left(-\tfrac{1}{b+1}, \tfrac{b}{2b+2}\right),\ \left(-\tfrac{1}{a+1}, \tfrac{a}{2a+2}\right)\right). \end{aligned}$$

Since $b < 0 \leq a$ the unique interior point of $\mathcal{C}_\kappa$ is $(0, 0)$. Then Construction 1.3.11 provides us with $\mathcal{B}_\kappa = \mathcal{C}_\kappa$. □

Lemma 3.5.40. *Consider a full intrinsic quadric surface X as in Construction 3.2.1 and let $\tilde{\tau}_0$ be as in Construction 1.3.11. Then the primitive generators of $\tilde{\tau}_0$ are given by*

$$u_1 \;:=\; (a, 0, -1), \quad u_2 \;:=\; (b, 0, -1), \quad u_3 \;:=\; (1, 1, 1).$$

Proof. By following Construction 1.3.11 we determine $\tilde{\tau}_0$. Let $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ and $\tilde{\sigma}$ be as in Remark 3.5.36. Set $e_0 \coloneqq (-1,-1,0,0)$. We have $\tilde{v}_1, \tilde{v}_2 \in \operatorname{lin}(e_0, e_3, e_4)$ and

$$\operatorname{cone}(\tilde{v}_3, \tilde{v}_4) \cap \operatorname{lin}(e_0, e_3, e_4) \; = \; \operatorname{cone}((2,2,2,2)).$$

Because of $b < a$ we obtain the following three pairwise distinct ray generators

$$\tilde{\sigma} \cap \operatorname{lin}(e_0, e_3, e_4) = \operatorname{cone}\left((-1,-1,a,0), (-1,-1,b,0), (1,1,1,1)\right).$$

Let $\eta_0 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$ be the map given by

$$\eta_0 \; = \; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence one gets

$$\begin{aligned} \tilde{\tau}_0 \; &= \; \eta_0^{-1}\left(\tilde{\sigma} \cap \operatorname{lin}(e_0, e_3, e_4)\right) \\ &= \; \operatorname{cone}\left(\eta_0^{-1}(-1,-1,a,0), \eta_0^{-1}(-1,-1,b,0), \eta_0^{-1}(1,1,1,1)\right) \\ &= \; \operatorname{cone}\left((a,0,-1), (b,0,-1), (2,2,2)\right). \end{aligned}$$

□

Lemma 3.5.41. *Consider a full intrinsic quadric surface X as in Construction 3.2.1 and let $\tilde{\tau}_0$ be as in Lemma 3.5.40. Then the primitive generators of $\tilde{\omega}_0 \coloneqq (\tilde{\tau}_0)^\vee$ are given by*

$$(0,1,0), \quad (-1, a+1, -a), \quad (1, -b-1, b).$$

Proof. In order to see that the above vectors are ray generators of $\tilde{\omega}_0$ it suffices to show that each of them cuts out a facet of $\tilde{\tau}_0$ and evaluates positively on the complementary ray. The generators u_1, u_2, u_3 of $\tilde{\tau}_0$ are given in Lemma 3.5.40. We obtain

$$\begin{aligned} (0,1,0) \cdot [u_1, u_2, u_3] &= (0,0,1)^\top, \\ (-1, a+1, -a) \cdot [u_1, u_2, u_3] &= (0, a-b, 0)^\top, \\ (1, -b-1, b) \cdot [u_1, u_2, u_3] &= (a-b, 0, 0)^\top. \end{aligned}$$

By Construction 3.2.1 we know that $b \le -2$ and $0 \le a \le -b-2$. Hence, the non zero entries in the terms on the right hand side are positive. □

Lemma 3.5.42. *Consider a full intrinsic quadric surface X as in Construction 3.2.1. Then the moment polytope $\mathcal{B}_0$ is given by*

$$\mathcal{B}_0 = \operatorname{conv}\left((0,0), \left(\tfrac{-1}{a+1}, \tfrac{-a}{a+1}\right), \left(\tfrac{-1}{b+1}, \tfrac{-b}{b+1}\right)\right).$$

Proof. The primitive generators of the cone $\tilde{\omega}_0$ are given by Lemma 3.5.41. Further, set

$$\iota\colon \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3, \quad u \mapsto (u_1, 1, u_2).$$

Then we have

$$\begin{aligned} \mathcal{C}_0 &:= \iota^{-1}\left(\operatorname{conv}(\tilde{\omega}_0)\right) \\ &= \iota^{-1}\left(\operatorname{conv}\left((0,1,0), \left(\tfrac{-1}{a+1}, 1, \tfrac{-a}{a+1}\right), \left(\tfrac{-1}{b+1}, 1, \tfrac{-b}{b+1}\right)\right)\right) \\ &= \operatorname{conv}\left(\iota^{-1}(0,1,0),\ \iota^{-1}\left(\tfrac{-1}{a+1}, 1, \tfrac{-a}{a+1}\right),\ \iota^{-1}\left(\tfrac{-1}{b+1}, 1, \tfrac{-b}{b+1}\right)\right) \\ &= \operatorname{conv}\left((0,0), \left(\tfrac{-1}{a+1}, \tfrac{-a}{a+1}\right), \left(\tfrac{-1}{b+1}, \tfrac{-b}{b+1}\right)\right). \end{aligned}$$

Furthermore, Lemma 3.5.35 yields that $\kappa = 0$ is not special. Hence, Construction 1.3.11 tells us that the moment polytope is given by $\mathcal{B}_0 = \mathcal{C}_0$. □

Proof of Proposition 3.5.34. For $\kappa = 1, 2$ the moment polytopes $\mathcal{B}_\kappa$ are given by Lemma 3.5.39. Then, in both cases we compute their barycenters b_κ as

$$\begin{aligned} \mathcal{B}_\kappa &= \operatorname{conv}\left(\left(0, -\tfrac{1}{2}\right), \left(\tfrac{-1}{1+a}, \tfrac{a}{2+2a}\right), \left(\tfrac{-1}{1+b}, \tfrac{b}{2+2b}\right)\right), \\ b_\kappa &= \left(-\tfrac{a+b+2}{3(a+1)(b+1)}, \tfrac{ab-1}{6(1+a)(1+b)}\right). \end{aligned}$$

Moreover, the moment polytope $\mathcal{B}_0$ is given by Lemma 3.5.42 and its barycenter can be computed as

$$\begin{aligned} \mathcal{B}_0 &= \operatorname{conv}\left((0,0), \left(\tfrac{-1}{a+1}, \tfrac{-a}{a+1}\right), \left(\tfrac{-1}{b+1}, \tfrac{-b}{b+1}\right)\right), \\ b_0 &= \left(-\tfrac{2+a+b}{3(1+a)(1+b)}, -\tfrac{a+b+2ab}{3(1+a)(1+b)}\right). \end{aligned}$$

Observe that $b_{\kappa,1}$ vanishes for all $\kappa = 0, 1, 2$ if and only if $b = -2 - a$. In this case, we have $b_{\kappa,2} = 1/6 > 0$ for $\kappa = 1, 2$. Finally, Proposition 3.5.33 tells us that $X(P)$ admits a Kähler-Einstein metric if and only if $b = -2 - a$. Comparing with Theorem 3.2.5 and inserting the condition $b = -2 - a$, the cases $S_{12}(1, \iota)$, $S_{21}(1, \iota)$ are ruled out since ι^+ and ι^- have to be both even or both odd. Hence we arrive at the shapes given by $S_{11}(1, \iota)$ and $S_{22}(1, \iota)$ and at $\iota^+ = \iota^-$. □

Proposition 3.5.43. *Let X be a complex full intrinsic quadric surface of Picard number two as in Construction 3.3.1. Then X doesn't admit a Kähler-Einstein metric.*

Lemma 3.5.44. *Consider a full intrinsic quadric surface X as in Construction 3.3.1. Then the fibers $\mathcal{X}_{\kappa,0}$ are normal for all $\kappa = 0, 1, 2$.*

Proof. This follows directly from $l_{01} = l_{02} = l_{11} = l_{12} = 1$, $l_{21} = 2$ and Proposition 1.3.7 (iv). □

Lemma 3.5.45. *Consider a full intrinsic quadric surface X as in Construction 3.3.1 and let $\tilde{\tau}_2$ be as in Construction 1.3.11. Then the primitive generators of $\tilde{\tau}_2$ are given by*

$$u_1 := (1, 1, -2), \quad u_2 := (a, 1, 1), \quad u_3 := (b + c, 1, 1).$$

Proof. We follow Construction 1.3.11 to determine $\tilde{\tau}_2$. Proposition 1.2.11 provides us with the generator matrix

$$\tilde{P} \;\coloneqq\; [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_5] \;\coloneqq\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

of the anticanonical cone of X. We denote by $\tilde{\sigma} \subseteq \mathbb{R}^4$ the cone over all five columns. First, we show

$$\sigma \;\coloneqq\; \operatorname{cone}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \cap \operatorname{lin}(e_2, e_3, e_4) \;=\; \operatorname{cone}((0,-1,a,1),(0,-1,b,1)). \quad (3.5.45.1)$$

We begin with the inclusion '$\subseteq$'. Let $w \coloneqq (w_1, w_2, w_3, w_4) \in \sigma$. Then one has a representation

$$w \;=\; \lambda_1\tilde{v}_1 + \lambda_2\tilde{v}_2 + \lambda_3\tilde{v}_3 + \lambda_4\tilde{v}_4 \in \mathbb{Z}^4$$

with $\lambda_i \in \mathbb{Q}_{\geq 0}$. Since $(w_1, w_2, w_3, w_4) \in \operatorname{lin}(e_2, e_3, e_4)$ we get $w_1 = 0$. Hence $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$. Substituting λ_4 according to the aforementioned equation yields

$$w \;=\; \lambda_1\tilde{v}_1 + \lambda_2\tilde{v}_2 + \lambda_3\tilde{v}_3 + (\lambda_1\tilde{v}_4 + \lambda_2\tilde{v}_4 - \lambda_3\tilde{v}_4)\,.$$

Because of $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$, we conclude that $\lambda_1 \geq \lambda_3$ or $\lambda_2 \geq \lambda_3$ holds. We may assume $\lambda_1 \geq \lambda_3$. Then, by rearranging the summands, one gets a linear combination with positive coefficients

$$\begin{aligned} w \;&=\; (\lambda_1 - \lambda_3)\tilde{v}_1 + \lambda_3\tilde{v}_1 + \lambda_2\tilde{v}_2 + \lambda_3\tilde{v}_3 + (\lambda_1\tilde{v}_4 + \lambda_2\tilde{v}_4 - \lambda_3\tilde{v}_4) \\ &=\; (\lambda_1 - \lambda_3)(\tilde{v}_1 + \tilde{v}_4) + \lambda_3(\tilde{v}_1 + \tilde{v}_3) + \lambda_2(\tilde{v}_2 + \tilde{v}_4). \end{aligned}$$

Altogether, any vector $w \in \sigma$ is a positive linear combination of $\tilde{v}_1 + \tilde{v}_4$, $\tilde{v}_1 + \tilde{v}_3$, $\tilde{v}_2 + \tilde{v}_4$. Analogously, $\lambda_2 \geq \lambda_3$ yields that any vector $w \in \sigma$ is a positive linear combination of $\tilde{v}_2 + \tilde{v}_4$, $\tilde{v}_2 + \tilde{v}_3$, $\tilde{v}_1 + \tilde{v}_4$. In other words, w is a positive linear combination of

$$\begin{aligned} \tilde{v}_1 + \tilde{v}_3 \;&=\; (0,-1,a,1), \quad & \tilde{v}_1 + \tilde{v}_4 \;&=\; (0,-1,a+c,1), \\ \tilde{v}_2 + \tilde{v}_3 \;&=\; (0,-1,b,1), \quad & \tilde{v}_2 + \tilde{v}_4 \;&=\; (0,-1,b+c,1). \end{aligned}$$

Construction 3.3.1 provides us with the conditions $b + c < c \leq a + c < a$. Whence the vectors $(0,-1,a+c,1)$ and $(0,-1,b,1)$ are contained in $\operatorname{cone}((0,-1,a,1),(0,-1,b+c,1))$. This yields the desired inclusion. The inclusion '$\supseteq$' is clear. Altogether, (3.5.45.1) follows. Further, we note $\tilde{v}_5 \in \operatorname{lin}(e_2, e_3, e_4)$. Hence

$$\tilde{\sigma} \cap \operatorname{lin}(e_2, e_3, e_4) \;=\; \operatorname{cone}\left((0,-1,a,1),\ (0,-1,b+c,1),\ (0,2,1,1)\right).$$

Let $\eta_2 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$ be the map given by

$$\eta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This yields

$$\begin{aligned} \tilde{\tau}_2 &:= \eta_2^{-1}\left(\tilde{\sigma} \cap \operatorname{lin}(e_2, e_3, e_4)\right) \\ &= \eta_2^{-1}\left(\operatorname{cone}\left((0,-1,a,1),\ (0,-1,b+c,1),\ (0,2,1,1)\right)\right) \\ &= \operatorname{cone}\left((a,1,1),\ (b+c,1,1),\ (1,1,-2)\right). \end{aligned}$$

Construction 3.3.1 yields the inequality $\det((a,1,1),(b+c,1,1),(1,1,-2)) = 3(b+c-a) < 0$, hence the cone is three-dimensional and requires all three generators. The assertion follows. $\square$

Lemma 3.5.46. *Consider a full intrinsic quadric surface X as in Construction 3.3.1 and let $\tilde{\tau}_2$ be given by Lemma 3.5.45. Then the dual cone $\tilde{\omega}_2 := \tilde{\tau}_2^\vee$ has the primitive generators*

$$(0,1,-1)\,, \quad \tfrac{1}{\gcd(3,1-b-c)}\left(3,-1-2b-2c,1-b-c\right), \quad \tfrac{1}{\gcd(3,a-1)}\left(-3,1+2a,a-1\right).$$

Proof. We prove that each of the listed ray generators cuts out a facet of $\tilde{\tau}_2$ and evaluates positively on the complementary ray. Let u_1, u_2, u_3 be as in Lemma 3.5.45. Then we check

$$\begin{aligned} (0,1,-1)\cdot[u_1,u_2,u_3] &= (3,0,0)^\top, \\ \tfrac{1}{\gcd(3,1-b-c)}\left(3,-1-2b-2c,1-b-c\right)\cdot[u_1,u_2,u_3] &= \tfrac{1}{\gcd(3,1-b-c)}(0,3a-3b-3c,0)^\top, \\ \tfrac{1}{\gcd(3,a-1)}\left(-3,1+2a,a-1\right)\cdot[u_1,u_2,u_3] &= \tfrac{1}{\gcd(3,a-1)}(0,0,3a-3b-3c)^\top. \end{aligned}$$

Moreover, Construction 3.3.1 yields $a-b-c>0$. Thus, all entries are positive or equal to zero. $\square$

Lemma 3.5.47. *Consider a full intrinsic quadric surface X as in Construction 3.3.1. Then the moment polytope $\mathcal{B}_2$ is given by*

$$\mathcal{B}_2 \;=\; \operatorname{conv}\left(\left(\tfrac{-3}{2a+1}, \tfrac{a-1}{2a+1}\right), \left(\tfrac{3}{-1-2b-2c}, \tfrac{1-b-c}{-1-2b-2c}\right), (0,-1)\right).$$

Proof. We set

$$\iota\colon \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3, \quad u \mapsto (u_1, 1, u_2).$$

Moreover, let $\tilde{\omega}_2$ be the cone given by Lemma 3.5.46. Then one gets

$$\begin{aligned} \mathcal{C}_2 &:= \iota^{-1}\left(\operatorname{conv}(\tilde{\omega}_2)\right) \\ &= \operatorname{conv}\left(\iota^{-1}\left(0,1,-1\right), \iota^{-1}\left(-\tfrac{3}{1+2b+2c}, 1, \tfrac{1-b-c}{-1-2b-2c}\right), \iota^{-1}\left(-\tfrac{3}{1+2a}, 1, \tfrac{a-1}{2a+1}\right)\right) \\ &= \operatorname{conv}\left(\left(0,-1\right), \left(\tfrac{3}{-1-2b-2c}, \tfrac{1-b-c}{-1-2b-2c}\right), \left(\tfrac{-3}{2a+1}, \tfrac{a-1}{2a+1}\right)\right). \end{aligned}$$

By Construction 3.2.1 we know that $-1-b-c \geq 0$ and $a \geq 0$. Hence the unique interior point of $\mathcal{C}_\kappa$ is $(0,0)$. Then, Construction 1.3.11 yields $\mathcal{B}_2 = \mathcal{C}_2$. $\square$

Proof of Proposition 3.5.43. The moment polytope $\mathcal{B}_2$ is given by Lemma 3.5.47. and its barycenter b_2 is

$$b_2 = \left(-2\tfrac{a+b+c+1}{(2a+1)(2b+2c+1)},\, -\tfrac{a+b+c+1}{(2a+1)(2b+2c+1)}\right).$$

In particular, we have $b_{2,1} = 0$ if and only if $b_{2,2} = 0$. Moreover, Proposition 1.3.7 (iv) tells us that $\kappa = 2$ is special. Hence, Proposition 3.5.33 yields that X doesn't admit a Kähler-Einstein metric. □

Proposition 3.5.48. *Let X be a complex full intrinsic quadric surface of Picard number three as in Construction 3.4.1 admitting a Kähler-Einstein metric. Then $X \cong X(P)$ for precisely one P from the following*

$$\begin{array}{ll} 2 \nmid \iota, \quad -2\iota \leq 2c+d, & -4\iota \leq 2c+d,\ c \leq d \leq -1, \\ c \leq d \leq -1,\ c+d \leq -\iota-1: & c+d \leq -2\iota-1: \\ \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota & -\iota-c-d & 0 & c & 0 & d \end{bmatrix}, & \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota & -2\iota-c-d & 0 & c & 0 & d \end{bmatrix}, \end{array}$$

where ι denotes the Gorenstein index of X. Conversely, each X with generator matrix P from the above list admits a Kähler-Einstein metric.

Lemma 3.5.49. *Consider a full intrinsic quadric surface X as in Construction 3.4.1. Then the fibers $\mathcal{X}_{\kappa,0}$ are normal for all $\kappa = 0, 1, 2$.*

Proof. This follows directly from $l_{ij} = 1$ for all $i = 0, 1, 2$, $j = 1, 2$ and Proposition 1.3.7 (iv). □

Remark 3.5.50. Consider a full intrinsic quadric surface X as in Construction 3.4.1. According to Proposition 1.2.11 the generator matrix of the anticanonical cone over X is

$$\tilde{P} := [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_5, \tilde{v}_6] := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

In the following we denote by $\tilde{\sigma}$ the cone over all six columns.

The proof of the following lemmata is completely analogous to Lemma 3.5.45, Lemma 3.5.46 and Lemma 3.5.47, but for convenience of the reader we give them in detail.

