# Stability of Trajectories of Classical, Weakly Interacting Particles

**Dissertation**

der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
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Tübingen
2024

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen.

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| Tag der mündlichen Qualifikation: | 13.03.2025 |
| Dekan: | Prof. Dr. Thilo Stehle |
| 1. Berichterstatter: | Prof. Dr. Peter Pickl |
| 2. Berichterstatterin: | Prof. boshi. Li Chen |

## Abstract

The Vlasov-Poisson equation describes the macroscopic time evolution of a system with a large number of particles interacting by a force, typically Coulomb or gravitational force.

Although the equation has established itself as a helpful tool yielding strong results, it still lacks a rigorous mathematical justification. So far, the mathematical results all either worked with a cut-off on the force, used a slightly weaker interaction all together, or assumed monokineticity of the initial data. In these settings, however, one was able to prove that the mean-field trajectories arising from the Vlasov-Poisson equation are typically a very good approximation for the real trajectories.

In this paper we consider a similar set-up where we can build on these previous results to help us answer a new but related question. We consider a system of $N$ initially i.i.d. particles with Coulomb interaction. If we move the initial position of one particle by a small $\delta \in \mathbb{R}^6$ in phase-space, what is the expected impact on the whole system in the time frame $[0, T]$? Due to the pair interaction, the effect is highly non-trivial. We prove that, under suitable conditions on the starting density and with a cut-off at approximately $N^{-\frac{1}{3}}$, the system as a whole typically stays stable. More precisely, for all the particles that we did not disturb, the distance between the original trajectory and the trajectory in the disturbed system stays smaller than $N^{-\alpha}|\delta|$, for suitable $\alpha > 0$. This $\alpha > 0$ depends on whether the particle has a collision with another particle and on its distance to the disturbed particle. Depending on these characteristics, $\alpha$ takes on values between $N^{-\frac{1}{18}+2\sigma}$ and $N^{-\frac{1}{3}+2\sigma}$. The best estimates are obtained for particles that are far away from the disturbed particle and do not have a collision.

We are further able to extract the leading order effect on the disturbed particle without referring to the $N$-body dynamics. The effect of moving the starting position of the particle in phase-space on the Newtonian trajectory can typically be estimated by the effect that the displacement has on the mean-field trajectory. The inaccuracy of this estimate is bounded by $N^{-\frac{1}{18}+4\sigma}|\delta|$. While previous results have proven that the mean-field trajectories and the Newtonian trajectories stay close, our result proves that the mean-field system is a good estimate of higher order.

This result can be interpreted as stability of the $N$-body system. We are able to prove that typically the closeness between the mean-field and Newtonian trajectories of the particles is preserved after moving the starting position of one particle.

Finally, we give an outlook on how our result can be used to obtain stronger estimates for probabilities. The ability to control probabilities, for example, the probability of collisions, is crucial in order to shrink our cut-off further. While our result still needs some improvements in order to estimate these collision probabilities, we believe that this application is an important proof of concept and motivates further research into this topic.

## Zusammenfassung

Die Vlasov-Poisson-Gleichung beschreibt die makroskopische zeitliche Entwicklung eines Vielteilchensystems, die über eine Kraft, typischerweise Coulomb- oder Gravitationskraft, miteinander wechselwirken.

Obwohl sich die Gleichung als nützliches Werkzeug etabliert hat, welches starke Ergebnisse liefert, fehlt ihr nach wie vor eine rigorose mathematische Begründung. Bisher basieren mathematische Resultate entweder auf einer Abschwächung der Wechselwirkung, einer Modifikation der Kraft durch einen Cut-off oder auf der Annahme von Monokinetizität der Anfangsdaten. Unter diesen Bedingungen konnte allerdings gezeigt werden, dass die Meanfield Trajektorien, die sich aus der Vlasov-Poisson-Gleichung ergeben, in der Regel eine sehr gute Näherung für die realen Bahnen darstellen.

In dieser Arbeit betrachten wir ein ähnliche Situation, das auf diese früheren Ergebnisse aufbaut, um eine neue, aber verwandte Fragestellung zu beantworten:

Was ist der zu erwartende Einfluss auf das gesamte System im Zeitraum $[0, T]$, wenn wir in einem $N$-Teilchen-System (von anfangs u.i.v. Teilchen) ein Teilchen um ein kleines $\delta \in \mathbb{R}^6$ im Phasenraum verschieben? Aufgrund der Paarwechselwirkung ist der Effekt auf das Gesamtsystem im Laufe der Zeit höchst kompliziert.