Lemma 3.5.51. *Consider a full intrinsic quadric surface X as in Construction 3.4.1 and let $\tilde{\tau}_0$ be as in Construction 1.3.11. Then the primitive generators of $\tilde{\tau}_0$ are given by*

$$u_1 := (0,2,1), \quad u_2 := (a,0,-1), \quad u_3 := (b,0,-1), \quad u_4 := (c+d,2,1).$$

Proof. We compute the generators of $\tilde{\tau}_0$ according to Construction 1.3.11. Let $\tilde{v}_i$, $i = 1, \ldots, 6$ and $\tilde{\sigma}$ be as in Remark 3.5.50. First, we prove

$$\sigma \;\coloneqq\; \operatorname{cone}(\tilde{v}_3, \tilde{v}_4, \tilde{v}_5, \tilde{v}_6) \cap \operatorname{lin}(e_0, e_3, e_4) \;=\; \operatorname{cone}\left((1,1,0,2), (1,1,c+d,2)\right). \quad (3.5.51.1)$$

We begin by showing '$\subseteq$'. Let $w \coloneqq (w_1, w_2, w_3, w_4) \in \sigma$. Then w is given as

$$w \;=\; \lambda_3\tilde{v}_3 + \lambda_4\tilde{v}_4 + \lambda_5\tilde{v}_5 + \lambda_6\tilde{v}_6 \in \mathbb{Z}^4$$

with $\lambda_i \in \mathbb{Q}_{\geq 0}$. Because of $(w_1, w_2, w_3, w_4) \in \operatorname{lin}(e_0, e_3, e_4)$ we obtain $w_1 = w_2$. Consequently, one has $\lambda_3 + \lambda_4 = \lambda_5 + \lambda_6$. By substituting λ_6 we get

$$w \;=\; \lambda_3\tilde{v}_3 + \lambda_4\tilde{v}_4 + \lambda_5\tilde{v}_5 + (\lambda_3\tilde{v}_6 + \lambda_4\tilde{v}_6 - \lambda_5\tilde{v}_6).$$

Since $\lambda_3 + \lambda_4 = \lambda_5 + \lambda_6$, we have $\lambda_3 \geq \lambda_5$ or $\lambda_4 \geq \lambda_5$. We may assume $\lambda_3 \geq \lambda_5$. Then, we rearrange the summands such that

$$\begin{aligned} w &= (\lambda_3 - \lambda_5)\tilde{v}_3 + \lambda_5\tilde{v}_3 + \lambda_4\tilde{v}_4 + \lambda_5\tilde{v}_5 + (\lambda_3\tilde{v}_6 + \lambda_4\tilde{v}_6 - \lambda_5\tilde{v}_6) \\ &= (\lambda_3 - \lambda_5)(\tilde{v}_3 + \tilde{v}_6) + \lambda_5(\tilde{v}_3 + \tilde{v}_5) + \lambda_4(\tilde{v}_4 + \tilde{v}_6). \end{aligned}$$

Note that this is a linear combination with positive coefficients. The case $\lambda_4 \geq \lambda_5$ results in a similar way in a representation of w as a positive linear combination of $\tilde{v}_4 + \tilde{v}_6$, $\tilde{v}_4 + \tilde{v}_5$ and $\tilde{v}_3 + \tilde{v}_6$. Thus, any vector $w \in \sigma$ is a positive linear combination of

$$\tilde{v}_3+\tilde{v}_5 \;=\; (1,1,0,2),\; \tilde{v}_3+\tilde{v}_6 \;=\; (1,1,d,2),\; \tilde{v}_4+\tilde{v}_5 \;=\; (1,1,c,2),\; \tilde{v}_4+\tilde{v}_6 \;=\; (1,1,c+d,2).$$

Construction 3.4.1 yields $c + d < c \leq d < 0$. Hence $\operatorname{cone}((1,1,0,2), (1,1,c+d,2))$ contains $(1,1,c,2)$ and $(1,1,d,2)$. The inclusion '$\supseteq$' is clear. This proves (3.5.51.1). Moreover, we have $\tilde{v}_1, \tilde{v}_2 \in \operatorname{lin}(e_0, e_3, e_4)$. This leads to

$$\tilde{\sigma} \cap \operatorname{lin}(e_0, e_3, e_4) \;=\; \operatorname{cone}\left((1,1,0,2),\; (1,1,c+d,2),\; (-1,-1,a,0),\; (-1,-1,b,0)\right).$$

Furthermore, let $\eta_0 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$ be the map given by

$$\eta_0 \;=\; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, we get

$$\begin{aligned} \tilde{\tau}_0 \;&\coloneqq\; \eta_0^{-1}\left(\tilde{\sigma} \cap \operatorname{lin}(e_2, e_3, e_4)\right) \\ &=\; \eta_0^{-1}\left(\operatorname{cone}\left((1,1,0,2),\; (1,1,c+d,2),\; (-1,-1,a,0),\; (-1,-1,b,0)\right)\right) \\ &=\; \operatorname{cone}\left((0,2,1),\; (c+d,2,1),\; (a,0,-1),\; (b,0,-1)\right). \end{aligned}$$

Lastly, due to Construction 3.4.1 we have $b < a$. Thus, each of the four generators of $\tilde{\tau}_0$ cannot be represented as a positive linear combination of the other three. ☐

Lemma 3.5.52. *Consider a full intrinsic quadric surface X as in Construction 3.4.1 and let $\tilde{\tau}_0$ be as in Lemma 3.5.51. Then the dual cone $\tilde{\omega}_0 := \tilde{\tau}_0^\vee$ has the primitive generators*

$$(0,1,0)\,, \quad \tfrac{1}{\gcd(2,a)}\,(-2,a,-2a)\,, \quad (0,1,-2)\,, \quad \tfrac{1}{\gcd(2,b-c-d)}\,(2,-b-c-d,2b)\,.$$

Proof. In order to see that the vectors listed above are ray generators of $\tilde{\omega}_0$ we show that each of them cuts out a facet of $\tilde{\tau}_0$ and evaluates positively on the two complementary rays. We denote by $\tilde{P}$ the matrix whose columns are the primitive generators of $\tilde{\tau}_0$ given by Lemma 3.5.51. Then we obtain

$$\begin{aligned} (0,1,0)\cdot \tilde{P} &= (2,0,0,2)^\top\,, \\ \tfrac{1}{\gcd(2,a)}\,(-2,a,-2a)\cdot \tilde{P} &= \tfrac{1}{\gcd(2,a)}\,(0,0,2a-2b,-2c-2d)^\top\,, \\ (0,1,-2)\cdot \tilde{P} &= (0,2,2,0)^\top\,, \\ \tfrac{1}{\gcd(2,b-c-d)}\,(2,-b-c-d,2b)\cdot \tilde{P} &= \tfrac{1}{\gcd(2,b-c-d)}\,(-2c-2d,2a-2b,0,0)^\top\,. \end{aligned}$$

We recall from Construction 3.4.1 that $a > b$ and $c, d < 0$ and hence the non zero entries are all positive. ☐

Lemma 3.5.53. *Consider a full intrinsic quadric surface X as in Construction 3.4.1. Then the moment polytope $\mathcal{B}_0$ is given by*

$$\mathcal{B}_0 \;=\; \operatorname{conv}\Big((0,1)\,, \Big(-\tfrac{2}{a},-1\Big)\,, (0,-1)\,, \Big(-\tfrac{2}{b+c+d},-\tfrac{2}{b+c+d}\Big)\Big)\,.$$

Proof. We set

$$\iota\colon \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3, \quad u \mapsto (u_1,1,u_2).$$

Moreover, let $\tilde{\omega}_0$ be the cone given by Lemma 3.5.52. Then one gets

$$\begin{aligned} \mathcal{C}_0 \;&:=\; \iota^{-1}\left(\operatorname{conv}(\tilde{\omega}_0)\right) \\ &=\; \iota^{-1}\big(\operatorname{conv}\big(\,(0,1,0)\,, \tfrac{1}{\gcd(2,a)}\,(-2,a,-2a)\,, (0,1,-2)\,, \tfrac{1}{\gcd(2,b-c-d)}\,(2,-b-c-d,2b)\,\big)\big) \\ &=\; \operatorname{conv}\Big((0,0)\,, \Big(-\tfrac{2}{a},-2\Big)\,, (0,-2)\,, \Big(-\tfrac{2}{b+c+d},-\tfrac{2b}{b+c+d}\Big)\Big)\,. \end{aligned}$$

Construction 3.4.1 yields $b+c+d < 0$ and $0 < a$. Hence the unique interior point of $\mathcal{C}_0$ is $(0,-1)$. It follows from Construction 1.3.11 that $\mathcal{B}_0 = \mathcal{C}_0 + (0,1)$. ☐

Lemma 3.5.54. *Consider a full intrinsic quadric surface X as in Construction 3.4.1 and let $\tilde{\tau}_1$ be as in Construction 1.3.11. Then the primitive generators of $\tilde{\tau}_1$ are given by*

$$u_1 \;:=\; (0,1,-1)\,, \quad u_2 \;:=\; (a,1,1)\,, \quad u_3 \;:=\; (b+d,1,1)\,, \quad u_4 \;:=\; (c,1,-1)\,.$$

Proof. We determine $\tilde{\tau}_1$ by following Construction 1.3.11. Let $\tilde{v}_i, i = 1, \ldots, 6$ and $\tilde{\sigma}$ be as in Remark 3.5.50. In a first step, we show

$$\sigma \; := \; \operatorname{cone}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_5, \tilde{v}_6) \cap \operatorname{lin}(e_1, e_3, e_4) \; = \; \operatorname{cone}\left((-1, 0, a, 1), (-1, 0, b + d, 1)\right).$$

We begin by showing '$\subseteq$'. Let $w := (w_1, w_2, w_3, w_4) \in \sigma$. Then w can be represented as

$$w \; = \; \lambda_1 \tilde{v}_1 + \lambda_2 \tilde{v}_2 + \lambda_5 \tilde{v}_5 + \lambda_6 \tilde{v}_6 \in \mathbb{Z}^4$$

with some $\lambda_i \in \mathbb{Q}_{\geq 0}$. Since $(w_1, w_2, w_3, w_4) \in \operatorname{lin}(e_1, e_3, e_4)$ we get $w_2 = 0$. Thus, we have $\lambda_1 + \lambda_2 = \lambda_5 + \lambda_6$. By substituting λ_6 accordingly we get

$$w \; = \; \lambda_1 \tilde{v}_1 + \lambda_2 \tilde{v}_2 + \lambda_5 \tilde{v}_5 + (\lambda_1 \tilde{v}_6 + \lambda_2 \tilde{v}_6 - \lambda_5 \tilde{v}_6).$$

Since $\lambda_1 + \lambda_2 = \lambda_5 + \lambda_6$, we have $\lambda_1 \geq \lambda_5$ or $\lambda_2 \geq \lambda_5$. We may assume $\lambda_1 \geq \lambda_5$. Further, we rearrange the summands such that

$$w \; = \; (\lambda_1 - \lambda_5)(\tilde{v}_1 + \tilde{v}_6) + \lambda_5(\tilde{v}_1 + \tilde{v}_5) + \lambda_2(\tilde{v}_2 + \tilde{v}_6).$$

Note that the linear combination has positive coefficients. Hence every $w \in \sigma$ is a positive linear combination of $\tilde{v}_1 + \tilde{v}_6$, $\tilde{v}_1 + \tilde{v}_5$, $\tilde{v}_2 + \tilde{v}_6$. Analogously, we represent w as a positive linear combination of $\tilde{v}_2 + \tilde{v}_6$, $\tilde{v}_2 + \tilde{v}_5$, $\tilde{v}_1 + \tilde{v}_6$ if we have $\lambda_2 \geq \lambda_5$. In other words, w is a positive linear combination of

$$\begin{aligned} \tilde{v}_1 + \tilde{v}_5 \; &= \; (-1, 0, a, 1), & \tilde{v}_1 + \tilde{v}_6 \; &= \; (-1, 0, a + d, 1), \\ \tilde{v}_2 + \tilde{v}_5 \; &= \; (-1, 0, b, 1), & \tilde{v}_2 + \tilde{v}_6 \; &= \; (-1, 0, b + d, 1). \end{aligned}$$

Construction 3.4.1 yields $b + d < b \leq a + d < a$. Hence $\operatorname{cone}((-1, 0, a, 1), (-1, 0, b + d, 1))$ contains $(-1, 0, a + d, 1)$ and $(-1, 0, b, 1)$. The inclusion '$\supseteq$' is clear. Moreover, we have $\tilde{v}_3, \tilde{v}_4 \in \operatorname{lin}(e_1, e_3, e_4)$. Therefore we obtain

$$\tilde{\sigma} \cap \operatorname{lin}(e_1, e_3, e_4) \; = \; \operatorname{cone}\left((-1, 0, a, 1),\ (-1, 0, b + d, 1),\ \tilde{v}_3,\ \tilde{v}_4\right).$$

Furthermore, let $\eta_1 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$ be the map given by

$$\eta_1 \; = \; \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence one obtains

$$\begin{aligned} \tilde{\tau}_1 \; &:= \; \eta_1^{-1} \left(\tilde{\sigma} \cap \operatorname{lin}(e_1, e_3, e_4)\right) \\ &= \; \eta_1^{-1} \left(\operatorname{cone}\left((-1, 0, a, 1),\ (-1, 0, b + d, 1),\ (1, 0, 0, 1),\ (1, 0, c, 1)\right)\right) \\ &= \; \operatorname{cone}\left((a, 1, 1),\ (b + d, 1, 1),\ (0, 1, -1),\ (c, 1, -1)\right). \end{aligned}$$

Lastly, we note $0 < c$. Thus, each of the four generators of $\tilde{\tau}_1$ cannot be represented as a positive linear combination of the other three. ☐

Lemma 3.5.55. *Consider a full intrinsic quadric surface X as in Construction 3.4.1 and let $\tilde{\tau}_1$ be as in Lemma 3.5.54. Then the dual cone $\tilde{\omega}_1 := \tilde{\tau}_1^\vee$ has the primitive generators*

$$(0,1,-1)\,, \quad \tfrac{1}{\gcd(2,a)}\,(-2,a,a)\,, \quad (0,1,1)\,, \quad \tfrac{1}{\gcd(2,c-b-d)}\,(2,-b-c-d,c-b-d)\,.$$

Proof. By showing that each of the vectors listed above cuts out a facet of $\tilde{\tau}_1$ and evaluates positively on the two complementary rays we prove that they are generators of $\tilde{\omega}_1$. Let $\tilde{P}$ be the matrix whose columns are the primitive generators of $\tilde{\tau}_1$. Then, we observe

$$\begin{aligned} (0,1,-1)\cdot\tilde{P} &= (2,0,0,2)^\top\,, \\ \tfrac{1}{\gcd(2,a)}\,(-2,a,a)\cdot\tilde{P} &= \tfrac{1}{\gcd(2,a)}\,(2a-2b-2d,0,0,-2c)^\top\,, \\ (0,1,1)\cdot\tilde{P} &= (0,2,2,0)^\top\,, \\ \tfrac{1}{\gcd(2,c-b-d)}\,(2,-b-c-d,c-b-d)\cdot\tilde{P} &= \tfrac{1}{\gcd(2,c-b-d)}\,(-2c,2a-2-b-2d,0,0)^\top\,. \end{aligned}$$

According to Construction 3.4.1 we have $a > b$ and $c,d < 0$. Thus, the entries are positive or equal to zero. ☐

Lemma 3.5.56. *Consider a full intrinsic quadric surface X as in Construction 3.4.1. Then the moment polytope $\mathcal{B}_1$ is given by*

$$\mathcal{B}_1 \;=\; \operatorname{conv}\Big((0,-1)\,,\Big(-\tfrac{2}{a},1\Big)\,,(0,1)\,,\Big(-\tfrac{2}{b+c+d},-\tfrac{c-b-d}{b+c+d}\Big)\Big)\,.$$

Proof. Let

$$\iota\colon \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3, \quad u \mapsto (u_1,1,u_2)$$

and let $\tilde{\omega}_1$ be the cone given by Lemma 3.5.55. This leads to

$$\begin{aligned} \mathcal{C}_1 &:= \iota^{-1}\,(\operatorname{conv}(\tilde{\omega}_1)) \\ &= \iota^{-1}(\operatorname{conv}(\,(0,1,-1)\,,\tfrac{1}{\gcd(2,a)}\,(-2,a,a)\,,(0,1,1)\,,\tfrac{1}{\gcd(2,b-c-d)}\,(2,-b-c-d,c-b-d)\,)) \\ &= \operatorname{conv}\Big((0,-1)\,,\Big(-\tfrac{2}{a},1\Big)\,,(0,1)\,,\Big(-\tfrac{2}{b+c+d},-\tfrac{c-b-d}{b+c+d}\Big)\Big)\,. \end{aligned}$$

Moreover, we have $b+c+d < 0$ and $0 < a$ due to Construction 3.4.1. Thus, the polytope $\mathcal{C}_1$ is such that its unique interior point is $(0,0)$. Hence, we derive $\mathcal{B}_1 = \mathcal{C}_1$ from Construction 1.3.11. ☐

Lemma 3.5.57. *Consider a full intrinsic quadric surface X as in Construction 3.4.1 and let $\tilde{\tau}_2$ be as in Construction 1.3.11. Then the primitive generators of $\tilde{\tau}_2$ are given by*

$$u_1 \;:=\; (0,1,-1)\,, \quad u_2 \;:=\; (a,1,1)\,, \quad u_3 \;:=\; (b+c,1,1)\,, \quad u_4 \;:=\; (d,1,-1)\,.$$

Proof. We determine $\tilde{\tau}_2$ by following Construction 1.3.11. Let $\tilde{v}_i, i = 1, \ldots, 6$ and $\tilde{\sigma}$ be as in Remark 3.5.50. We begin by showing

$$\sigma \ := \ \operatorname{cone}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) \cap \operatorname{lin}(e_2, e_3, e_4) \ = \ \operatorname{cone}\left((0, -1, a, 1), (0, -1, b + c, 1)\right).$$

First of all, we prove '$\subseteq$'. Let $w := (w_1, w_2, w_3, w_4) \in \sigma$. Then w can be represented as

$$w \ = \ \lambda_1 \tilde{v}_1 + \lambda_2 \tilde{v}_2 + \lambda_3 \tilde{v}_3 + \lambda_4 \tilde{v}_4 \in \mathbb{Z}^4$$

with some $\lambda_i \in \mathbb{Q}_{\geq 0}$. Since $(w_1, w_2, w_3, w_4) \in \operatorname{lin}(e_2, e_3, e_4)$ we get $w_1 = 0$. Thus, we have $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$. We substitute λ_4 to obtain

$$w \ = \ \lambda_1 \tilde{v}_1 + \lambda_2 \tilde{v}_2 + \lambda_3 \tilde{v}_3 + (\lambda_1 \tilde{v}_4 + \lambda_2 \tilde{v}_4 - \lambda_3 \tilde{v}_4).$$

Since $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$, we have $\lambda_1 \geq \lambda_3$ or $\lambda_2 \geq \lambda_3$. We may assume $\lambda_1 \geq \lambda_3$. Further, we rearrange the summands such that

$$w \ = \ (\lambda_1 - \lambda_3)(\tilde{v}_1 + \tilde{v}_4) + \lambda_3(\tilde{v}_1 + \tilde{v}_3) + \lambda_2(\tilde{v}_2 + \tilde{v}_4).$$

This yields a linear combination with positive coefficients. Hence every $w \in \sigma$ is a positive linear combination of $\tilde{v}_1 + \tilde{v}_4$, $\tilde{v}_1 + \tilde{v}_3$, $\tilde{v}_2 + \tilde{v}_4$. For $\lambda_2 \geq \lambda_3$ rearranging analogously shows that w is a positive linear combination of $\tilde{v}_2 + \tilde{v}_4$, $\tilde{v}_2 + \tilde{v}_3$, $\tilde{v}_1 + \tilde{v}_4$. In other words, w is a positive linear combination of

$$\begin{array}{rclrcl} \tilde{v}_1 + \tilde{v}_3 & = & (0, -1, a, 1), & \tilde{v}_1 + \tilde{v}_4 & = & (0, -1, a + c, 1), \\ \tilde{v}_2 + \tilde{v}_3 & = & (0, -1, b, 1), & \tilde{v}_2 + \tilde{v}_4 & = & (0, -1, b + c, 1). \end{array}$$

Construction 3.4.1 yields $b + c < b \leq a + c < a$. Hence $\operatorname{cone}((0, -1, a, 1), (0, -1, b + c, 1))$ contains $(0, -1, a + c, 1)$ and $(0, -1, b, 1)$. The inclusion '$\supseteq$' is clear. Moreover, we have $\tilde{v}_5, \tilde{v}_6 \in \operatorname{lin}(e_2, e_3, e_4)$. It follows

$$\tilde{\sigma} \cap \operatorname{lin}(e_2, e_3, e_4) \ = \ \operatorname{cone}\left((0, -1, a, 1),\ (0, -1, b + c, 1),\ \tilde{v}_5,\ \tilde{v}_6\right).$$

Moreover, let $\eta_2 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4$ be the map given by

$$\eta_2 \ = \ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We put this together to

$$\begin{aligned} \tilde{\tau}_2 \ &:= \ \eta_2^{-1}\left(\tilde{\sigma} \cap \operatorname{lin}(e_2, e_3, e_4)\right) \\ &= \ \eta_2^{-1}\left(\operatorname{cone}\left((0, -1, a, 1),\ (0, -1, b + c, 1),\ (0, 1, 0, 1),\ (0, 1, d, 1)\right)\right) \\ &= \ \operatorname{cone}\left((a, 1, 1),\ (b + c, 1, 1),\ (0, 1, -1),\ (d, 1, -1)\right). \end{aligned}$$

Further, Construction 3.4.1 provides us with $0 > d$. Thus, none of the four generators of $\tilde{\tau}_2$ lies inside the cone spanned by the other three generators. □

Lemma 3.5.58. *Consider a full intrinsic quadric surface X as in Construction 3.4.1 and let $\tilde{\tau}_2$ be as in Lemma 3.5.57. Then the dual cone $\tilde{\omega}_2 \coloneqq \tilde{\tau}_2^\vee$ has the primitive generators*

$$(0,1,-1)\,, \quad \tfrac{1}{\gcd(2,a)}\,(-2,a,a)\,, \quad (0,1,1)\,, \quad \tfrac{1}{\gcd(2,d-b-c)}\,(2,-b-c-d,d-b-c)\,.$$

Proof. In order to see that the vectors listed above are ray generators of $\tilde{\omega}_2$, we show that each of them cuts out a facet of $\tilde{\tau}_2$ and evaluates positively on the two complementary rays. Denote by $\tilde{P}$ the matrix which columns are the primitive generators of $\tilde{\tau}_2$. Then we check

$$\begin{aligned}
(0,1,-1)\cdot\tilde{P} &= (2,0,0,2)^\top\,,\\
\tfrac{1}{\gcd(2,a)}\,(-2,a,a)\cdot\tilde{P} &= \tfrac{1}{\gcd(2,a)}\,(2a-2b-2c,0,0,-2d)^\top\,,\\
(0,1,1)\cdot\tilde{P} &= (0,2,2,0)^\top\,,\\
\tfrac{1}{\gcd(2,d-b-c)}\,(2,-b-c-d,d-b-c)\cdot\tilde{P} &= \tfrac{1}{\gcd(2,d-b-c)}\,(-2d,2a-2-b-2c,0,0)^\top\,.
\end{aligned}$$

According to Construction 3.4.1 we get $a > b$ and $c, d < 0$. This implies that the non zero entries are all positive. $\square$

Lemma 3.5.59. *Consider a full intrinsic quadric surface X as in Construction 3.4.1. Then the moment polytope $\mathcal{B}_2$ is given by*

$$\mathcal{B}_2 \;=\; \operatorname{conv}\Big((0,-1)\,, \Big(-\tfrac{2}{a},1\Big)\,, (0,1)\,, \Big(-\tfrac{2}{b+c+d}, -\tfrac{d-b-c}{b+c+d}\Big)\Big)\,.$$

Proof. Set

$$\iota\colon \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3, \quad u \mapsto (u_1,1,u_2).$$

Further, let $\tilde{\omega}_2$ be the cone given by Lemma 3.5.58. Then one gets

$$\begin{aligned}
\mathcal{C}_2 &\coloneqq \iota^{-1}\left(\operatorname{conv}(\tilde{\omega}_2)\right)\\
&= \iota^{-1}\big(\operatorname{conv}\big(\,(0,1,-1)\,, \tfrac{1}{\gcd(2,a)}\,(-2,a,a)\,, (0,1,1)\,, \tfrac{1}{\gcd(2,b-c-d)}\,(2,-b-c-d,d-b-c)\,\big)\big)\\
&= \operatorname{conv}\Big((0,-1)\,, \Big(-\tfrac{2}{a},1\Big)\,, (0,1)\,, \Big(-\tfrac{2}{b+c+d}, -\tfrac{d-b-c}{b+c+d}\Big)\Big)\,.
\end{aligned}$$

Construction 3.4.1 provides us with $b+c+d<0$ and $0<a$. Consequently, the unique interior point of $\mathcal{C}_2$ is $(0,0)$. Then, we infer from Construction 1.3.11 that $\mathcal{B}_2 = \mathcal{C}_2$. $\square$

Proof of Proposition 3.5.48. Lemma 3.5.53, Lemma 3.5.56 and Lemma 3.5.59 yield the following moment polytopes and barycenters

$$\begin{array}{rcl}
\mathcal{B}_0 & = & \operatorname{conv}\left((0,-1),\, \left(-\tfrac{2}{b+c+d}, \tfrac{c+d-b}{b+c+d}\right),\, (0,1),\, \left(-\tfrac{2}{a}, -1\right)\right), \\
b_0 & = & \left(-\tfrac{2(a+b+c+d)}{3a(b+c+d)},\ \tfrac{(b+2c+2d-a)b+(c+a+d)(c+d))}{3(a-b-c-d)(b+c+d)}\right), \\[1em]
\mathcal{B}_1 & = & \operatorname{conv}\left((0,-1),\, \left(-\tfrac{2}{b+c+d}, \tfrac{b+d-c}{b+c+d}\right),\, (0,1),\, \left(-\tfrac{2}{a}, 1\right)\right), \\
b_1 & = & \left(-\tfrac{2(a+b+c+d)}{3a(b+c+d)},\ \tfrac{(a-b-2c-2d)b-(a+c+2d)c+(a-d)d)}{3(a-b-c-d)(b+c+d)}\right), \\[1em]
\mathcal{B}_2 & = & \operatorname{conv}\left((0,-1),\, \left(-\tfrac{2}{b+c+d}, \tfrac{b+c-d}{b+c+d}\right),\, (0,1),\, \left(-\tfrac{2}{a}, 1\right)\right), \\
b_2 & = & \left(-\tfrac{2(a+b+c+d)}{3a(b+c+d)},\ \tfrac{(a-b-2c-2d)b+(a-c-2d)c-(a+d)d}{3(a-b-c-d)(b+c+d)}\right).
\end{array}$$

We conclude that $X(P)$ admits a Kähler-Einstein metric if and only if we have $d = -a-b-c$, reflecting $b_{\kappa,1} = 0$, and conditions

$$\begin{array}{ll}
b > 0, & \text{if } \kappa = 0, \\
b + d < 0, & \text{if } \kappa = 1, \\
a + d > 0, & \text{if } \kappa = 2,
\end{array}$$

reflecting $b_{\kappa,2} > 0$. Substituting $d = -a-b-c$, and a, b with the corresponding entries from Theorem 3.4.5, we arrive at

$$\iota^+ = \iota^-, \qquad \iota^+ = 2\iota^-, \qquad 2\iota^+ = \iota^-, \qquad \iota^+ = \iota^-,$$

according to the shapes defined by $S_{11}(3,\iota)$, $S_{12}(3,\iota)$, $S_{21}(3,\iota)$ and $S_{22}(3,\iota)$. Note that $S_{12}(3,\iota)$, $S_{21}(3,\iota)$ are ruled out by ι^+, ι^- being odd, respectively. ☐

Remark 3.5.60. Let $X = X(P)$ arise from Construction 3.2.1, 3.3.1 or 3.4.1. Set $\rho = \rho(X)$. Then, for $(\rho, \kappa) \neq (1,0)$, the toric degeneration $\mathcal{X}_{\kappa,0}$ is a normal projective toric del Pezzo surface and, according to the constellations (ρ, κ), the generator matrix of its defining complete fan Δ_κ is given by

(1,1)

$$\begin{bmatrix} 1 & 1{+}2a & 1{+}2b \\ -2 & 2 & 2 \end{bmatrix}$$

(1,2)

$$\begin{bmatrix} 1 & 1{+}2a & 1{+}2b \\ -2 & 2 & 2 \end{bmatrix}$$

(2,0)

$$\begin{bmatrix} a & b & 1 & 2c{+}1 \\ -1 & -1 & 2 & 2 \end{bmatrix}$$

(2,1)

$$\begin{bmatrix} 0 & c & 2a{+}1 & 2b{+}1 \\ -1 & -1 & 2 & 2 \end{bmatrix}$$

(2,2)

$$\begin{bmatrix} 1 & a & b{+}c \\ -2 & 1 & 1 \end{bmatrix}$$

(3,0)

$$\begin{bmatrix} 0 & c{+}d & b & a \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

(3,1)

$$\begin{bmatrix} a & b{+}d & c & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

(3,2)

$$\begin{bmatrix} a & b{+}c & d & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

The above presentation of the primitive ray generators of Δ_κ in $\mathbb{Z}^2$ is done with respect to the *antitropical coordinates* introduced in Construction 1.3.4. Note that the columns of the above matrices are precisely the vertices of the associated Fano polytopes $\mathcal{A}_\kappa$; see also Lemma 3.5.45, Lemma 3.5.51, Lemma 3.5.54 and Lemma 3.5.57.

Remark 3.5.61. Among the surfaces discussed in Example 3.2.6, 3.3.6, 3.4.6 only the surface X from Example 3.2.6 admits a Kähler-Einstein metric, due to Theorem 3.5.32. Let $w_X \in \mathrm{Cl}(X)$ be the anticanonical class. Then the anticanoncial ring of X is a Veronese subalgebra of its Cox ring:

$$\mathcal{S}(X) \ \coloneqq\ \bigoplus_{k\in\mathbb{Z}} \Gamma(X, -\mathcal{K}_X) \ \cong\ \bigoplus_{k\in\mathbb{Z}} \mathcal{R}_{kw_X}(X).$$

Using, for instance, the software package [23], we can explicitly compute a minimal homogeneous generator system that has the generator degrees $1,1,1,2$ and a single defining relation in degree 4. This identifies X from Example 3.2.6 as the second sporadic case in the list of [33, Thm. 8].

3.6 On the anticanoncial embedding

The main result of this section provides the anticanonical embedding for full intrinsic quadric surfaces with Kähler-Einstein metric in Picard number one and odd Gorenstein index; the Gorenstein case was already mentioned in Remark 3.5.61.

Theorem 3.6.1. *Let X be a full intrinsic quadric surface of Picard number one and of Gorenstein index ι admitting a Kähler-Einstein metric.*

(i) *If $\iota = 1$, then X is anticanonically embedded as a surface into $\mathbb{P}(1,1,1,2)$ with defining equation g of the degree 4 given in homogeneous coordinates as*

$$g \ =\ -U_1U_2U_4 + U_3^4 + 2U_3^2U_4 + U_4^2.$$

(ii) *If $\iota \geq 3$ is odd, then X is anticanonically embedded as a complete intersection into $\mathbb{P}(1,2,2,\iota,\iota)$ with two defining equations, g of degree 4 and h degree 2ι, given in homogeneous coordinates as*

$$\begin{aligned} g \ &=\ U_1^4 - U_2U_3, \\ h \ &=\ \Big(2U_1^2 + U_2 + U_3\Big) \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} U_2^k U_3^{\iota-1-k} \\ &\quad - (U_2 - U_3)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} U_2^k U_3^{\iota-2-k} - U_4U_5. \end{aligned}$$

For the case of full intrinsic quadric surfaces of Picard number one and even Gorenstein index a series of computer experiments leads us to the following.

Conjecture 3.6.2. *Let X be a full intrinsic quadric surface of Picard number one admitting a Kähler-Einstein metric. If the Gorenstein index ι of X is even, then X is anticanonically embedded as a non-complete intersection into the weighted projective space*

$$\mathbb{P}\left(1,2,2,\tfrac{\iota}{2}+1,\tfrac{\iota}{2}+1,\tfrac{\iota}{2}+1,\tfrac{\iota}{2}+1,\iota,\iota\right)$$

with twenty defining homogeneous equations: one in degree 4, four in degree $\frac{\iota}{2}+3$, ten in degree $\iota+2$, four in degree $\frac{3\iota}{2}+3$ and one in degree 2ι.

Let us give an outline of the proof. Theorem 3.6.1 (i) is about one particular case which can be explicitly computed using for instance the software package [23]. For Theorem 3.6.1 (ii) we provide an explicit isomorphism. Let $w_X \in \mathrm{Cl}(X)$ be the anticanonical class. Then the anticanoncial ring of X is the Veronese subalgebra of its Cox ring given by

$$\mathcal{S}(X) \ \coloneqq\ \bigoplus_{d\geq 0} \mathcal{R}(X)_{dw_X} \ \subseteq\ \mathcal{R}(X).$$

Consider a complex full intrinsic quadric surface X of Picard number one and of odd Gorenstein index $\iota \geq 3$ admitting a Kähler-Einstein metric. Let $T_1, T_2, T_3, T_4 \in \mathcal{R}(X)$ be the generators of the Cox ring. Then, we establish the following isomorphism of graded rings

$$\begin{aligned}
\psi\colon \mathbb{K}[U_1,U_5,U_6,U_7,U_8]/\left\langle g,\, h\right\rangle &\longrightarrow \mathcal{S}(X),\\
U_1 &\longmapsto \tilde{U}_1 \coloneqq T_3T_4,\\
U_5 &\longmapsto \tilde{U}_5 \coloneqq T_3^4,\\
U_6 &\longmapsto \tilde{U}_6 \coloneqq T_4^4,\\
U_7 &\longmapsto \tilde{U}_7 \coloneqq T_1^{2\iota},\\
U_8 &\longmapsto \tilde{U}_8 \coloneqq T_2^{2\iota},
\end{aligned}$$

where the polynomials g, h are as in Theorem 3.6.1 (ii) and the $\mathbb{Z}$-grading for both rings is given by

$$[\deg(U_i)] \ =\ \left[\deg(\tilde{U}_i)\right] \ =\ [1,2,2,\iota,\iota]\,.$$

Subsection 3.6.1: We provide a toric embedding $\tilde{X} \subseteq \tilde{Z}$ of the anticanonical cone over $X \subseteq Z$ and give an explicit description of the weight cone $\tilde{\omega}$ of $\tilde{Z}$.

Subsection 3.6.2: We determine the Hilbert basis $\mathcal{H}(\tilde{\omega}) = \{u_1, \ldots, u_8\}$ of the additive monoid $\tilde{\omega} \cap \mathbb{Z}^4$. This provides us with a surjective ring homomorphism

$$\jmath^*\colon \mathbb{K}[U_1,\ldots,U_8] \longrightarrow \mathbb{K}[\tilde{\omega}\cap\mathbb{Z}^4] \ =\ \mathcal{O}(\tilde{Z}), \qquad U_i \mapsto \chi^{\tilde{u}_i}.$$

Subsection 3.6.3: We establish a commutative diagram of epimorphisms which provides us in particular with generators of the anticanonical ring:

$$\begin{tikzcd}
\mathbb{K}[U_1,\dots,U_8] \arrow[r, "\varphi"] \arrow[d, two heads, "U_i\mapsto\chi^{\tilde{u}_i}"', "\jmath^*"] & \mathcal{S}(X) \arrow[r, equal] & \bigoplus_{n\geq 0}\mathcal{R}(X)_{nw_X} \\
\mathbb{K}[\tilde{\omega}\cap\mathbb{Z}^4] \arrow[r, "\cong", "\tilde{p}^*\colon \chi^u\mapsto\chi^{\tilde{P}^*u}"'] & \mathcal{S}(Z) \arrow[u, two heads, "h\mapsto h"'] \arrow[r, equal] & \bigoplus_{n\geq 0}\mathcal{R}(Z)_{nw_X}
\end{tikzcd}$$

Subsection 3.6.4: In a last step, we determine generators $g_1,\dots,g_5$ of the ideal of relations between the generators U_i.

$$g_1 \coloneqq 2U_3+U_4+U_5-U_6,$$
$$g_2 \coloneqq 2U_2+U_4-U_5+U_6,$$
$$g_3 \coloneqq 2U_1^2-U_4+U_5+U_6,$$
$$g_4 \coloneqq U_4^2-2U_4U_5+U_5^2-2U_4U_6-2U_5U_6+U_6^2,$$
$$g_5 \coloneqq \Big(2U_1^2+U_5+U_6\Big)\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}U_5^kU_6^{\iota-1-k}-(U_5-U_6)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}U_5^kU_6^{\iota-2-k}-U_7U_8.$$

Then Theorem 3.6.1 follows by eliminating the variables U_2, U_3 and U_4 and renumbering the remaining ones.

3.6.1. The embedded anticanonical cone • In Proposition 3.6.4 we explicitly determine the embedded anticanonical cone $\tilde{X}\subseteq\tilde{Z}$ over the embedded $X\subseteq Z$ for any full intrinsic quadric of Picard number one. Moreover, in Proposition 3.6.5 we provide first relations between the generators of the anticanonical ring. We begin by fixing the setting.

Setting 3.6.3. Let X be a full intrisic quadric of Picard number one arising from Construction 3.2.1, that means $X = X(P)$ with

$$P \;\coloneqq\; [v_1,v_2,v_3,v_4] \;\coloneqq\; \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}, \qquad b\leq -2,\; 0\leq a\leq -b-2.$$

Let Z be the toric variety arising from the projective fan in $\mathbb{Z}^3$ with generator matrix P. Then we obtain a normal, non-toric, rational, projective surface

$$X \;\coloneqq\; X(P) \;\coloneqq\; \overline{V(h)} \;\subseteq\; Z, \qquad h \;\coloneqq\; 1+U_1+U_2 \;\in\mathcal{O}(\mathbb{T}^3).$$

Let P^* be the transpose of P and set $K \coloneqq \mathbb{Z}^4/\mathrm{im}(P^*)$. For the divisor class group of X, we have

$$\mathrm{Cl}(X) \;\cong\; K \;\cong\; \mathbb{Z}\oplus\mathbb{Z}\,/\,2\gcd(2a+2,\,a-b)\mathbb{Z}.$$

Moreover, denoting by $Q\colon \mathbb{Z}^4\to K$ the projection, we obtain the following description of the Cox ring of X as a graded algebra:

$$\mathcal{R}(X) \;\cong\; \mathbb{K}[T_1,\dots,T_4]/\langle T_1T_2+T_3^2+T_4^2\rangle, \qquad w_i \;\coloneqq\; \deg(T_i) \;=\; Q(e_i) \;=\; [D_i^X],$$

where $D_i^X \subseteq X$ is the prime divisor on X obtained by intersecting X with the toric prime divisor of Z given by the ray through v_i and $[D_i^X] \in \mathrm{Cl}(X)$ denotes its class.

Proposition 3.6.4. *Let $X = X(P)$ be as in Setting 3.6.3. Then the anticanonical cone over X is the 3-dimensional affine variety $\tilde{X}$ with 2-torus action given by the defining matrix*

$$\tilde{P} \;:=\; [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4] \;:=\; \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The columns $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ of $\tilde{P}$ are the primitive generators of a polyhedral cone $\tilde{\sigma}$ of full dimension in $\mathbb{Z}^4$ and we have

$$\tilde{X} \;=\; \overline{V(1 + S_1 + S_2)} \;\subseteq\; \tilde{Z},$$

where $\tilde{Z}$ is the affine toric 4-fold associated with $\tilde{\sigma}$. Moreover, the dual cone $\tilde{\omega} := \tilde{\sigma}^\vee$ has the primitive generators

$$\tfrac{1}{\gcd(b,2)}[b, b, 2, -2b-2], \quad \tfrac{1}{\gcd(a,2)}[-a, -a, -2, 2a+2], \quad [1, -1, 0, 2], \quad [-1, 1, 0, 2].$$

Proof. The statement that the anticanonical cone over X is given by $\tilde{P}$ is a direct consequence of Construction 1.3.11.

We turn to the statement on the dual cone $\tilde{\omega}$ of $\tilde{\sigma}$. First note that $\tilde{P}$ has determinant $4a - 4b > 0$ and hence $\tilde{\sigma}$ is a simplicial cone. Next, observe that the four vectors listed in the assertion are indeed primitive. Now in order to see that they are ray generators of $\tilde{\omega}$ we show that each of them cuts out a facet of $\tilde{\sigma}$ and evaluates positively on the remaining ray:

$$\begin{aligned} [b, b, 2, -2b-2] \cdot \tilde{P} &= [-2b+4, 0, 0, 0]^T, \\ [-a, -a, -2, 2a+2] \cdot \tilde{P} &= [0, 6a+4, 0, 0]^T, \\ [1, -1, 0, 2] \cdot \tilde{P} &= [0, 0, a-b+2, 0]^T, \\ [-1, 1, 0, 2] \cdot \tilde{P} &= [0, 0, 0, 2]^T. \end{aligned}$$

Setting 3.6.3 yields $b \le -2$ and $0 \le a \le -b-2$. Consequently, the non-zero entries on the right hand side are positive. □

Proposition 3.6.5. *Consider X as in Setting 3.6.3, its anticanonical class w_X and the anticanonical algebra*

$$\mathcal{S}(X) \;=\; \bigoplus_{d \ge 0} \mathcal{R}(X)_{dw_X} \;\subseteq\; \mathcal{R}(X).$$

Then, in terms of the Cox ring generators $T_1, T_2, T_3, T_4 \in \mathcal{R}(X)$, we find the following elements of $\mathcal{S}(X)$ which are the generators of $\mathcal{R}(X)_{dw_X}$:

$d = 1: \quad \tilde{U}_1 := T_3 T_4,$

$d = 2: \quad \tilde{U}_2 := T_1 T_2 T_4^2, \quad \tilde{U}_3 := T_1 T_2 T_3^2, \quad \tilde{U}_4 := T_1^2 T_2^2, \quad \tilde{U}_5 := T_3^4, \quad \tilde{U}_6 := T_4^4.$

Among the elements $\tilde{U}_1, \ldots, \tilde{U}_6$, we find the following trinomal relations in degree $d = 2$ of the anticaonical ring:

$$\tilde{U}_2 + \tilde{U}_3 + \tilde{U}_4 \ = \ 0, \qquad \tilde{U}_1^2 + \tilde{U}_2 + \tilde{U}_6 \ = \ 0, \qquad \tilde{U}_1^2 + \tilde{U}_3 + \tilde{U}_5 \ = \ 0.$$

Moreover, in the anticanonical degree $d = 4$, we encounter the following three binomial relations

$$\tilde{U}_1^2\tilde{U}_4 \ = \ \tilde{U}_2\tilde{U}_3, \qquad \tilde{U}_1^2\tilde{U}_3 \ = \ \tilde{U}_2\tilde{U}_5, \qquad \tilde{U}_1^2\tilde{U}_2 \ = \ \tilde{U}_3\tilde{U}_6.$$

Lemma 3.6.6. *Let $X = X(P)$ arise from Setting 3.6.3. Then the degree $\mu_X \in K$ of the defining relation $T_1T_2 + T_3^2 + T_4^2$ and the anticanonical class $w_X \in K$ are given by*

$$\mu_X \ = \ w_1 + w_2 \ = \ 2w_3 = \ 2w_4, \qquad w_X \ = \ w_1 + w_2 + w_3 + w_4 - \mu_X \ = \ w_3 + w_4.$$

Furthermore, the relation degree $\mu_X \in K$ differs from the anticanonical class $w_X \in K$.

Proof. The representations of μ_X are due to the homogeneity of the defining quadric and the representations of the anticanonical class follow from Remark 1.2.10.

To show the second claim, assume $\mu_X = w_X$. Then we get $2w_3 = w_3 + w_4$ and $w_3 - w_4 = 0$. Further, we denote by $Q\colon \mathbb{Z}^4 \to \mathbb{Z}^4/\mathrm{im}(P^*) = \mathrm{Cl}(X)$ the projection from Setting 3.6.3. Hence we have $Q(e_3 - e_4) = w_3 - w_4 = 0$. In other words $e_3 - e_4$ belongs to the kernel of Q which in turn is spanned by the rows of the matrix P. On the other hand, $P^* \cdot x = e_3 - e_4$ is uniquely solved by $(1/2, -1/2, 0)$. Consequently, $e_3 - e_4$ cannot be presented as an integer linear combination over the rows of P and thus does not lie in the kernel. $\square$

Lemma 3.6.7. *Let X arise from Setting 3.6.3 and let w_X be the anticanonical class. Then all positive linear combinations of $2w_X \in K$ in terms of w_1, w_2, w_3, w_4 are given by*

$$2w_X \ = \ w_1 + w_2 + 2w_3 \ = \ w_1 + w_2 + 2w_4 \ = \ 2w_1 + 2w_2 \ = \ 4w_3 \ = \ 4w_4.$$

Proof. We have to insert each of the equations

$$\mu_X \ = \ w_1 + w_2 \ = \ 2w_3 = \ 2w_4,$$

from Lemma 3.6.6 into

$$2w_X \ = \ 2w_1 + 2w_2 + 2w_3 + 2w_4 - \mu_X$$

and combine the equations with each other in all the possible ways that we can. Then, there are exactly five combinations with positive coefficients. $\square$

Proof of Proposition 3.6.5. Using the representations from Lemma 3.6.6 we immediately see $\tilde{U}_1 := T_3T_4 \in \mathcal{R}(X)_{w_X}$. Moreover, Lemma 3.6.6 provides us with the following equations

$$\begin{aligned}
\deg(\tilde{U}_2) &= \deg(T_1T_2T_3^2) &&= w_1 + w_2 + 2w_3 &&= 2w_4 + 2w_3 &&= 2w_X,\\
\deg(\tilde{U}_3) &= \deg(T_1T_2T_4^2) &&= w_1 + w_2 + 2w_4 &&= 2w_3 + 2w_4 &&= 2w_X,\\
\deg(\tilde{U}_4) &= \deg(T_1^2T_2^2) &&= 2w_1 + 2w_2 &&= 2w_3 + 2w_4 &&= 2w_X,\\
\deg(\tilde{U}_5) &= \deg(T_4^4) &&= 4w_4 &&= 2w_3 + 2w_4 &&= 2w_X,\\
\deg(\tilde{U}_6) &= \deg(T_3^4) &&= 4w_3 &&= 2w_3 + 2w_4 &&= 2w_X,
\end{aligned}$$

showing $\tilde{U}_2 \ldots, \tilde{U}_6 \in \mathcal{R}(X)_{2w_X}$. Moreover, Lemma 3.6.7 yields that every element inside $\mathcal{R}(X)_{2w_X}$ is a linear combination of the monomials $\tilde{U}_2 \ldots, \tilde{U}_6$. Hence, $\tilde{U}_2 \ldots, \tilde{U}_6$ generate $\mathcal{R}(X)_{2w_X}$. The binomial relations among the $\tilde{U}_i$ follow directly from their definitions. As we have $T_1T_2 + T_3^2 + T_4^2 = 0$ in $\mathcal{R}(X)$, we obtain the desired trinomial relations as follows

$$\begin{aligned}
\tilde{U}_2 + \tilde{U}_3 + \tilde{U}_4 &= T_1T_2T_4^2 + T_1T_2T_3^2 + T_1^2T_2^2 &&= (T_1T_2 + T_3^2 + T_4^2)T_1T_2 &&= 0,\\
\tilde{U}_1^2 + \tilde{U}_2 + \tilde{U}_6 &= T_3^2T_4^2 + T_1T_2T_4^2 + T_4^4 &&= (T_1T_2 + T_3^2 + T_4^2)T_4^2 &&= 0,\\
\tilde{U}_1^2 + \tilde{U}_3 + \tilde{U}_5 &= T_3^2T_4^2 + T_1T_2T_3^2 + T_3^4 &&= (T_1T_2 + T_3^2 + T_4^2)T_3^2 &&= 0.
\end{aligned}$$

□

3.6.2. Computing the Hilbert basis • Now we enter the Kähler-Einstein case. The main results of this section are Proposition 3.6.10, where we determine the grading matrix of X, and Proposition 3.6.12, where we compute the Hilbert basis of the dual cone $\tilde{\omega}$ of the cone $\tilde{\sigma}$ from Proposition 3.6.4. First, we adapt Setting 3.6.3 and Proposition 3.6.4 to the Kähler-Einstein case.

Setting 3.6.8. Let $X = X(P)$ be a complex full intrinsic quadric surface of Picard number one as in Setting 3.6.3 admitting a Kähler-Einstein metric and let the Gorenstein index ι of X be odd and differ from one. By Theorem 3.5.32 we can work with a defining matrix of the form

$$P \;=\; [v_1, v_2, v_3, v_4] \;=\; \begin{bmatrix} -1 & -1 & 2 & 0\\ -1 & -1 & 0 & 2\\ \iota - 1 & -\iota - 1 & 1 & 1 \end{bmatrix}$$

and the divisor class group of X is given by

$$\mathrm{Cl}(X) \;=\; \mathbb{Z}^4/\mathrm{im}(P^*) \;=\; \mathbb{Z} \oplus \mathbb{Z}\,/\,4\iota\mathbb{Z}.$$

Moreover, we recall

$$\tilde{P} \;=\; [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4] \;=\; \begin{bmatrix} -1 & -1 & 2 & 0\\ -1 & -1 & 0 & 2\\ \iota - 1 & -\iota - 1 & 1 & 1\\ 0 & 0 & 1 & 1 \end{bmatrix},$$

the cone $\tilde{\sigma} := \operatorname{cone}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4)$ and $\tilde{\omega} := \tilde{\sigma}^\vee$ from Proposition 3.6.4. We have a corresponding 3-dimensional affine variety

$$\tilde{X} \ = \ \overline{V(1+S_1+S_2)} \ \subseteq \ \tilde{Z},$$

where $\tilde{Z}$ is the affine toric 4-fold associated with $\tilde{\sigma}$. Further, the cone $\tilde{\omega}$ has the primitive generators

$$\tfrac{1}{2}[-\iota-1,-\iota-1,2,2\iota], \quad \tfrac{1}{2}[1-\iota,1-\iota,-2,2\iota], \quad [1,-1,0,2], \quad [-1,1,0,2].$$

Remark 3.6.9. Consider a degree matrix Q in $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ and a 3×4 projective generator matrix P sharing the same fake weight vector. Then Lemma 2.6.1 says that Q and P correspond to each other if and only if Q annihilates the rows of P.

Proposition 3.6.10. *Let $X = X(P)$ be as in Setting 3.6.8 and consider the projection the projection $Q\colon \mathbb{Z}^4 \to \mathbb{Z}^4/\operatorname{im}(P^*) = \mathbb{Z} \oplus \mathbb{Z}/4\iota\mathbb{Z}$. Then after applying a suitable automorphism we obtain*

$$Q \ = \ \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \bar{2}\bar{\iota}+\bar{1} \end{bmatrix}.$$

Proof. We use $l_1 = l_2 = 2$ and $d_0 = -2\iota, d_1 = 1, d_2 = 2\iota - 1$ to determine the fake weight vector $w(P)$ as in Proposition 1.4.3. That is

$$w(P) \ = \ (4\iota, 4\iota, 4\iota, 4\iota).$$

We derive from Remark 1.1.8 that $\operatorname{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/4\iota\mathbb{Z}$ and from Proposition 2.2.1 that Q is of the form

$$Q \ = \ [w_1, w_2, w_3, w_4] \ := \ \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{\eta}_0 & \bar{\eta}_1 & \bar{\eta}_2 & \bar{\eta}_3 \end{bmatrix}.$$

According to Remark 2.2.9 we can add multiples of the upper to the lower row and scale the lower row by units without changing the isomorphy type. By adding multiples of the first row to the second we can achieve $\bar{\eta}_0 = \bar{0}$. Then, Remark 3.6.9 yields the equations

$$\bar{\eta}_1 \ = \ 2\bar{\eta}_2, \qquad \bar{\eta}_1 \ = \ 2\bar{\eta}_3, \qquad (\bar{\iota}+\bar{1})\,\bar{\eta}_1 \ = \ \bar{\eta}_2 + \bar{\eta}_3. \tag{3.6.10.1}$$

Consequently, we get

$$Q \ = \ \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2}\bar{\eta}_2 & \bar{\eta}_2 & \bar{\eta}_3 \end{bmatrix}.$$

Since Q is a degree matrix, three of the columns w_i generate $\operatorname{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/4\iota\mathbb{Z}$. Thus, one of the $\bar{\eta}_i$ must be a unit. We have $\bar{\eta}_1 = \bar{2}\bar{\eta}_2 \notin (\mathbb{Z}/4\iota\mathbb{Z})^*$ since $\gcd(\eta_1, 4\iota) \geq 2$ holds for any representant η_1. Further, we may assume $\bar{\eta}_2 \in (\mathbb{Z}/4\iota\mathbb{Z})^*$. By multiplying the second row with a unit, we obtain

$$Q \ = \ \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \bar{\eta}_3 \end{bmatrix}.$$

Then, the last equation from (3.6.10.1) provides us with

$$\bar{1} + \bar{\eta}_3 \;=\; \left(\bar{\iota} + \bar{1}\right)\bar{2} \;=\; \overline{2\iota} + \bar{2}.$$

Hence, $\bar{\eta}_3 = \overline{2\iota} + \bar{1}$ and the claim follows. $\square$

Corollary 3.6.11. *Let $X = X(P)$ be as in Setting 3.6.8. Then the anticanonical class $w_X \in \mathbb{Z} \oplus \mathbb{Z}/4\iota\mathbb{Z}$ satisfies*

$$w_X \;=\; \left(2, \overline{2\iota} + \bar{2}\right), \qquad \iota \cdot w_X \;=\; \left(2\iota, \bar{0}\right).$$

Proof. According to Proposition 3.6.10 the projection $Q\colon \mathbb{Z}^4 \to \mathbb{Z}^4/\mathrm{im}(P^*) = \mathbb{Z} \oplus \mathbb{Z}/4\iota\mathbb{Z}$ is given as

$$Q \;=\; [w_1, w_2, w_3, w_4] \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \overline{2\iota} + \bar{1} \end{bmatrix}.$$

Then Lemma 3.6.6 yields the anticanonical class

$$w_X \;=\; w_3 + w_4 \;=\; \left(2, \overline{2\iota} + \bar{2}\right).$$

Since ι is odd, we conclude $\bar{\iota} \cdot \overline{2\iota} = \overline{2\iota} \in \mathbb{Z}/4\iota\mathbb{Z}$. Consequently, one gets

$$\iota \cdot w_X \;=\; \left(2\iota, \bar{2}\left(\bar{\iota}\right)^2 + \overline{2\iota}\right) \;=\; \left(2\iota, \overline{2\iota} + \overline{2\iota}\right) \;=\; \left(2\iota, \bar{0}\right).$$

$\square$

Consider a pointed lattice cone ω in $\mathbb{Z}^n$ and the associated monoid $S(\omega) := \omega \cap \mathbb{Z}^n$. A non-zero element $u \in S(\omega)$ is called *indecomposable*, if $u = v + v'$ with $v, v' \in S(\omega)$ implies $v = u$ or $v' = u$. The *Hilbert basis* $\mathcal{H}(\omega)$ is the set of all indecomposable elements of $S(\omega)$. Recall that $\mathcal{H}(\omega)$ is the unique minimal generator system of the monoid $S(\omega)$.

Proposition 3.6.12. *Let $\tilde{\omega}$ be as in Setting 3.6.8. Then the Hilbert basis $\mathcal{H}\left(\tilde{\omega}\right)$ of $S(\tilde{\omega})$ is the set of the column vectors of:*

$$G \;:=\; [\tilde{u}_1, \ldots, \tilde{u}_8] \;:=\; \begin{bmatrix} 0 & -1 & 0 & -1 & 1 & -1 & -\frac{\iota+1}{2} & -\frac{\iota-1}{2} \\ 0 & 0 & -1 & -1 & -1 & 1 & -\frac{\iota+1}{2} & -\frac{\iota-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 2 & 2 & 2 & 2 & \iota & \iota \end{bmatrix}.$$

Proof. In order to show that $\mathcal{H}\left(\tilde{\omega}\right) = \{\tilde{u}_1, \ldots, \tilde{u}_8\}$ is the Hilbert basis of $S(\tilde{\omega})$ it suffices to present any lattice point of the form

$$w \;=\; (w_1, w_2, w_3, w_4) \;=\; \lambda_5\tilde{u}_5 + \lambda_6\tilde{u}_6 + \lambda_7\tilde{u}_7 + \lambda_8\tilde{u}_8 \;\in\; \mathbb{Z}^4 \text{ with } 0 \le \lambda_i \le 1$$

as an integer positive linear combination over $\mathcal{H}\left(\tilde{\omega}\right)$. Note that $w_3 = \lambda_7 - \lambda_8 \in \mathbb{Z}$. Thus $\lambda_7, \lambda_8 \in [0, 1]$ implies $w_3 = -1, 0, 1$. We proceed according to these three cases.

Case 1: $w_3 = -1$. Then we get $\lambda_7 = 0$ and $\lambda_8 = 1$. So, the vector w can be represented as

$$w \;=\; \left(-\tfrac{\iota-1}{2} - \lambda_6 + \lambda_5,\; -\tfrac{\iota-1}{2} + \lambda_6 - \lambda_5,\; -1,\; \iota + 2\lambda_5 + 2\lambda_6\right) \;\in\; \mathbb{Z}^4.$$

Since w_1, $\frac{\iota-1}{2} \in \mathbb{Z}$ and $\lambda_5, \lambda_6 \in [0,1]$ hold, we conclude $\lambda_5 - \lambda_6 = -1, 0, 1$. This leads to the following subcases.

Case 1.1: $\lambda_5 - \lambda_6 = 1$. We obtain $\lambda_5 = 1,\ \lambda_6 = 0$. This implies $w = \tilde{u}_5 + \tilde{u}_8$ which is the desired presentation as an integer positive linear combination.

Case 1.2: $\lambda_5 - \lambda_6 = 0$. Then $4\lambda_5 = w_4 - \iota \in \mathbb{Z}$, hence $\lambda_5 = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ holds. In total, this yields the desired presentation as integer positive linear combination $w = \tilde{u}_8 + 4\lambda_5\tilde{u}_1$.

Case 1.3: $\lambda_5 - \lambda_6 = -1$. So we get $\lambda_5 = 0$ and $\lambda_6 = 1$. Therefore $w = \tilde{u}_6 + \tilde{u}_8$.

Case 2: $w_3 = 1$. Then we obtain $\lambda_7 = 1$ and $\lambda_8 = 0$. Thus, the vector w can be represented as

$$w \;=\; \left(-\tfrac{\iota+1}{2} - \lambda_6 + \lambda_5,\; -\tfrac{\iota+1}{2} + \lambda_6 - \lambda_5,\; 1,\; \iota + 2\lambda_5 + 2\lambda_6\right) \;\in\; \mathbb{Z}^4.$$

Because of w_1, $\frac{\iota+1}{2} \in \mathbb{Z}$ and $\lambda_5, \lambda_6 \in [0,1]$, we get $\lambda_5 - \lambda_6 = -1, 0, 1$. Again, we distinguish between these three cases.

Case 2.1: $\lambda_5 - \lambda_6 = 1$. We see $\lambda_5 = 1$ and $\lambda_6 = 0$. In total, we arrive at $w = \tilde{u}_5 + \tilde{u}_7$.

Case 2.2: $\lambda_5 - \lambda_6 = 0$. Then we have $4\lambda_5 = w_4 - \iota \in \mathbb{Z}$ and hence $\lambda_5 = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. Thus we receive the desired positive integer linear combinations $w = \tilde{u}_7 + 4\lambda_5\tilde{u}_1$.

Case 2.3: $\lambda_5 - \lambda_6 = -1$. One obtains $\lambda_6 = 1,\ \lambda_5 = 0$ and finally $w = \tilde{u}_6 + \tilde{u}_7$.

Case 3: $w_3 = 0$. Then one gets $\lambda_7 = \lambda_8$. It follows that w can be represented as

$$w \;=\; (-\lambda_7\iota - \lambda_6 + \lambda_5,\; -\lambda_7\iota + \lambda_6 - \lambda_5,\; 0,\; 2\lambda_7\iota + 2\lambda_5 + 2\lambda_6) \;\in\; \mathbb{Z}^4.$$

Hence we have $w_2 - w_1 = 2(\lambda_6 - \lambda_5) \in \mathbb{Z}$ and $2w_1 + w_4 = 4\lambda_5 \in \mathbb{Z}$. Consequently, one gets $\lambda_5, \lambda_6 = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. We proceed according to these cases.

Case 3.1: $\lambda_5 = \lambda_6$. We obtain $w_1 = -\lambda_7\iota \in \mathbb{Z}$. Because of $\lambda_5 = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, we obtain a presentation with positive integer coefficients $w = 4\lambda_5\tilde{u}_1 + \lambda_7\iota\tilde{u}_4$.

Case 3.2: $\lambda_6 = 0$. Then $2\lambda_5 = w_1 - w_2 \in \mathbb{Z}$ holds, whence $\lambda_5 = 0, \frac{1}{2}, 1$. So we distinguish between these cases. It suffices to treat the cases $\lambda_5 \neq \lambda_6$.

Case 3.2.1: $\lambda_5 = \frac{1}{2}$. Then we have $-\lambda_7\iota + \frac{1}{2} = w_1 \in \mathbb{Z}_{\leq 0}$ and

$$w \;=\; (w_1,\, w_1 - 1,\, 0,\, -2w_1 + 2) \;=\; (0,-1,0,2) - w_1(-1,-1,0,2) \;=\; \tilde{u}_3 - w_1\tilde{u}_4,$$

which is the desired presentation with positive coefficients.

Case 3.2.2: $\lambda_5 = 1$. One obtains $-\lambda_7\iota + 1 = w_1 \in \mathbb{Z}_{\leq 1}$ and hence

$$\begin{aligned} w &= (w_1,\, w_1 - 2,\, 0,\, -2w_1 + 4) \\ &= (1 - w_1)(-1, -1, 0, 2) + (1, -1, 0, 2) \\ &= (1 - w_1)\tilde{u}_4 + \tilde{u}_5. \end{aligned}$$

Case 3.3: $\lambda_6 = \frac{1}{4}$. Then we get $\frac{1}{2} - 2\lambda_5 = w_2 - w_1 \in \mathbb{Z}$. We conclude $\lambda_5 = \frac{1}{4}, \frac{3}{4}$. It suffices to treat the case $\lambda_5 \neq \lambda_6$, i.e. let $\lambda_5 = \frac{3}{4}$. Then the desired positive linear combination is given by

$$\begin{aligned} w &= (w_1,\, w_1 - 1,\, 0,\, -2w_1 + 3) \\ &= (0, 0, 0, 1) + (0, -1, 0, 2) - w_1(-1, -1, 0, 2) \\ &= \tilde{u}_1 + \tilde{u}_3 - w_1\tilde{u}_4. \end{aligned}$$

Case 3.4: $\lambda_6 = \frac{1}{2}$. Then $1 - 2\lambda_5 = w_2 - w_1 =\in \mathbb{Z}$ holds. This yields $\lambda_5 = 0, \frac{1}{2}, 1$. We treat the cases $\lambda_5 \neq \lambda_6$.

Case 3.4.1: $\lambda_5 = 0$. We obtain $-\lambda_7\iota - \frac{1}{2} = w_1 \in \mathbb{Z}_{\leq -1}$ and a representation

$$\begin{aligned} w &= (w_1,\, w_1 + 1,\, 0,\, -2w_1) \\ &= (-1, 0, 0, 2) + (-w_1 - 1)(-1, -1, 0, 2) \\ &= \tilde{u}_2 + (-w_1 - 1)\tilde{u}_4. \end{aligned}$$

Case 3.4.2: $\lambda_5 = 1$. Then we have $-\lambda_7\iota + \frac{1}{2} = w_1 \in \mathbb{Z}_{\leq 0}$ and the representation

$$\begin{aligned} w &= (w_1,\, w_1 - 1,\, 0,\, -2w_1 + 4) \\ &= (0, 0, 0, 2) + (0, -1, 0, 2) - w_1(-1, -1, 0, 2) \\ &= 2\tilde{u}_1 + \tilde{u}_3 - w_1\tilde{u}_4. \end{aligned}$$

Case 3.5: $\lambda_6 = \frac{3}{4}$. Then we have $-2\lambda_5 + \frac{3}{2} = w_2 - w_1 \in \mathbb{Z}$. This implies $\lambda_5 = \frac{1}{4}, \frac{3}{4}$. It suffices to proceed with $\lambda_5 \neq \lambda_6$, that is $\lambda_5 = \frac{1}{4}$. Thus, we have $w_1 = -\lambda_7\iota - \frac{1}{2} \leq -1$. Consequently, we get a positive linear combination

$$\begin{aligned} w &= (w_1,\, w_1 + 1,\, 0,\, -2w_1 + 1) \\ &= (0, 0, 0, 1) + (-1, 0, 0, 2) + (-w_1 - 1)(-1, -1, 0, 2) \\ &= \tilde{u}_1 + \tilde{u}_2 + (-w_1 - 1)\tilde{u}_4. \end{aligned}$$

Case 3.6: $\lambda_6 = 1$. We obtain $2\lambda_5 - 2 = w_1 - w_2 \in \mathbb{Z}$. This yields $\lambda_5 = 0, \frac{1}{2}, 1$. We only deal with the case $\lambda_5 \neq \lambda_6$.

Case 3.6.1: $\lambda_5 = 0$. Then we get $w_1 = -\lambda_7\iota - 1 \leq -1$ and the representation

$$\begin{aligned} w &= (w_1,\ w_1 + 2,\ 0,\ -2w_1) \\ &= (-w_1 - 1)(-1, -1, 0, 2) + (-1, 1, 0, 2) \\ &= (-w_1 - 1)\tilde{u}_4 + \tilde{u}_6. \end{aligned}$$

Case 3.6.2: $\lambda_5 = \frac{1}{2}$. Then $w_1 = -\lambda_7\iota - \frac{1}{2} \leq -1$ holds and hence we get the desired representation of w as

$$\begin{aligned} w &= (w_1,\ w_1+1,\ 0,\ -2w_1+2) \\ &= (0,0,0,2) + (-1,0,0,2) + (-w_1-1)(-1,-1,0,2) \\ &= 2\tilde{u}_1 + \tilde{u}_2 + (-w_1-1)\tilde{u}_4. \end{aligned}$$

Lastly, we show that $\tilde{u}_1, \ldots, \tilde{u}_8$ are indecomposable. The elements $\tilde{u}_5, \tilde{u}_6, \tilde{u}_7, \tilde{u}_8$ are indecomposable since they are the primitive ray generators of $\tilde{\omega}$. So it suffices to show the claim for $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4$.

In order to show that $\tilde{u}_1$ is indecomposable, we show that $G \cdot x = \tilde{u}_1$ has only one solution in $\mathbb{Z}^8_{\geq 0}$, namely $x = e_1$. The solutions in $\mathbb{K}^8$ are the vectors x of the form

$$(1 + 2t_4 - 2t_3 - 2t_2 + 2\iota t_1,\ -t_4 + t_3 - t_2 - \iota t_1,\ -t_4 - t_3 + t_2 - \iota t_1,\ t_4,\ t_3,\ t_2,\ t_1,\ t_1),$$

where the parameters $t_1, t_2, t_3, t_4 \in \mathbb{K}$ can be freely chosen. Now we determine the solutions such that $x = (x_1, \ldots, x_8) \in \mathbb{Z}^8_{\geq 0}$. This yields

$$0 \ \leq\ x_2 + x_3 \ =\ -2t_4 - 2\iota t_1, \qquad 0 \ \leq\ x_8 \ =\ t_1, \qquad 0 \ \leq\ x_4 \ =\ t_4.$$

We conclude $t_1 = t_4 = 0$. Further, we have

$$0 \ \leq\ x_1 \ =\ 1 - 2t_3 - 2t_2, \qquad 0 \ \leq\ x_6 \ =\ t_2, \qquad 0 \ \leq\ x_5 \ =\ t_3.$$

Therefore, $t_2 = t_3 = 0$. In total, the vector $x = (1, 0, \ldots, 0)$ is the only solution of $G \cdot x = \tilde{u}_1$ such that $x = (x_1, \ldots, x_8) \in \mathbb{Z}^8_{\geq 0}$. Thus, $\tilde{u}_1$ is indecomposable.

Analogously, the solutions in $\mathbb{K}^8$ of $G \cdot x = \tilde{u}_2$ are the vectors x of the form

$$(2t_4 - 2t_3 - 2t_2 + 2\iota t_1, 1 - t_4 + t_3 - t_2 - \iota t_1, -t_4 - t_3 + t_2 - \iota t_1, t_4, t_3, t_2, t_1, t_1),$$

where the parameters $t_1, t_2, t_3, t_4 \in \mathbb{K}$ can be freely chosen. If $x = (x_1, \ldots, x_8) \in \mathbb{Z}^8_{\geq 0}$ then we obtain

$$0 \ \leq\ x_2 + x_3 \ =\ -2t_4 - 2\iota t_1, \qquad 0 \ \leq\ t_1, t_4.$$

This implies $t_1 = t_4 = 0$. Moreover, we have

$$0 \ \leq\ x_1 \ =\ 1 - 2t_3 - 2t_2, \qquad 0 \ \leq\ t_2, t_3.$$

Consequently, we get $t_2 = t_3 = 0$. That means $x = (0, 1, 0 \ldots, 0)$ and $\tilde{u}_2$ is indecomposable.

The solutions in $\mathbb{K}^8$ of $G \cdot x = \tilde{u}_3$ are the vectors x of the form

$$(2t_4 - 2t_3 - 2t_2 + 2\iota t_1, -t_4 + t_3 - t_2 - \iota t_1, 1 - t_4 - t_3 + t_2 - \iota t_1, t_4, t_3, t_2, t_1, t_1),$$

where the parameters $t_1, t_2, t_3, t_4 \in \mathbb{K}$ can be freely chosen. We pick out the solutions such that $x = (x_1, \dots, x_8) \in \mathbb{Z}^8_{\geq 0}$. This leads to

$$0 \leq x_2 + x_3 = -2t_4 - 2\iota t_1, \qquad 0 \leq x_8 = t_1, \qquad 0 \leq x_4 = t_4.$$

We conclude $t_1 = t_4 = 0$. Further, the conditions

$$0 \leq x_1 = 1 - 2t_3 - 2t_2, \qquad 0 \leq t_2, t_3$$

then yield $t_2 = t_3 = 0$. Hence, $x = (0, 0, 1, 0 \dots, 0)$ and $\tilde{u}_3$ is indecomposable.

Lastly, the solutions in $\mathbb{K}^8$ of $G \cdot x = \tilde{u}_4$ are the vectors x of the form

$$(-2 + 2t_4 - 2t_3 - 2t_2 + 2\iota t_1, 1 - t_4 + t_3 - t_2 - \iota t_1, 1 - t_4 - t_3 + t_2 - \iota t_1, t_4, t_3, t_2, t_1, t_1)$$

where the parameters $t_1, t_2, t_3, t_4 \in \mathbb{K}$ can be freely chosen. We demand $x = (x_1, \dots, x_8) \in \mathbb{Z}^8_{\geq 0}$. This implies

$$0 \leq x_1 + x_2 + x_3 = -2t_2 - 2t_3, \qquad 0 \leq x_6 = t_2, \qquad 0 \leq x_5 = t_3.$$

Thus, we get $t_2 = t_3 = 0$. Further, one has

$$0 \leq x_2 + x_3 = 2 - 2t_4 - 2\iota t_1, \qquad 0 \leq t_1, t_4.$$

Since $\iota \geq 3$, this yields $t_1 = 0$ and $t_4 = 1$. We conclude that $x = (0, 0, 0, 1, 0 \dots, 0)$ is the only solution of $G \cdot x = \tilde{u}_4$ with positive integer entries. That means, $\tilde{u}_4$ is irreducible. □

Corollary 3.6.13. *Consider the cone $\tilde{\omega}$ as in Setting 3.6.8 and the Hilbert basis $\mathcal{H}(\tilde{\omega})$ of $S(\tilde{\omega})$ as in Proposition 3.6.12. Then, we obtain an epimorphism*

$$\jmath^* \colon \mathbb{K}[U_1, \dots, U_8] \longrightarrow \mathbb{K}[S(\tilde{\omega})], \qquad U_i \mapsto \chi^{\tilde{u}_i}.$$

Proof. The $\tilde{u}_1, \dots, \tilde{u}_8$ generate $S(\tilde{\omega})$ as a monoid due to Proposition 3.6.12. Hence the $\chi^{\tilde{u}_1}, \dots, \chi^{\tilde{u}_8}$ generate the affine algebra $\mathbb{K}[\tilde{\omega} \cap \mathbb{Z}^4]$. □

3.6.3. Generators for the anticanonical ring • The results of this section are Proposition 3.6.14, where we provide an epimorphism $\mathbb{K}[U_1, \dots, U_8] \to \mathcal{S}(X)$, Proposition 3.6.20 and Remark 3.6.21, where we identify the generators of $\mathcal{S}(X)$ with restrictions of character functions $\chi^{\tilde{P}^* u}$ and Proposition 3.6.22, where we identify a further relation in the anticanonical ring.

Proposition 3.6.14. *Let $X \subseteq Z$ and $\tilde{X} \subseteq \tilde{Z}$ arise from Setting 3.6.8 and let ι be the Gorenstein index of X. Consider the $\mathbb{Z}$-graded polynomial algebra*

$$\mathbb{K}[U_1, \dots, U_8], \qquad [\deg(U_1), \dots \deg(U_8)] = [1, 2, 2, 2, 2, 2, \iota, \iota].$$

Further, let $\tilde{\omega}$ be as in Setting 3.6.8 and denote by $\{\tilde{u}_1, \dots, \tilde{u}_8\}$ the Hilbert basis of $S(\tilde{\omega})$ given by Proposition 3.6.12. Then we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{K}[U_1,\ldots,U_8] \arrow[r, "\varphi"] \arrow[d, two heads, "U_i\mapsto\chi^{\tilde{u}_i}"', "\jmath^*"] & \mathcal{S}(X) \arrow[r, equal] & \bigoplus_{n\geq 0}\mathcal{R}(X)_{nw_X} \\
\mathbb{K}[S(\tilde{\omega})] \arrow[r, "\cong", "\tilde{p}^*\colon \chi^u\mapsto\chi^{\tilde{P}^*u}"'] & \mathcal{S}(Z) \arrow[u, two heads, "h\mapsto h"'] \arrow[r, equal] & \bigoplus_{n\geq 0}\mathcal{R}(Z)_{nw_X}
\end{tikzcd}$$

where we endow $\mathbb{K}[S(\tilde{\omega})]$ *with the* $\mathbb{Z}$*-grading* $\deg(\chi^u) = u_4$ *and where* φ *is a surjection given by*

$$\begin{array}{llllllll}
\varphi(U_1) = T_3T_4, & \varphi(U_2) = T_1T_2T_4^2, & \varphi(U_3) = T_1T_2T_3^2, & \varphi(U_4) = T_1^2T_2^2, \\
\varphi(U_5) = T_3^4, & \varphi(U_6) = T_4^4, & \varphi(U_7) = T_1^{2\iota}, & \varphi(U_8) = T_2^{2\iota}.
\end{array}$$

Construction 3.6.15. Consider $X \subseteq Z$ arising from the generator matrix P and $\tilde{X} \subseteq \tilde{Z}$ arising from the generator matrix $\tilde{P}$ as in Setting 3.6.8. Then we have commutative diagrams

$$\begin{tikzcd}
\mathbb{Z}^4 \arrow[rr, "\tilde{P}"] \arrow[dr, "P"'] & & \mathbb{Z}^4 \arrow[dl, "\mathrm{PR}"', "(z_1,z_2,z_3)\mapsfrom z"] \\
& \mathbb{Z}^3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{Z} \arrow[rr, "\tilde{p}", "/\!\!/\tilde{H}"'] \arrow[dr, "p", "/\!\!/H"'] & & \tilde{Z} \arrow[dl, "\mathrm{pr}"', "/\!\!/H/\tilde{H}"] \\
& Z &
\end{tikzcd}$$

where $p, \tilde{p}$ are the quotients by the actions of the characteristic quasitori $H, \tilde{H}$ respectively, where $\tilde{Z} \to Z$ is the cone over Z for the ample divisor $D_3^Z + D_4^Z$ on Z, and where $\tilde{X} \to X$ is the cone over X for the anticanonical divisor $D_3^X + D_4^X$ on X.

Proposition 3.6.16. *Consider* $X \subseteq Z$ *and* $\tilde{p}\colon \hat{Z} \to \tilde{Z}$ *as in Construction 3.6.15 and let* w_X *be the anticanonical class of* X*. Further, let* $\tilde{\omega}$ *be as in Setting 3.6.8 and denote by* $\{\tilde{u}_1,\ldots,\tilde{u}_8\}$ *the Hilbert basis of* $S(\tilde{\omega})$ *given by Proposition 3.6.12. Then we have*

$$\tilde{p}^*\left(\mathbb{K}[S(\tilde{\omega})]\right) \;=\; \bigoplus_{n\in\mathbb{Z}}\mathcal{R}(Z)_{nw_X}.$$

and we obtain

$$\begin{aligned}
\tilde{p}^*\left(\chi^{\tilde{u}_1}\right) \;&\in\; \mathcal{R}(Z)_{1\cdot w_X}, \\
\tilde{p}^*\left(\chi^{\tilde{u}_2}\right), \tilde{p}^*\left(\chi^{\tilde{u}_3}\right), \tilde{p}^*\left(\chi^{\tilde{u}_4}\right), \tilde{p}^*\left(\chi^{\tilde{u}_5}\right), \tilde{p}^*\left(\chi^{\tilde{u}_6}\right) \;&\in\; \mathcal{R}(Z)_{2\cdot w_X}, \\
\tilde{p}^*\left(\chi^{\tilde{u}_7}\right), \tilde{p}^*\left(\chi^{\tilde{u}_8}\right) \;&\in\; \mathcal{R}(Z)_{\iota\cdot w_X}.
\end{aligned}$$

Lemma 3.6.17. *Consider* $X \subseteq Z, p\colon \hat{Z} \to Z$ *and* P *as in Construction 3.6.15 and and let* w_X *be the anticanonical class of* X*. We set* $D^Z := D_3^Z + D_4^Z$ *and*

$$\begin{aligned}
\varphi\colon \Gamma(Z, \mathcal{O}(n\cdot D^Z)) \;&\longrightarrow\; \mathcal{R}(Z)_{nw_X} \\
\chi^u \;&\longmapsto\; p^*(\chi^u)\, T_3^n T_4^n, \\
\chi^w \;&\longleftarrow\!\!\shortmid\; T^u,
\end{aligned}$$

where $w \in \mathbb{Z}^3$ *is the unique element such that* $P^*w = u - ne_3 - ne_4$*. Then* φ *is a well-defined isomorphism.*

Proof. We check the well-definedness of both assignments. Let $\chi^u \in \Gamma(Z, \mathcal{O}(n \cdot D^Z))$ be a character. Then, we obtain

$$\begin{aligned} p^*\left(\operatorname{div}(\chi^u) + n \cdot D^Z\right) &= p^*(\operatorname{div}(\chi^u)) + n \cdot p^*\left(D^Z\right) \\ &= \operatorname{div}(p^*(\chi^u)) + n \cdot p^*\left(D_3^Z\right) + n \cdot p^*\left(D_4^Z\right). \end{aligned}$$

Since $p^*\left(D_i^Z\right) = V_{\hat{Z}}(T_i) = \operatorname{div}(T_i)$, we get

$$\begin{aligned} \operatorname{div}(p^*(\chi^u)) + n \cdot p^*\left(D_3^Z\right) + n \cdot p^*\left(D_4^Z\right) &= \operatorname{div}(p^*(\chi^u)) + n \cdot \operatorname{div}(T_3) + n \cdot \operatorname{div}(T_4) \\ &= \operatorname{div}(p^*(\chi^u) T_3^n T_4^n). \end{aligned}$$

Because of $\chi^u \in \Gamma(Z, \mathcal{O}(n \cdot D^Z))$, we obtain

$$\operatorname{div}(\chi^u) + n \cdot D^Z \ \geq \ 0.$$

This yields $p^*\left(\operatorname{div}(\chi^u) + n \cdot D^Z\right) \geq 0$. Hence $\operatorname{div}(p^*(\chi^u) T_3^n T_4^n) \geq 0$. This is equivalent to $p^*(\chi^u) T_3^n T_4^n \in \mathcal{O}(\hat{Z})$. Moreover, let P be as in Construction 3.6.15 and let $Q\colon \mathbb{Z}^4 \to \mathbb{Z}^4/\mathrm{im}(P^*)$ be the the projection. Then, we have $p^*(\chi^u) = \chi^{P^*u}$ and

$$Q(P^*u + ne_3 + ne_4) \ = \ Q(ne_3 + ne_4) \ = \ nw_X.$$

Since Q is the grading matrix of $\hat{Z}$ one obtains $p^*(\chi^u) T_3^n T_4^n \in \mathcal{O}(\hat{Z})_{nw_X}$. Due to $\mathcal{R}(Z) = \mathcal{O}(\hat{Z})$ we get $p^*(\chi^u) T_3^n T_4^n \in \mathcal{R}(Z)_{nw_X}$.

Next, consider $T^u \in \mathcal{R}(Z)_{nw_X}$ with $u \in \mathbb{Z}^4$. Then we have $Q(u) = nw_X$. We set

$$T^{\hat{u}} \coloneqq \frac{T^u}{T_3^n T_4^n}$$

and $\hat{u} \coloneqq u - ne_3 - ne_4$. Because of

$$Q(u - ne_3 - ne_4) \ = \ Q(u) - nQ(e_3 + e_4) \ = \ nw_X - nw_X \ = \ 0$$

and since $\ker(Q) = P^*(\mathbb{Z}^3)$, there exists an $w \in \mathbb{Z}^3$ such that $T^{P^*w} = T^{\hat{u}}$. Because P^* is injective, w is unique with this property. Hence, the assignment $T^u \mapsto \chi^w$ is well defined. Moreover, because of $T^u \in \mathcal{R}(Z) = \mathcal{O}(\hat{Z})$ we have

$$\operatorname{div}(T^u) \ = \ \operatorname{div}\left(T^{\hat{u}} T_3^n T_4^n\right) \ = \ \operatorname{div}\left(T^{\hat{u}}\right) + \operatorname{div}(T_3^n) + \operatorname{div}(T_4^n) \ \geq \ 0.$$

In other words, one gets

$$p^*(\operatorname{div}(\chi^w)) \ = \ \operatorname{div}\left(T^{P^*w}\right) \ = \ \operatorname{div}\left(T^{\hat{u}}\right) \ \geq \ -\operatorname{div}(T_3^n) - \operatorname{div}(T_4^n) \ = \ -np^*\left(D_3^Z + D_4^Z\right),$$

where we used $p^*\left(D_i^Z\right) = V_{\hat{Z}}(T_i)$. Then, we obtain

$$\operatorname{div}(\chi^w) \geq -D_3^Z - D_4^Z.$$

Thus, $\chi^w \in \Gamma(Z, \mathcal{O}(n \cdot D^Z))$ and the inverse function is well-defined as well. Since we explicitly specify the inverse function of φ, the statement about bijectivity is clear. $\square$

Lemma 3.6.18. *Consider $\tilde{P}$ from Setting 3.6.8 and the Hilbert basis $\{\tilde{u}_1, \dots, \tilde{u}_8\}$ of $S(\tilde{\omega})$ from Proposition 3.6.12. Then we have*

$$\begin{array}{llllllll} \tilde{P}^*\tilde{u}_1 & = (0,0,1,1), & \tilde{P}^*\tilde{u}_2 & = (1,1,0,2), & \tilde{P}^*\tilde{u}_3 & = (1,1,2,0), & \tilde{P}^*\tilde{u}_4 & = (2,2,0,0), \\ \tilde{P}^*\tilde{u}_5 & = (0,0,4,0), & \tilde{P}^*\tilde{u}_6 & = (0,0,0,4), & \tilde{P}^*\tilde{u}_7 & = (2\iota,0,0,0), & \tilde{P}^*\tilde{u}_8 & = (0,2\iota,0,0). \end{array}$$

Proof. This assertion results from a direct computation. Recall that $\tilde{P}^*$ from Setting 3.6.8 and $\tilde{u}_i$ from Proposition 3.6.12 are given by

$$\begin{bmatrix} -1 & -1 & \iota-1 & 0 \\ -1 & -1 & -\iota-1 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}, \quad [\tilde{u}_1, \dots, \tilde{u}_8] = \begin{bmatrix} 0 & -1 & 0 & -1 & 1 & -1 & -\frac{\iota+1}{2} & -\frac{\iota-1}{2} \\ 0 & 0 & -1 & -1 & -1 & 1 & -\frac{\iota+1}{2} & -\frac{\iota-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & 2 & 2 & 2 & 2 & \iota & \iota \end{bmatrix}.$$

□

Proof of Proposition 3.6.16. We begin by showing '$\subseteq$'. By using the projection $Q\colon \mathbb{Z}^4 \to \mathbb{Z} \oplus \mathbb{Z}/4\iota\mathbb{Z}$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \bar{2}\bar{\iota}+\bar{1} \end{bmatrix}$$

from Proposition 3.6.10 and by using the identities for $\tilde{P}^*\tilde{u}_i$ from Lemma 3.6.18, we obtain

$$\begin{gathered} Q \circ \tilde{P}^*(\tilde{u}_1) = (2, \bar{2}\bar{\iota}+\bar{2}), \\ Q \circ \tilde{P}^*(\tilde{u}_2) = Q \circ \tilde{P}^*(\tilde{u}_3) = Q \circ \tilde{P}^*(\tilde{u}_4) = Q \circ \tilde{P}^*(\tilde{u}_5) = Q \circ \tilde{P}^*(\tilde{u}_6) = (4, \bar{4}), \\ Q \circ \tilde{P}^*(\tilde{u}_7) = Q \circ \tilde{P}^*(\tilde{u}_8) = (2\iota, \bar{0}). \end{gathered}$$

Moreover, we derive $w_X = (2, \bar{2}\bar{\iota}+\bar{2})$, $2w_X = (4,\bar{4})$ and $\iota \cdot w_X = (2\iota, \bar{0})$ from Corollary 3.6.11.

Furthermore, Construction 3.6.15 provides us with the good quotient of affine varieties $\tilde{p}\colon \hat{Z} \to \tilde{Z}$. Then, we get the injective ring homomorphism

$$\tilde{p}^*\colon \mathbb{K}[S(\tilde{\omega})] \longrightarrow \mathcal{O}(\hat{Z}) \qquad \chi^u \longmapsto \chi^{\tilde{P}^*u}.$$

Thus, we have

$$\begin{aligned} \tilde{p}^*\left(\chi^{\tilde{u}_1}\right) &\in \mathcal{R}(Z)_{1\cdot w_X}, \\ \tilde{p}^*\left(\chi^{\tilde{u}_2}\right), \tilde{p}^*\left(\chi^{\tilde{u}_3}\right), \tilde{p}^*\left(\chi^{\tilde{u}_4}\right), \tilde{p}^*\left(\chi^{\tilde{u}_5}\right), \tilde{p}^*\left(\chi^{\tilde{u}_6}\right) &\in \mathcal{R}(Z)_{2\cdot w_X}, \\ \tilde{p}^*\left(\chi^{\tilde{u}_7}\right), \tilde{p}^*\left(\chi^{\tilde{u}_8}\right) &\in \mathcal{R}(Z)_{\iota\cdot w_X}. \end{aligned}$$

Next, we show the inclusion '$\supseteq$'. Let $D^Z := D_3^Z + D_4^Z$. We get the commutative diagram

$$\begin{tikzcd}
\mathcal{R}(Z)_{nw_X} & & \mathcal{O}(\tilde{Z})_n = \Gamma(\tilde{Z}, \mathcal{O}(n \cdot \mathrm{pr}^* D^Z)) \arrow[ll, "\chi^{\tilde{P}^* u} \mapsfrom \chi^u", "\tilde{p}^*"'] \\
& \Gamma(Z, \mathcal{O}(n \cdot D^Z)) \arrow[ul, "\varphi"', "p^*(\chi^u) T_3^n T_4^n \mapsfrom \chi^u"] \arrow[ur, "\mathrm{pr}^*", "\chi^u \mapsto \chi^{\mathrm{PR}^* u}"'] &
\end{tikzcd}$$

where φ is the isomorphism from Lemma 3.6.17. The diagram is commutative since the corresponding diagram of matrices and quotients from Construction 3.6.15 is commutative. Since φ is an isomorphism, for any $T^u \in \mathcal{R}(Z)_{nw_X}$ there exists a $w \in \mathbb{Z}^3$ such that $T^u = \tilde{p}^*(\chi^w)$ with $\chi^w \in \mathcal{O}(\tilde{Z}) = \mathbb{K}[\tilde{\omega} \cap \mathbb{Z}^4]$. □

Corollary 3.6.19. *Consider $X \subseteq Z$ and $\tilde{\omega}$ as in Setting 3.6.8 and let $\{\tilde{u}_1, \ldots \tilde{u}_8\}$ be the Hilbert basis of $S(\tilde{\omega})$ from Proposition 3.6.12. Then we have*

$$\begin{aligned}
\tilde{p}^*\left(\chi^{\tilde{u}_1}\right) &\in \mathcal{R}(X)_{1 \cdot w_X}, \\
\tilde{p}^*\left(\chi^{\tilde{u}_2}\right), \tilde{p}^*\left(\chi^{\tilde{u}_3}\right), \tilde{p}^*\left(\chi^{\tilde{u}_4}\right), \tilde{p}^*\left(\chi^{\tilde{u}_5}\right), \tilde{p}^*\left(\chi^{\tilde{u}_6}\right), &\in \mathcal{R}(X)_{2 \cdot w_X}, \\
\tilde{p}^*\left(\chi^{\tilde{u}_7}\right), \tilde{p}^*\left(\chi^{\tilde{u}_8}\right) &\in \mathcal{R}(X)_{\iota \cdot w_X}.
\end{aligned}$$

Proof. The closed embeddings $\hat{X} \hookrightarrow \hat{Z}$ of the characteristic spaces induce a diagram

$$\begin{tikzcd}
\bigoplus_{[D] \in \mathrm{Cl}(Z)} \mathcal{R}(Z)_{[D]} \arrow[r, equal] \arrow[d, "h \mapsto h"] & \mathcal{O}(\hat{Z}) \arrow[d, "h \mapsto h|_{\hat{X}}"'] \\
\bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}(X)_{[D]} \arrow[r, equal] & \mathcal{O}(\hat{X})
\end{tikzcd}$$

where the downwards arrows are surjective. Then the claim follows from Proposition 3.6.16. □

Proposition 3.6.20. *Consider X, $\tilde{\omega}$ and $\tilde{P}$ from Setting 3.6.8 and the Hilbert basis $\{\tilde{u}_1, \ldots, \tilde{u}_8\}$ of $S(\tilde{\omega})$ from Proposition 3.6.12. Then $\tilde{U}_1, \ldots, \tilde{U}_6 \in \mathcal{S}(X)$ from Proposition 3.6.5 are given by*

$$\tilde{U}_i = \chi^{\tilde{P}^* \tilde{u}_i}.$$

Proof. In Lemma 3.6.18 we computed $\tilde{P}^* \tilde{u}_i$ for $i = 1, \ldots, 6$. Using the latter, we obtain

$$\begin{array}{llllllll}
\chi^{\tilde{P}^* \tilde{u}_1} = \chi^{(0,0,1,1)} & = T_3 T_4 & = \tilde{U}_1, & \chi^{\tilde{P}^* \tilde{u}_2} = \chi^{(1,1,0,2)} & = T_1 T_2 T_4^2 & = \tilde{U}_2, \\
\chi^{\tilde{P}^* \tilde{u}_3} = \chi^{(1,1,2,0)} & = T_1 T_2 T_3^2 & = \tilde{U}_3, & \chi^{\tilde{P}^* \tilde{u}_4} = \chi^{(2,2,0,0)} & = T_1^2 T_2^2 & = \tilde{U}_4, \\
\chi^{\tilde{P}^* \tilde{u}_5} = \chi^{(0,0,4,0)} & = T_3^4 & = \tilde{U}_5, & \chi^{\tilde{P}^* \tilde{u}_6} = \chi^{(0,0,0,4)} & = T_4^4 & = \tilde{U}_6.
\end{array}$$

□

Remark 3.6.21. We set $\tilde{U}_7 \coloneqq \chi^{\tilde{P}^*\tilde{u}_7}$ and $\tilde{U}_8 \coloneqq \chi^{\tilde{P}^*\tilde{u}_8}$. Then, Lemma 3.6.18 provides us with

$$\tilde{U}_7 \;=\; \chi^{\tilde{P}^*\tilde{u}_7} \;=\; \chi^{(2\iota,0,0,0)} \;=\; T_1^{2\iota}, \qquad \tilde{U}_8 \;=\; \chi^{\tilde{P}^*\tilde{u}_8} \;=\; \chi^{(0,2\iota,0,0)} \;=\; T_2^{2\iota}.$$

Proposition 3.6.22. *Consider X, the Gorenstein index ι of X as in Setting 3.6.8 and let $\{\tilde{u}_1, \dots \tilde{u}_8\}$ be the Hilbert basis from Proposition 3.6.12 and let $\tilde{U}_i = \chi^{\tilde{P}^*\tilde{u}_i}$. Then we obtain the following relation in $\mathcal{R}(X)_{4\iota \cdot w_X}$:*

$$\tilde{U}_1^{2\iota}\tilde{U}_7\tilde{U}_8 - \tilde{U}_2^{\iota}\tilde{U}_3^{\iota} \;=\; 0.$$

Proof. We derive the degrees of $\chi^{\tilde{P}^*\tilde{u}_i}$ from Corollary 3.6.19 and obtain

$$\begin{array}{lll} \deg(\tilde{U}_1) \;=\; w_X, & \deg(\tilde{U}_2) \;=\; 2w_X, & \deg(\tilde{U}_3) \;=\; 2w_X, \\ \deg(\tilde{U}_7) \;=\; \iota w_X, & \deg(\tilde{U}_8) \;=\; \iota w_X. & \end{array}$$

This implies $\tilde{U}_2^{\iota}\tilde{U}_3^{\iota} \in \mathcal{R}(X)_{4\iota \cdot w_X}$ and $\tilde{U}_1^{2\iota}\tilde{U}_7\tilde{U}_8 \in \mathcal{R}(X)_{4\iota \cdot w_X}$. Moreover, with the $\tilde{u}_i$ given as in Proposition 3.6.12 we directly compute

$$2\iota\tilde{u}_1 + \tilde{u}_7 + \tilde{u}_8 \;=\; \iota\tilde{u}_2 + \iota\tilde{u}_3.$$

Consequently, we get

$$\tilde{U}_1^{2\iota}\tilde{U}_7\tilde{U}_8 - \tilde{U}_2^{\iota}\tilde{U}_3^{\iota} \;=\; \left(\chi^{\tilde{P}^*\tilde{u}_1}\right)^{2\iota} \cdot \chi^{\tilde{P}^*\tilde{u}_7} \cdot \chi^{\tilde{P}^*\tilde{u}_8} - \left(\chi^{\tilde{P}^*\tilde{u}_2}\right)^{\iota} \cdot \left(\chi^{\tilde{P}^*\tilde{u}_3}\right)^{\iota} \;=\; 0.$$

Thus, the claim follows. □

Proof of Proposition 3.6.14. The commutativity of the diagram follows from Proposition 3.6.20 and from Remark 3.6.21.

Due to Corollary 3.6.13 the map $\jmath^*$ is surjective. According to Proposition 3.6.16 the map $\tilde{p}^*$ from the diagram is surjective. Further, it is clear that $\mathcal{S}(Z) \to \mathcal{S}(X)$ is surjective. Hence, the commutative diagram yields that φ is surjective.

□

3.6.4. Relations of the anticanonical ring • We continue to examine the map φ from Proposition 3.6.14. The main result of this section is Proposition 3.6.24, where we determine the generators of the ideal $\ker(\varphi)$.

Setting 3.6.23. Consider $X = X(P)$, the cone $\tilde{\omega} \subseteq \mathbb{Q}^4$ and the monoid $S(\tilde{\omega}) = \tilde{\omega} \cap \mathbb{Z}^4$ as in Setting 3.6.8. The Hilbert basis $\mathcal{H}(\tilde{\omega}) = \{\tilde{u}_1, \dots, \tilde{u}_8\}$ of $S(\tilde{\omega})$ from Proposition 3.6.12 yields the epimorphism

$$\jmath^* \colon \mathbb{K}[U_1, \dots, U_8] \;\longrightarrow\; \mathbb{K}[S(\tilde{\omega})], \qquad U_i \;\mapsto\; \chi^{\tilde{u}_i}.$$

Let $\iota \in \mathbb{Z}_{\geq 0}$. Then, we set

$$\begin{aligned}
g_1 &\coloneqq 2U_3 + U_4 + U_5 - U_6,\\
g_2 &\coloneqq 2U_2 + U_4 - U_5 + U_6,\\
g_3 &\coloneqq 2U_1^2 - U_4 + U_5 + U_6,\\
g_4 &\coloneqq U_4^2 - 2U_4U_5 + U_5^2 - 2U_4U_6 - 2U_5U_6 + U_6^2,
\end{aligned}$$

$$g_5 \coloneqq \left(2U_1^2 + U_5 + U_6\right)\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}U_5^kU_6^{\iota-1-k} - (U_5-U_6)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}U_5^kU_6^{\iota-2-k} - U_7U_8$$

and define the ideals

$$\begin{aligned}
I &\coloneqq \left\langle U_1^2U_4 - U_2U_3,\ U_1^2U_3 - U_2U_5,\ U_1^2U_2 - U_3U_6,\ U_2 + U_3 + U_4\right\rangle,\\
\hat{I} &\coloneqq \langle g_1, g_2, g_3, g_4\rangle,\\
J &\coloneqq \hat{I}\ +\ \left\langle U_1^{2\iota}U_7U_8 - U_2^{\iota}U_3^{\iota}\right\rangle,\\
K &\coloneqq \hat{I} + \langle g_5\rangle
\end{aligned}$$

and their extensions

$$\begin{aligned}
U^{-1}I, U^{-1}\hat{I}, U^{-1}K, U^{-1}J &\ \trianglelefteq\ \mathbb{K}[U_1^{\pm1},\dots,U_8^{\pm1}],\\
\hat{U}^{-1}I, \hat{U}^{-1}\hat{I}, \hat{U}^{-1}K, \hat{U}^{-1}J &\ \trianglelefteq\ \left(U_1^2 + U_3\right)^{-1}\mathbb{K}[U_1^{\pm1},\dots,U_8^{\pm1}].
\end{aligned}$$

Proposition 3.6.24. *Consider the ideal K as in Setting 3.6.23 and consider the epimorphism $\varphi\colon \mathbb{K}[U_1,\dots,U_8] \to \mathcal{S}(X)$ from Proposition 3.6.14. Then we have $K = \ker(\varphi)$.*

The proof of Proposition 3.6.24 is given at the end of this section. In a first major preparation step we show $\hat{U}^{-1}K \subseteq \hat{U}^{-1}\ker(\varphi)$. The second step is Proposition 3.6.33, where we prove $V_{\mathbb{K}^8}(K) = V_{\mathbb{K}^8}(\ker(\varphi))$, and the last step is Corollary 3.6.36, where we show that K is a prime ideal.

Proposition 3.6.25. *Consider the epimorphism $\jmath^*$ and the ideals $J, K \trianglelefteq \mathbb{K}[U_1,\dots,U_8]$ from Setting 3.6.23 and the epimorphism φ from Proposition 3.6.14. Then the following statements hold:*

(i) *In the localization $\left(U_1^2 + U_3\right)^{-1}\mathbb{K}[U_1^{\pm},\dots,U_8^{\pm}]$ we have $\hat{U}^{-1}K \subseteq \hat{U}^{-1}J$.*
(ii) *In $\mathbb{K}[U_1^{\pm},\dots,U_6^{\pm},U_7,U_8]$ we have $(U_1\cdots U_6)^{-1}J \subseteq (U_1\cdots U_6)^{-1}\ker(\jmath^*)$.*
(iii) *In $\mathbb{K}[U_1,\dots,U_8]$ we have $\ker(\jmath^*) \subseteq \ker(\varphi)$.*

Lemma 3.6.26. *Consider the homomorphism $\jmath^*$ and the ideals $I, \hat{I} \trianglelefteq \mathbb{K}[U_1,\dots,U_6]$ from Setting 3.6.23. Set $\tilde{U} \coloneqq U_1\cdots U_6$. Then $\hat{I}$ is the saturation of I with respect to $\langle\tilde{U}\rangle$ and we have*

$$\tilde{U}^{-1}I\ =\ \tilde{U}^{-1}\hat{I}\ \subseteq\ \tilde{U}^{-1}\ker(\jmath^*)\ \trianglelefteq\ \mathbb{K}[U_1^{\pm1},\dots,U_6^{\pm1},U_7,U_8].$$

Proof. The fact that $\hat{I}$ is a saturation of I can be directly checked by means of a computer algebra system, for instance `Singular`. In other words, we have

$$\tilde{U}^{-1}I \;=\; \tilde{U}^{-1}\hat{I}$$

in $\mathbb{K}[U_1^{\pm 1}, \dots, U_6^{\pm 1}]$. Furthermore, Proposition 3.6.5 provides us with $I \subseteq \ker(\jmath^*)$. Then the claim follows from

$$\tilde{U}^{-1}\hat{I} \;=\; \tilde{U}^{-1}I \;\subseteq\; \tilde{U}^{-1}\ker(\jmath^*).$$

□

Proposition 3.6.27. *Consider the ideal $J \trianglelefteq \mathbb{K}[U_1, \dots, U_8]$ and the polynomial g_5 as in Setting 3.6.23. Then in the localization $(U_1^2 + U_3)^{-1}\,\mathbb{K}[U_1^{\pm}, \dots, U_8^{\pm}]$ we have $g_5 \in \hat{U}^{-1}J$.*

The proof of this proposition is a consequence of the following five lemmata. We successively substitute the variables U_5, U_6, U_7 and U_2 in g_5 with rational functions taken from $\hat{U}^{-1}J$.

Lemma 3.6.28. *Let $\iota \in \mathbb{Z}_{\geq 0}$. Then in $\mathbb{K}[U_1^{-1}, U_3]$ the following holds*

$$\left(\sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k} - \left(2+\tfrac{U_3}{U_1^2}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k}\right) \;=\; \left(\tfrac{U_3}{U_1^2}\right)^{2\iota-2}.$$

Proof. With $x := 1 + U_3U_1^{-2}$ we reformulate the equation from the assertion as follows:

$$\sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - (1+x)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}x^{2k} \;=\; (x-1)^{2\iota-2}.$$

For the left hand side of this equation we obtain

$$\begin{aligned}
&\sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - (1+x)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}x^{2k} \\
=\;& \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - (1+2x+x^2) \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}x^{2k} \\
=\;& \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}(x^{2k}+x^{2k+2}) - \sum_{k=0}^{\iota-2} \tbinom{2\iota-2}{2k+1}x^{2k+1}.
\end{aligned}$$

For the right hand side of the reformulated equation from the assertion we have

$$(x-1)^{2\iota-2} \;=\; \sum_{k=0}^{2\iota-2} \tbinom{2\iota-2}{k}x^k(-1)^{2\iota-2-k} \;=\; \sum_{k=0}^{\iota-1} \tbinom{2\iota-2}{2k}x^{2k} - \sum_{k=0}^{\iota-2} \tbinom{2\iota-2}{2k+1}x^{2k+1}.$$

So it remains to show

$$\sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}(x^{2k}+x^{2k+2}) \;=\; \sum_{k=0}^{\iota-1} \tbinom{2\iota-2}{2k}x^{2k}.$$

Then, for the left hand side of this equation we get

$$\begin{aligned} & \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}(x^{2k}+x^{2k+2}) \\ = \; & \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}x^{2k} - \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}x^{2k+2} \\ = \; & \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}x^{2k} - \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}x^{2k} - \sum_{k=1}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota-2}{2k-1}x^{2k}. \end{aligned}$$

Hence it suffices to verify the following three identities

$$\begin{aligned} \tfrac{1}{2}\tbinom{2\iota}{2k+1} - \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} \;&=\; \tbinom{2\iota-2}{2k}, && k=0, \\ \tfrac{1}{2}\tbinom{2\iota}{2k+1} - \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} - \tfrac{1}{2}\tbinom{2\iota-2}{2k-1} \;&=\; \tbinom{2\iota-2}{2k}, && k=1,\ldots,\iota-2, \\ \tfrac{1}{2}\tbinom{2\iota}{2k+1} - \tfrac{1}{2}\tbinom{2\iota-2}{2k-1} \;&=\; \tbinom{2\iota-2}{2k}, && k=\iota-1. \end{aligned}$$

The first and the third identity are clear by definition. Moreover, for $k=1,\ldots,\iota-2$ we have

$$\begin{aligned} \tbinom{2\iota}{2k+1} \;&=\; \tfrac{(2\iota)!}{(2k+1)!(2\iota-2k-1)!} \;=\; \tfrac{2\iota(2\iota-1)}{(2k+1)(2\iota-2k-1)}\tfrac{(2\iota-2)!}{(2k)!(2\iota-2k-2)!} \;=\; \tfrac{2\iota(2\iota-1)}{(2k+1)(2\iota-2k-1)}\tbinom{2\iota-2}{2k}, \\ \tbinom{2\iota-2}{2k+1} \;&=\; \tfrac{(2\iota-2)!}{(2k+1)!(2\iota-2k-3)!} \;=\; \tfrac{2\iota-2k-2}{2k+1}\tfrac{(2\iota-2)!}{(2k)!(2\iota-2k-2)!} \;=\; \tfrac{2\iota-2k-2}{2k+1}\tbinom{2\iota-2}{2k}, \\ \tbinom{2\iota-2}{2k-1} \;&=\; \tfrac{(2\iota-2)!}{(2k-1)!(2\iota-2k-1)!} \;=\; \tfrac{2k}{2\iota-2k-1}\tfrac{(2\iota-2)!}{(2k)!(2\iota-2k-2)!} \;=\; \tfrac{2k}{2\iota-2k-1}\tbinom{2\iota-2}{2k}. \end{aligned}$$

Thus, the second identity follows from

$$\tfrac{2\iota(2\iota-1)}{(2k+1)(2\iota-2k-1)} - \tfrac{2\iota-2k-2}{2k+1} - \tfrac{2k}{2\iota-2k-1} \;=\; 2.$$

□

Lemma 3.6.29. *In the localization* $(U_1^2+U_3)^{-1}\,\mathbb{K}[U_1^{\pm},\ldots,U_8^{\pm}]$ *we have*

$$\begin{aligned} 0 \;=\; -\tfrac{U_1^{4\iota-4}U_3^2}{(U_1^2+U_3)^\iota}\Bigg(& \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k} \\ & - \left(2+\tfrac{U_3}{U_1^2}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k}\Bigg) + \tfrac{U_3^{2\iota}}{(U_1^2+U_3)^\iota}. \end{aligned}$$

Proof. Obviously, we have

$$-\tfrac{U_1^{4\iota-4}U_3^2}{\left(U_1^2+U_3\right)^\iota}\left(\tfrac{U_3}{U_1^2}\right)^{2\iota-2} + \tfrac{U_3^{2\iota}}{\left(U_1^2+U_3\right)^\iota} \;=\; 0.$$

Lemma 3.6.28 provides us with the identity

$$\left(\sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k} - \left(2+\tfrac{U_3}{U_1^2}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k}\right) \;=\; \left(\tfrac{U_3}{U_1^2}\right)^{2\iota-2}.$$

Substituting this in the first equation gives the assertion. ☐

Lemma 3.6.30. *Consider the ideal $J \trianglelefteq \mathbb{K}[U_1,\dots,U_8]$ as in Setting 3.6.23 and*

$$\begin{aligned} f \;:=\; U_1^{2\iota}\tfrac{U_2^{\iota-2}}{U_3^\iota}\Bigg((U_2+U_3)^2 \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k} \\ - \left(\tfrac{U_3^2-U_2^2}{U_2}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k}\Bigg) - \tfrac{U_2^\iota U_3^\iota}{U_1^{2\iota}}. \end{aligned}$$

Then in the localization $\left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^\pm,\dots,U_8^\pm]$ we have $f \in \hat{U}^{-1}J$.

Proof. Lemma 3.6.26 yields $U^{-1}I = U^{-1}\hat{I}$ in $\mathbb{K}[U_1^{\pm1},\dots,U_8^\pm]$. In particular, we obtain $\hat{U}^{-1}I = \hat{U}^{-1}\hat{I}$ in the localization $\left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^\pm,\dots,U_8^\pm]$. By definition we have $J = \hat{I} + \langle U_1^{2\iota}U_7U_8 - U_2^\iota U_3^\iota\rangle$ and thus obtain

$$\hat{U}^{-1}I \;=\; \hat{U}^{-1}\hat{I} \;\subseteq\; \hat{U}^{-1}J.$$

Recall $U_1^2U_2 - U_3U_6 \in I$ from Setting 3.6.23. In particular, we have $U_1^2U_2 - U_3U_6 \in \hat{U}^{-1}I$ and consequently

$$\tfrac{U_1^2U_2}{U_3} - U_6 \;\in\; \hat{U}^{-1}J.$$

Further, we recall from Setting 3.6.23 that

$$g_2 \;=\; 2U_2 + U_4 - U_5 + U_6 \;\in\; I, \qquad g_3 \;=\; 2U_1^2 - U_4 + U_5 + U_6 \;\in\; I.$$

Hence, $U_1^2 + U_2 + U_6 \;=\; \frac{1}{2}\left(g_2+g_3\right) \in \hat{U}^{-1}I \subseteq \hat{U}^{-1}J$. Writing $U_6 = \frac{U_1^2U_2}{U_3} + h_6$ with $h_6 \in \hat{U}^{-1}J$ and substituting U_6 accordingly in $U_1^2 + U_2 + U_6$ we arrive at

$$U_1^2U_3 + U_2U_3 + U_1^2U_2 \;\in\; \hat{U}^{-1}J.$$

This implies

$$U_2 + \tfrac{U_1^2U_3}{U_1^2+U_3} \in \hat{U}^{-1}J.$$

Let $h_2 \in \hat{U}^{-1}J$ be such that $-\frac{U_1^2U_3}{U_1^2+U_3} = U_2 + h_2$. Using this identity and using that ι is odd we obtain the following

$$\begin{aligned}
-\frac{U_1^{2\iota-4}}{\left(U_1^2+U_3\right)^{\iota-2}U_3^2} \;&=\; -\frac{U_1^{2\iota-4}U_3^{\iota-2}}{\left(U_1^2+U_3\right)^{\iota-2}U_3^{\iota}} \;=\; \frac{U_2^{\iota-2}}{U_3^{\iota}}, && (3.6.30.1)\\
\frac{U_3}{U_1^2+U_3} \;&=\; -\frac{U_1^2U_3}{U_1^2+U_3} + \frac{U_3\left(U_1^2+U_3\right)}{U_1^2+U_3} \;=\; U_2 + U_3, && (3.6.30.2)\\
\left(1+\frac{U_3}{U_1^2}\right)^2 \;&=\; \left(\frac{U_3\left(U_1^2+U_3\right)}{U_1^2U_3}\right)^2 \;=\; \left(\frac{U_3}{U_2}\right)^2, && (3.6.30.3)\\
-\frac{U_3^{2\iota}}{\left(U_1^2+U_3\right)^{\iota}} \;&=\; \frac{-U_1^{2\iota}U_3^{\iota}U_3^{\iota}}{U_1^{2\iota}\left(U_1^2+U_3\right)^{\iota}} \;=\; \frac{U_2^{\iota}U_3^{\iota}}{U_1^{2\iota}} && (3.6.30.4)
\end{aligned}$$

in the factor ring $(U_1^2+U_3)^{-1}\mathbb{K}[U_1^{\pm 1},\dots,U_8^{\pm 1}]/\hat{U}^{-1}J$. Moreover, in this factor ring we have

$$\begin{aligned}
\left(\frac{U_3^2-U_2^2}{U_2}\right)^2 &= \left(\frac{U_3^2-\left(-\frac{U_1^2U_3}{U_1^2+U_3}\right)^2}{-\frac{U_1^2U_3}{U_1^2+U_3}}\right)^2\\
&= \left(\left(U_3^2-\left(-\frac{U_1^2U_3}{U_1^2+U_3}\right)^2\right)\frac{-(U_1^2+U_3)}{U_1^2U_3}\right)^2\\
&= \left(U_3\frac{-(U_1^2+U_3)}{U_1^2}+\left(\frac{U_1^2U_3}{U_1^2+U_3}\right)^2\frac{(U_1^2+U_3)}{U_1^2U_3}\right)^2\\
&= \left(U_3\frac{-(U_1^2+U_3)}{U_1^2}+\frac{U_1^2U_3}{U_1^2+U_3}\right)^2\\
&= \left(-U_3-\frac{U_3^2}{U_1^2}+\frac{U_1^2U_3}{U_1^2+U_3}\right)^2\\
&= \left(\frac{-U_3U_1^2\left(U_1^2+U_3\right)}{U_1^2\left(U_1^2+U_3\right)}-\frac{U_3^2\left(U_1^2+U_3\right)}{U_1^2\left(U_1^2+U_3\right)}+\frac{U_1^2U_3U_1^2}{U_1^2\left(U_1^2+U_3\right)}\right)^2\\
&= \frac{\left(-U_3U_1^2\left(U_1^2+U_3\right)-U_3^2\left(U_1^2+U_3\right)+U_1^4U_3\right)^2}{U_1^4\left(U_1^2+U_3\right)^2}\\
&= \frac{\left(-U_3U_1^4-U_3^2U_1^2-U_3^2U_1^2-U_3^3+U_1^4U_3\right)^2}{U_1^4\left(U_1^2+U_3\right)^2}\\
&= \frac{\left(U_3^2\left(-U_3-2U_1^2\right)\right)^2}{U_1^4\left(U_1^2+U_3\right)^2}\\
&= \frac{U_3^4\left(2U_1^2+U_3\right)^2}{U_1^4\left(U_1^2+U_3\right)^2}\\
&= \frac{U_3^4}{\left(U_1^2+U_3\right)^2}\frac{\left(2U_1^2+U_3\right)^2}{U_1^4}\\
&= \frac{U_3^4}{\left(U_1^2+U_3\right)^2}\left(2+\frac{U_3}{U_1^2}\right)^2. \qquad (3.6.30.5)
\end{aligned}$$

Further, we recall from Lemma 3.6.29

$$\begin{aligned} g \;=\; & -\tfrac{U_1^{4\iota-4}U_3^2}{\left(U_1^2+U_3\right)^\iota}\Bigg(\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k} \\ & \qquad -\left(2+\tfrac{U_3}{U_1^2}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k}\Bigg)+\tfrac{U_3^{2\iota}}{\left(U_1^2+U_3\right)^\iota} \;\in\; \hat{U}^{-1}J. \end{aligned}$$

By moving the term $\frac{U_3^4}{\left(U_1^2+U_3\right)^2}$ into the parenthesis we bring g into the form

$$\begin{aligned} g \;=\; & -U_1^{2\iota}\tfrac{U_1^{2\iota-4}}{\left(U_1^2+U_3\right)^{\iota-2}U_3^2}\Bigg(\left(\tfrac{U_3}{U_1^2+U_3}\right)^2\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k} \\ & \qquad -\tfrac{U_3^4}{\left(U_1^2+U_3\right)^2}\left(2+\tfrac{U_3}{U_1^2}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(1+\tfrac{U_3}{U_1^2}\right)^{2k}\Bigg)+\tfrac{U_3^{2\iota}}{\left(U_1^2+U_3\right)^\iota}. \end{aligned}$$

This in turn allows us to plug in directly the identities (3.6.30.1) to (3.6.30.5) into g, which yields

$$\begin{aligned} 0 \;=\; g \;=\; f \;=\; & U_1^{2\iota}\tfrac{U_2^{\iota-2}}{U_3^\iota}\Bigg(\left(U_2+U_3\right)^2\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k} \\ & \qquad -\left(\tfrac{U_3^2-U_2^2}{U_2}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k}\Bigg)-\tfrac{U_2^\iota U_3^\iota}{U_1^{2\iota}} \\ \in\; & \left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^{\pm1},\ldots,U_8^{\pm1}]/\hat{U}^{-1}J. \end{aligned}$$

Consequently, we get $f \in \hat{U}^{-1}J$. ☐

Lemma 3.6.31. *Consider the ideal $J \trianglelefteq \mathbb{K}[U_1,\ldots,U_8]$ as in Setting 3.6.23 and*

$$\begin{aligned} f \;\coloneqq\; & U_1^{2\iota}\tfrac{U_2^{\iota-2}}{U_3^\iota}\Bigg(\left(U_2+U_3\right)^2\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k} \\ & \qquad -\left(\tfrac{U_3^2-U_2^2}{U_2}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k}\Bigg)-U_7U_8. \end{aligned}$$

Then in the localization $\left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^\pm,\ldots,U_8^\pm]$ we have $f \in \hat{U}^{-1}J$.

Proof. By defintion we have $J = \hat{I} + \langle U_1^{2\iota}U_7U_8 - U_2^\iota U_3^\iota\rangle$. Hence, we get

$$U_1^{2\iota}U_7U_8 - U_2^\iota U_3^\iota \;\in\; \hat{U}^{-1}J$$

in the localization $\left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^\pm,\ldots,U_8^\pm]$. This implies $U_7U_8 - \frac{U_2^\iota U_3^\iota}{U_1^{2\iota}} \in \hat{U}^{-1}J$. This provides us with $h_{78} \in \hat{U}^{-1}J$ such that $\frac{U_2^\iota U_3^\iota}{U_1^{2\iota}} = U_7U_8 + h_{78}$. Hence, by accordingly substituting $\frac{U_2^\iota U_3^\iota}{U_1^{2\iota}}$ in the Laurent polynomial from Lemma 3.6.30, we obtain the desired claim. ☐

Lemma 3.6.32. *Consider the ideal* $J \trianglelefteq \mathbb{K}[U_1, \ldots, U_8]$ *as in Setting 3.6.23 and the Laurent polynomial*

$$\begin{aligned} f \;&:=\; \left(2U_1^2 + \tfrac{U_1^2U_3}{U_2} + U_6\right) \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} \left(\tfrac{U_1^2U_3}{U_2}\right)^k U_6^{\iota-1-k} \\ &\qquad - \left(\tfrac{U_1^2U_3}{U_2} - U_6\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} \left(\tfrac{U_1^2U_3}{U_2}\right)^k U_6^{\iota-2-k} - U_7U_8. \end{aligned}$$

Then in the localization $\left(U_1^2 + U_3\right)^{-1} \mathbb{K}[U_1^{\pm}, \ldots, U_8^{\pm}]$ *we have* $f \in \hat{U}^{-1}J$.

Proof. Lemma 3.6.26 provides us with $U^{-1}I = U^{-1}\hat{I}$ in $\mathbb{K}[U_1^{\pm1}, \ldots, U_8^{\pm1}]$. In particular, we obtain $\hat{U}^{-1}I = \hat{U}^{-1}\hat{I}$ in the localization $\left(U_1^2 + U_3\right)^{-1} \mathbb{K}[U_1^{\pm}, \ldots, U_8^{\pm}]$. By definition we have $J = \hat{I} + \langle U_1^{2\iota}U_7U_8 - U_2^{\iota}U_3^{\iota}\rangle$. Thus, we get

$$\hat{U}^{-1}I \;=\; \hat{U}^{-1}\hat{I} \;\subseteq\; \hat{U}^{-1}J.$$

Recall $U_1^2U_2 - U_3U_6 \in I$ from Setting 3.6.23. In particular, on has $U_1^2U_2 - U_3U_6 \in \hat{U}^{-1}J$. Consequently, we get

$$\tfrac{U_1^2U_2}{U_3} - U_6 \;\in\; \hat{U}^{-1}J.$$

Then, there exists $h_6 \in \hat{U}^{-1}J$ such that $\frac{U_1^2U_2}{U_3} = U_6 + h_6$. Furthermore, we simply rearrange the terms of the following Laurent polynomial

$$\begin{aligned} &U_1^{2\iota}\tfrac{U_2^{\iota-2}}{U_3^{\iota}} \left((U_2+U_3)^2 \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} - \left(\tfrac{U_3^2-U_2^2}{U_2}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} \right) - U_7U_8 \\ &= U_1^{2\iota} \left(\tfrac{U_2}{U_3}\right)^{\iota-2} \left(\tfrac{(U_2+U_3)^2}{U_3^2} \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} - \left(\tfrac{U_3^2-U_2^2}{U_2U_3}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} \right) - U_7U_8 \\ &= U_1^{2\iota} \left(\tfrac{U_2}{U_3}\right)^{\iota-2} \left(\tfrac{(U_2+U_3)^2}{U_2U_3}\tfrac{U_2}{U_3} \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} - \left(\tfrac{U_3^2-U_2^2}{U_2U_3}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} \right) - U_7U_8 \\ &= U_1^{2\iota-2+2}\tfrac{(U_2+U_3)^2}{U_2U_3} \left(\tfrac{U_2}{U_3}\right)^{\iota-1} \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} \\ &\qquad - U_1^{2\iota-4+4} \left(\tfrac{U_2}{U_3}\right)^{\iota-2} \left(\tfrac{U_3^2-U_2^2}{U_2U_3}\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} \left(\tfrac{U_3}{U_2}\right)^{2k} - U_7U_8 \end{aligned}$$

$$\begin{aligned}
&= U_1^{2\iota-2}\left(\tfrac{U_2}{U_3}\right)^{\iota-1}\left(\tfrac{2U_1^2U_2U_3+U_1^2U_3^2+U_1^2U_2^2}{U_2U_3}\right)\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k}\\
&\qquad - U_1^{2\iota-4}\left(\tfrac{U_2}{U_3}\right)^{\iota-2}\left(\tfrac{U_1^2U_3^2-U_1^2U_2^2}{U_2U_3}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k} - U_7U_8\\
&= \left(\tfrac{2U_1^2U_2U_3+U_1^2U_3^2+U_1^2U_2^2}{U_2U_3}\right)\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(\tfrac{U_1^2U_3}{U_2}\right)^{k}\left(\tfrac{U_1^2U_2}{U_3}\right)^{\iota-1-k}\\
&\qquad - \left(\tfrac{U_1^2U_3^2-U_1^2U_2^2}{U_2U_3}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(\tfrac{U_1^2U_3}{U_2}\right)^{k}\left(\tfrac{U_1^2U_2}{U_3}\right)^{\iota-2-k} - U_7U_8\\
&= \left(2U_1^2+\tfrac{U_1^2U_3}{U_2}+\tfrac{U_1^2U_2}{U_3}\right)\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(\tfrac{U_1^2U_3}{U_2}\right)^{k}\left(\tfrac{U_1^2U_2}{U_3}\right)^{\iota-1-k}\\
&\qquad - \left(\tfrac{U_1^2U_3}{U_2}-\tfrac{U_1^2U_2}{U_3}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(\tfrac{U_1^2U_3}{U_2}\right)^{k}\left(\tfrac{U_1^2U_2}{U_3}\right)^{\iota-2-k} - U_7U_8.
\end{aligned}$$

Moreover, according to Lemma 3.6.31 we have

$$\begin{aligned}
&U_1^{2\iota}\tfrac{U_2^{\iota-2}}{U_3^{\iota}}\Bigg((U_2+U_3)^2\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k}\\
&\qquad\qquad - \left(\tfrac{U_3^2-U_2^2}{U_2}\right)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}\left(\tfrac{U_3}{U_2}\right)^{2k}\Bigg) - U_7U_8 \;\in\; \hat{U}^{-1}J.
\end{aligned}$$

By substituting $\frac{U_1^2U_2}{U_3} = U_6 + h_6$ in the rearranged polynomial from above, the claim follows. ☐

Proof of Proposition 3.6.27. Due to Lemma 3.6.26 we have $U^{-1}I = U^{-1}\hat{I}$ inside the ring $\mathbb{K}[U_1^{\pm 1},\ldots,U_8^{\pm 1}]$. Hence $\hat{U}^{-1}I = \hat{U}^{-1}\hat{I}$ in the localization $\left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^{\pm},\ldots,U_8^{\pm}]$. By definition, we have $J = \hat{I} + \langle U_1^{2\iota}U_7U_8 - U_2^{\iota}U_3^{\iota}\rangle$. Then, we get

$$\hat{U}^{-1}I \;=\; \hat{U}^{-1}\hat{I} \;\subseteq\; \hat{U}^{-1}J.$$

Since $U_1^2U_3 - U_2U_5$ is a generator of I, we obtain $U_1^2U_3 - U_2U_5 \in \hat{U}^{-1}J$ inside the ring $\mathbb{K}[U_1^{\pm 1},\ldots,U_8^{\pm 1}]$. This implies $\frac{U_1^2U_3}{U_2} - U_5 \in \hat{U}^{-1}J$. Hence there is a $h_5 \in \hat{U}^{-1}J$ such that

$$\tfrac{U_1^2U_3}{U_2} \;=\; U_5 + h_5.$$

Further, due to Lemma 3.6.32 we have

$$\begin{aligned} f \;:=\; & \left(2U_1^2 + \tfrac{U_1^2U_3}{U_2} + U_6\right) \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} \left(\tfrac{U_1^2U_3}{U_2}\right)^k U_6^{\iota-1-k} \\ & - \left(\tfrac{U_1^2U_3}{U_2} - U_6\right)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} \left(\tfrac{U_1^2U_3}{U_2}\right)^k U_6^{\iota-2-k} - U_7U_8 \;\in\; \hat{U}^{-1}J \end{aligned}$$

in the localization $\left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^{\pm},\ldots,U_8^{\pm}]$. By substituting $\frac{U_1^2U_3}{U_2} = U_5 + h_5$ accordingly in the polynomial f, we obtain that the desired polynomial

$$g_5 = \left(2U_1^2 + U_5 + U_6\right) \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} U_5^k U_6^{\iota-1-k} - (U_5-U_6)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} U_5^k U_6^{\iota-2-k} - U_7U_8$$

lies in the ideal $\hat{U}^{-1}J$. □

Proof of Proposition 3.6.25. We recall $J = \hat{I} + \langle U_1^{2\iota}U_7U_8 - U_2^{\iota}U_3^{\iota}\rangle$ and $K = \hat{I} + \langle g_5\rangle$ from Setting 3.6.23. According to Proposition 3.6.27 we have $g_5 \in \hat{U}^{-1}J$ in the localization $\left(U_1^2+U_3\right)^{-1}\mathbb{K}[U_1^{\pm},\ldots,U_8^{\pm}]$. This implies

$$\hat{U}^{-1}K \;\subseteq\; \hat{U}^{-1}J$$

and we obtain (i). We proceed showing (ii). We set $\tilde{U} := (U_1\cdots U_6)$. Due to Lemma 3.6.26 we have

$$\tilde{U}^{-1}\hat{I} \;\subseteq\; \tilde{U}^{-1}\ker(\jmath^*)$$

in $\mathbb{K}[U_1^{\pm1},\ldots,U_6^{\pm1},U_7,U_8]$. It is left to check $U_1^{2\iota}U_7U_8 - U_2^{\iota}U_3^{\iota} \;\in\; \tilde{U}^{-1}\ker(\jmath^*)$. With the $\tilde{u}_i$ given as in Proposition 3.6.12 we directly compute

$$2\iota\tilde{u}_1 + \tilde{u}_7 + \tilde{u}_8 \;=\; \iota\tilde{u}_2 + \iota\tilde{u}_3.$$

From Setting 3.6.23 we recall the epimorphism

$$\jmath^*\colon \mathbb{K}[U_1,\ldots,U_8] \longrightarrow \mathbb{K}[S(\tilde{\omega})], \qquad U_i \;\mapsto\; \chi^{\tilde{u}_i}.$$

Then, we get

$$U_1^{2\iota}U_7U_8 - U_2^{\iota}U_3^{\iota} \;\in\; \ker(\jmath^*).$$

In particular, we have

$$\tilde{U}^{-1}J \;=\; \tilde{U}^{-1}\hat{I} \;+\; \tilde{U}^{-1}\left\langle U_1^{2\iota}U_7U_8 - U_2^{\iota}U_3^{\iota}\right\rangle \;\subseteq\; \tilde{U}^{-1}\ker(\jmath^*)$$

in $\mathbb{K}[U_1^{\pm1},\ldots,U_6^{\pm1},U_7,U_8]$ and the statement (ii) follows. The statement (iii), that is $\ker(\jmath) \subseteq \ker(\varphi)$, follows directly from the commutative diagram in Proposition 3.6.14. □

Proposition 3.6.33. *Let the ideal* $K \trianglelefteq \mathbb{K}[U_1,\ldots,U_8]$ *be as in Setting 3.6.23 and let* $\varphi\colon \mathbb{K}[U_1,\ldots,U_8] \to \mathcal{S}(X)$ *be as in Proposition 3.6.14. Then we have*

$$V_{\mathbb{K}^8}(K) \;=\; V_{\mathbb{K}^8}(\ker(\varphi)).$$

Lemma 3.6.34. *Consider the ideal the ideal* $K = \langle g_1, g_2, g_3, g_4, g_5 \rangle \trianglelefteq \mathbb{K}[U_1, \dots, U_8]$ *with*

$$\begin{aligned}
g_1 &= 2U_3 + U_4 + U_5 - U_6, \\
g_2 &= 2U_2 + U_4 - U_5 + U_6, \\
g_3 &= 2U_1^2 - U_4 + U_5 + U_6, \\
g_4 &= U_4^2 - 2U_4U_5 + U_5^2 - 2U_4U_6 - 2U_5U_6 + U_6^2,
\end{aligned}$$

$$g_5 = \left(2U_1^2 + U_5 + U_6\right) \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} U_5^k U_6^{\iota-1-k} - (U_5 - U_6)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} U_5^k U_6^{\iota-2-k} - U_7U_8.$$

as in Setting 3.6.23. Then, with $g \coloneqq U_1^4 - U_5U_6$, *we get an isomorphism of graded rings*

$$\mathbb{K}[U_1, \dots, U_8]/K \;\cong\; \mathbb{K}[U_1, U_5, U_6, U_7, U_8]/\left\langle g,\, g_5 \right\rangle,$$

where the degrees of the remaining U_i *are as before.*

Proof. We consider the following epimorphisms

$$\begin{aligned}
\varphi_1\colon \mathbb{K}[U_1, \dots, U_8] &\longrightarrow \mathbb{K}[U_1, U_2, U_3, U_5, U_6, U_7, U_8], \\
U_i &\longmapsto \begin{cases} U_i, & i \neq 4, \\ -U_2 - U_3, & i = 4, \end{cases} \\
\varphi_2 : \mathbb{K}[U_1, U_2, U_3, U_5, U_6, U_7, U_8] &\longrightarrow \mathbb{K}[U_1, U_3, U_5, U_6, U_7, U_8], \\
U_i &\longmapsto \begin{cases} U_i, & i \neq 2, \\ U_3 + U_5 - U_6, & i = 2, \end{cases} \\
\varphi_3 : \mathbb{K}[U_1, U_3, U_5, U_6, U_7, U_8] &\longrightarrow \mathbb{K}[U_1, U_5, U_6, U_7, U_8], \\
U_i &\longmapsto \begin{cases} U_i, & i \neq 3, \\ -U_1^2 - U_5, & i = 3. \end{cases}
\end{aligned}$$

We observe

$$\begin{aligned}
\ker(\varphi_1) &= \langle U_2 + U_3 + U_4 \rangle, & \ker(\varphi_2) &= \langle U_2 - U_3 - U_5 + U_6 \rangle, \\
\ker(\varphi_3) &= \left\langle U_3 + U_1^2 + U_5 \right\rangle.
\end{aligned}$$

Next, we set

$$f_1 \coloneqq U_2 + U_3 + U_4, \qquad f_2 \coloneqq U_2 - U_3 - U_5 + U_6, \qquad f_3 \coloneqq U_3 + U_1^2 + U_5.$$

Then we get an isomorphism

$$\mathbb{K}[U_1, \dots, U_8]/\langle f_1, f_2, f_3 \rangle \longrightarrow \mathbb{K}[U_1, U_5, U_6, U_7, U_8].$$

This provides us with an isomorphism

$$\mathbb{K}[U_1, \dots, U_8]/\left\langle f_1, f_2, f_3, U_1^4 - U_5U_6 \right\rangle \longrightarrow \mathbb{K}[U_1, U_5, U_6, U_7, U_8]/\left\langle U_1^4 - U_5U_6 \right\rangle.$$

Further, using the computer algebra system `Singular` we compute the Gröbnerbasis of $\langle f_1, f_2, f_3, U_1^4 - U_5U_6\rangle$. It is given by (f_1, g_1, g_3, g_4). Because of

$$2f_1 - g_1 \;=\; 2\,(U_2 + U_3 + U_4) - 2U_3 - U_4 - U_5 + U_6 \;=\; 2U_2 + U_4 - U_5 + U_6 \;=\; g_2$$

we obtain

$$\Big\langle f_1, f_2, f_3, U_1^4 - U_5U_6 \Big\rangle \;=\; \langle f_1, g_1, g_3, g_4\rangle \;=\; \langle g_1, g_2, g_3, g_4\rangle\,.$$

Hence, we have an isomorphism

$$\mathbb{K}[U_1, \ldots, U_8]/\left\langle g_1, g_2, g_3, g_4\right\rangle \;\longrightarrow\; \mathbb{K}[U_1, U_5, U_6, U_7, U_8]/\Big\langle U_1^4 - U_5U_6 \Big\rangle\,.$$

Since g_5 does only depend on the variables U_1, U_5, U_6, U_7 and U_8, this yields the claim. □

Proposition 3.6.35. *Consider the polynomials*

$$g \;=\; U_1^4 - U_5U_6,$$

$$g_5 = \Big(2U_1^2 + U_5 + U_6\Big)\sum_{k=0}^{\iota-1}\tfrac{1}{2}\tbinom{2\iota}{2k+1}U_5^kU_6^{\iota-1-k} - (U_5 - U_6)^2\sum_{k=0}^{\iota-2}\tfrac{1}{2}\tbinom{2\iota-2}{2k+1}U_5^kU_6^{\iota-2-k} - U_7U_8.$$

Then the zero set $V_{\mathbb{K}^5}(g, g_5)$ is normal, irreducible and a complete intersection.

Proof. In a first step we show that the zero set $Y := V_{\mathbb{K}^5}(g, g_5)$ is connected and that each irreducible component contains the origin. Lemma 3.6.34 provides us with the grading of the ring $\mathbb{K}[U_1, U_5, U_6, U_7, U_8]$

$$[\deg(U_1), \deg(U_5), \deg(U_6), \deg(U_7), \deg(U_8)] \;=\; [1, 2, 2, \iota, \iota]\,.$$

The corresponding $\mathbb{K}^*$-action on $\mathbb{K}^5$ is given as

$$t\cdot(z_1, \ldots, z_5) \;=\; \Big(tz_1, t^2z_2, t^2z_3, t^\iota z_4, t^\iota z_5\Big)\,.$$

The polynomials g and g_5 are homogeneous and thus the $\mathbb{K}^*$-action restricts to Y. Moreover, every irreducible component of Y is $\mathbb{K}^*$-invariant and the closure of every $\mathbb{K}^*$-orbit contains zero. Consequently, every irreducible component contains zero and Y must be connected.

In a second step, we show that Y is a normal complete intersection that means all local rings $\mathcal{O}_{Y,y}$, where $y \in Y$, are normal and that $I(Y) = \langle g, g_5\rangle$. The Jacobian of (g, g_5) is

$$J_{(g,g_5)} = \begin{bmatrix} 4U_1^3 & -U_6 & -U_5 & 0 & 0 \\ \frac{\partial g_5}{\partial U_1} & \frac{\partial g_5}{\partial U_5} & \frac{\partial g_5}{\partial U_6} & -U_8 & -U_7 \end{bmatrix}.$$

Let $z \in Y$. If we have $\mathrm{rk}J_{(g,g_5)}(z) < 2$ then every 2×2 minor of $J_{(g,g_5)}(z)$ has to have determinant equal to zero. This yields

$$\begin{aligned}\left\{z \in V_{\mathbb{K}^5}(g, g_5);\, \mathrm{rk}J_{(g,g_5)} < 2\right\} \;\subseteq\; V(g, g_5) \cap \Bigg(& \Big(V(U_5) \cup V(U_7, U_8)\Big) \cap \\ & \Big(V(U_6) \cup V(U_7, U_8)\Big) \cap \\ & \Big(V(U_1) \cup V(U_7, U_8)\Big)\Bigg).\end{aligned}$$

Thus, the set of all points where the Jacobian is not of full rank is of codimension ≥ 2. Then Serre's criterion for normality [35, AI, Sec. 6.2] yields that Y is a normal complete intersection, that is $I(Y) = \langle g, g_5 \rangle$. Consequently, we get

$$\langle g, g_5 \rangle \;=\; I(Y) \;=\; \sqrt{\langle g, g_5 \rangle}.$$

Finally, observe that Y is irreducible. Otherwise, at least two irreducible components contain the origin. Thus, $\mathcal{O}_{Y,0}$ can't be a normal ring. This contradicts the second step. Consequently Y is irreducible and $I(Y) = \langle g, g_5 \rangle$ is a prime ideal. ☐

Corollary 3.6.36. *Let* $K \trianglelefteq \mathbb{K}[U_1, \ldots, U_8]$ *be as in Setting 3.6.23. Then* $V_{\mathbb{K}^8}(K)$ *is normal, irreducible and a complete intersection.*

Proof. Let $g = U_1^4 - U_5U_6$. Then, Lemma 3.6.34 yields that $V_{\mathbb{K}^8}(K)$ is isomorphic to $V_{\mathbb{K}^5}(g, g_5)$. Hence Proposition 3.6.35 yields the assertion. ☐

Lemma 3.6.37. *Let the ideal* $K \trianglelefteq \mathbb{K}[U_1, \ldots, U_8]$ *be as in Setting 3.6.23. Then we have* $\dim(V_{\mathbb{K}^8}(K)) = 3$.

Proof. Due to Corollary 3.6.36 the null set $V_{\mathbb{K}^8}(K)$ is an irreducible, normal, complete intersection. Consequently, we obtain $\dim(V_{\mathbb{K}^8}(K)) = 3$. ☐

Proof of Proposition 3.6.33. We derive from Proposition 3.6.14 that φ is surjective. Hence, the map $\mathrm{Spec}(\varphi)\colon \tilde{X} \hookrightarrow \mathbb{K}^8$ is a closed embedding. Thus, we obtain $\ker(\varphi) = I_{\mathbb{K}^8}(\tilde{X})$. This implies

$$V_{\mathbb{K}^8}(\ker(\varphi)) \;=\; V_{\mathbb{K}^8}(I_{\mathbb{K}^8}(\tilde{X})) \;=\; \mathrm{Spec}(\varphi)\left(\tilde{X}\right).$$

By using Lemma 3.6.37 one obtains

$$\dim(V_{\mathbb{K}^8}(K)) \;=\; 3 \;=\; \dim\left(\tilde{X}\right) \;=\; \dim(V_{\mathbb{K}^8}(\ker(\varphi))).$$

Due to Proposition 3.6.25 we have $\hat{U}^{-1}K \subseteq \hat{U}^{-1}\ker(\varphi)$. This provides us with the following inclusion on an open subset of $\mathbb{K}^8$

$$V_{\mathbb{T}^8 \setminus V\left(U_1^2 + U_3\right)}(\ker(\varphi)) \;\subseteq\; V_{\mathbb{T}^8 \setminus V\left(U_1^2 + U_3\right)}(K).$$

Hence, we get $V_{\mathbb{K}^8}(\ker(\varphi)) \subseteq V_{\mathbb{K}^8}(K)$. Altogether, since the two closed sets are contained in each other and have the same dimension, we conclude $V_{\mathbb{K}^8}(K) = V_{\mathbb{K}^8}(\ker(\varphi))$. $\square$

Proof of Proposition 3.6.24. According to Proposition 3.6.33 we have $V_{\mathbb{K}^8}(K) = V_{\mathbb{K}^8}(\ker(\varphi))$. This implies

$$\sqrt{K} \;=\; I\left(V_{\mathbb{K}^8}(K)\right) \;=\; I\left(V_{\mathbb{K}^8}(\ker(\varphi))\right) \;=\; \sqrt{\ker(\varphi)} \;=\; \ker(\varphi).$$

Further, Corollary 3.6.36 tells us that K is a prime ideal. Then we obtain $K = \ker(\varphi)$. $\square$

Proof of Theorem 3.6.1. For (i), we use `MDSpackage` [23] to compute a minimal homogeneous generator system for $\mathcal{S}(X)$, see that the generator degrees are $1, 1, 1, 2$ and that the ideal of relations is generated by a single polynomial of degree 4 as in (i). We show the second assertion. Consider the $\mathbb{Z}$-graded polynomial algebra

$$\mathbb{K}[U_1, \ldots, U_8], \qquad [\deg(U_1), \ldots \deg(U_8)] \;=\; [1, 2, 2, 2, 2, 2, \iota, \iota]\,.$$

According to Proposition 3.6.14 we have an epimorphism of graded rings onto the anticanonical ring $\varphi\colon \mathbb{K}[U_1, \ldots, U_8] \to \mathcal{S}(X)$, given by

$$\begin{array}{llllllll} \varphi(U_1) &= T_3T_4, & \varphi(U_2) &= T_1T_2T_4^2, & \varphi(U_3) &= T_1T_2T_3^2, & \varphi(U_4) &= T_1^2T_2^2, \\ \varphi(U_5) &= T_3^4, & \varphi(U_6) &= T_4^4, & \varphi(U_7) &= T_1^{2\iota}, & \varphi(U_8) &= T_2^{2\iota}. \end{array}$$

This induces an isomorphism of graded rings $\mathbb{K}[U_1, \ldots, U_8]/\ker(\varphi) \cong \mathcal{S}(X)$. Due to Proposition 3.6.24, the main result of Subsection 3.6.4, the kernel of φ is generated by the polynomials

$$\begin{aligned} g_1 &\coloneqq 2U_3 + U_4 + U_5 - U_6, \\ g_2 &\coloneqq 2U_2 + U_4 - U_5 + U_6, \\ g_3 &\coloneqq 2U_1^2 - U_4 + U_5 + U_6, \\ g_4 &\coloneqq U_4^2 - 2U_4U_5 + U_5^2 - 2U_4U_6 - 2U_5U_6 + U_6^2, \end{aligned}$$

$$g_5 \coloneqq \left(2U_1^2 + U_5 + U_6\right) \sum_{k=0}^{\iota-1} \tfrac{1}{2}\tbinom{2\iota}{2k+1} U_5^k U_6^{\iota-1-k} - (U_5 - U_6)^2 \sum_{k=0}^{\iota-2} \tfrac{1}{2}\tbinom{2\iota-2}{2k+1} U_5^k U_6^{\iota-2-k} - U_7U_8.$$

Lemma 3.6.34 allows us to eliminate the variables U_2, U_3 and U_4 such that we arrive at an isomorphism of graded rings

$$\mathbb{K}[U_1, U_5, U_6, U_7, U_8]/\left\langle g,\, g_5\right\rangle \;\cong\; \mathbb{K}[U_1, \ldots, U_8]/\ker(\varphi) \;\cong\; \mathcal{S}(X),$$

where $g = U_1^4 - U_5U_6$ and where the degrees of the remaining U_i are as before. After suitably renumbering the variables, we obtain the assertion. $\square$

BIBLIOGRAPHY

[1] V. Alexeev, *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810. ↑131

[2] C. Araujo, *Kähler-Einstein Metrics for some Quasi-Smooth Log Del Pezzo Surfaces*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4303–4312. ↑139

[3] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. ↑9, 11, 12, 13, 15, 16, 17, 18, 24, 32, 45, 75, 93, 94, 95, 96, 97, 98, 102, 109, 115, 116, 119, 122, 124, 125, 126, 127, 128, 129, 130, 131

[4] F. Berchtold and J. Hausen, *Cox rings and combinatorics*, Trans. Amer. Math. Soc. **359** (2007), no. 3, 1205–1252. ↑5, 11, 93

[5] B. Bechtold, J. Hausen, E. Huggenberger, and M. Nicolussi, *On terminal Fano 3-folds with 2-torus action*, Int. Math. Res. Not. IMRN **5** (2016), 1563–1602. ↑131

[6] D. Bourqui, *La conjecture de Manin géométrique pour une famille de quadriques intrinsèques*, Manuscripta Math. **135** (2011), no. 1-2, 1–41 (French, with English and French summaries). ↑5

[7] I. Cheltsov, J. Park, and C. Shramov, *Delta invariants of singular del Pezzo surfaces*, J. Geom. Anal. **31** (2021), no. 3, 2354–2382. ↑139

[8] I. Cheltsov, J. Park, and C. Shramov, *Exceptional del Pezzo hypersurfaces*, J. Geom. Anal. **20** (2010), no. 4, 787–816. ↑139

[9] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. ↑11, 12

[10] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. ↑9, 11, 12, 13, 44, 57

[11] V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247 (Russian). ↑9

[12] A. Fahrner and J. Hausen, *On intrinsic quadrics*, Canad. J. Math. **72** (2020), no. 1, 145–181. ↑5, 94, 96

[13] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. ↑9

[14] Y. Gyoda and K. Matsushita, *Generalization of Markov Diophantine equation via generalized cluster algebra*, Electron. J. Combin. **30** (2023), no. 4, Paper No. 4.10, 20. ↑37

[15] P. Hacking and Y. Prokhorov, *Smoothable del Pezzo surfaces with quotient singularities*, Compos. Math. **146** (2010), no. 1, 169–192. ↑3, 57, 60

[16] D. Hättig, B. Hafner, J. Hausen, and J. Springer, *Del Pezzo surfaces of Picard number one admitting a torus action* (2022), available at `arXiv:2207.14790v1`. ↑27

[17] D. Hättig, J. Hausen, and J. Springer, *Classifying log del Pezzo surfaces with torus action* (2023), available at `arXiv:2302.03095`. ↑27, 57, 98, 99, 100, 101, 103, 106, 107, 108, 110, 113, 114, 115, 116, 119, 122, 126, 127, 128, 129, 130, 132

[18] D. Hättig, J. Hausen, and H. Süß, *Log del Pezzo $\mathbb{C}^*$-surfaces, Kähler-Einstein metrics, Kähler-Ricci solitons and Sasaki-Einstein metrics*, Michigan Mathematical Journal (to appear), available at `arXiv:2306.03796`. ↑18, 19, 20, 21, 22, 23, 73, 76, 85, 86, 139, 140

[19] J. Hausen, *Cox rings and combinatorics. II*, Mosc. Math. J. **8** (2008), no. 4, 711–757, 847. ↑11

[20] J. Hausen and E. Herppich, *Factorially graded rings of complexity one*, Torsors, étale homotopy and applications to rational points, London Math. Soc. Lecture Note Ser., vol. 405, Cambridge Univ. Press, Cambridge, 2013, pp. 414–428. ↑15, 18, 93

[21] J. Hausen, E. Herppich, and H. Süß, *Multigraded factorial rings and Fano varieties with torus action*, Doc. Math. **16** (2011), 71–109. ↑133

[22] J. Hausen, C. Hische, and M. Wrobel, *On torus actions of higher complexity*, Forum Math. Sigma **7** (2019), e38. ↑15, 17, 18, 99, 103, 110

[23] J. Hausen and S. Keicher, *A software package for Mori dream spaces*, LMS J. Comput. Math. **18** (2015), no. 1, 647–659. ↑97, 155, 156, 184

[24] J. Hausen, S. Keicher, and A. Laface, *Computing Cox rings*, Math. Comp. **85** (2016), no. 297, 467–502. ↑96, 97

[25] J. Hausen and K. Király, *Full intrinsic quadrics of dimension two*, New York J. Math. **30** (2024), 1768–1802, available at `https://nyjm.albany.edu/j/2024/30-75.html`. ↑93

[26] J. Hausen and K. Király, *$\mathbb{K}^*$-surfaces of Picard number one and integral degree* (2024), available at `arXiv:2411.15079v1`. ↑37

[27] J. Hausen, K. Király, and M. Wrobel, *On degenerations of the projective plane*, available at `arXiv:2405.04862v4`. ↑9, 13, 25, 26, 27, 37, 44, 57, 73, 78

[28] J. Hausen and H. Süß, *The Cox ring of an algebraic variety with torus action*, Adv. Math. **225** (2010), no. 2, 977–1012. ↑15, 18, 93

[29] J. Hausen and M. Wrobel, *Non-complete rational T-varieties of complexity one*, Math. Nachr. **290** (2017), no. 5-6, 815–826. ↑15, 75

[30] C. Hische, *On canonical Fano intrinsic quadrics*, Glasg. Math. J. **65** (2023), no. 2, 288–309. ↑5

[31] C. Hische and M. Wrobel, *On the anticanonical complex*, available at `arxiv.org/abs/1808.01997`. ↑131

[32] A. Hurwitz, *Über eine Aufgabe der unbestimmten Analysis*, In: Mathematische Werke, Band II: Zahlentheorie, Algebra und Geometrie (1963), 410–421. ↑37, 43

[33] J. M. Johnson and J. Kollár, *Kähler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 1, 69–79 (English, with English and French summaries). ↑7, 139, 155

[34] B. V. Karpov and D. Y. Nogin, *Three-block exceptional sets on del Pezzo surfaces*, Izv. Ross. Akad. Nauk Ser. Mat. **62** (1998), no. 3, 3–38 (Russian, with Russian summary); English transl., Izv. Math. **62** (1998), no. 3, 429–463. ↑43

[35] H.-P. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics, Viehweg und Teubner Verlag Wiesbaden, 1984. ↑183

[36] P. Orlik and P. Wagreich, *Algebraic surfaces with k^*-action*, Acta Math. **138** (1977), no. 1-2, 43–81. ↑24, 25, 124

[37] J. Springer, *The Picard index of a surface with torus action*, Collect. Math., posted on 2024, DOI 10.1007/s13348-024-00443-x. published online. ↑133

[38] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), no. 1, 101–172. ↑139

[39] G. Tian and S.-T. Yau, *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$*, Comm. Math. Phys. **112** (1987), no. 1, 175–203. ↑139