Wir beweisen, dass das System unter geeigneten Bedingungen an die Anfangsdichte und mit einem Cut-off bei $\approx N^{-\frac{1}{3}}$ typischerweise stabil bleibt. Das bedeutet, dass für alle Teilchen, die wir nicht gestört haben, der Abstand zwischen der ursprünglichen Trajektorie und der Bahn im gestörten System kleiner als $N^{-\alpha}|\delta|$ bleibt, wobei $\alpha > 0$ geeignet gewählt ist. Dieser Parameter $\alpha > 0$ hängt davon ab, ob das Teilchen eine Kollision mit einem anderen Teilchen hat, und von seiner Distanz zum gewackelten Teilchen. Je nachdem nimmt $\alpha$ Werte zwischen $N^{-\frac{1}{18}+2\sigma}$ und $N^{-\frac{1}{3}+2\sigma}$ an, wobei $\sigma > 0$ beliebig klein ist. Die besten Abschätzungen erhalten wir für Teilchen, die weit vom gewackelten Teilchen entfernt sind und keine Kollisionen haben.

Darüber hinaus können wir den führenden Effekt auf das gestörte Teilchen bestimmen, ohne auf die $N$-Körper-Dynamik zurückzugreifen. Der Effekt, den eine Verschiebung der Anfangsposition eines Teilchens im Phasenraum auf die Newtonsche Bahn hat, kann in der Regel durch den Einfluss der Verschiebung auf die Meanfield Trajektorie angenährt werden. Der Fehler, der durch diese Abschätzung entsteht ist dabei kleiner als $N^{-\frac{1}{18}+4\sigma}|\delta|$. Während frühere Ergebnisse gezeigt haben, dass die Meanfield Trajektorien und die Newtonschen Trajektorien nah beieinander bleiben, beweist unser Resultat, dass das Meanfield System auch in höherer Ordnung eine gute Näherung darstellt.

Dieses Ergebnis kann als Stabilität des $N$-Teilchen-Systems interpretiert werden. Wir können zeigen, dass die Nähe zwischen den Meanfield Trajetorien und den Newtonschen Trajektorien der Teilchen typischerweise erhalten bleibt, auch nachdem die Anfangsposition eines Teilchens verschoben wurde.

Abschließend geben wir einen Ausblick, wie unser Ergebnis genutzt werden kann, um starke Abschätzungen für Wahrscheinlichkeiten zu erhalten. Die Abschätzung von Wahrscheinlichkeiten, wie z.B. der Kollisionswahrscheinlichkeit, ist entscheidend, um den Cut-off weiter zu verkleinern. Auch wenn unser Ergebnis noch verbessert werden muss, um solche Kollisionswahrscheinlichkeiten abschätzen zu können, sehen wir diese Anwendung als wichtigen Proof of Concept, der weitere Forschung in dieses Thema rechtfertigt.

## Acknowledgment

I am honored that Prof. Dr. Peter Pickl agreed to be my doctoral supervisor.

*Peter, I am grateful for your guidance and your faith in me. You understood and supported my decision to stay in Munich and you taught me a great deal about math, physics, academia and the kind of teacher (or maybe even professor) I would like to be.*

I am glad that I got the chance to work with Dr. Heribert Zenk, Prof. Dr. Sebastian Hensel and Prof. Dr. Karin Binder and Dr. Erwin Schörner at LMU.

*Thank you all for your council and your support; for making me feel needed and valued. Thanks to you, I always felt welcome and at home at LMU.*

I am inspired by the incredible tutors I got to work with at LMU and Tübingen.

*Thank you for your contagious dedication to high-quality education for the students. I am looking forward to the great things you will accomplish in research and teaching.*

I am humbled by my students whose criticism and praise motivated me to do my best.

*I hope you guys have learned as much from me as I have learned from you. Thank you for forgiving my mistakes, appreciating my efforts and making me a better teacher.*

I am grateful for the time Fabian Darabi Far and Manuela Feistl spent refining this thesis.

*Manu, thank you for paving the way. Every step I had to take was easier as I just had to follow your foot steps. Fabian, thank you for your interest in my research. Your feedback allowed me to make formulations more precise and explanations easier to follow.*

I am blessed to have the support of my loving partner Anne Reif.

*Thank you for being my fiercest critic and my most passionate supporter at the same time. With you by my side, I am curious what the future holds.*

I am privileged to have a family as loving and encouraging as mine.

*Thank you for having my back no matter what. For always lending a sympathetic ear and supporting me in all possible areas of life.*


## Dedication

*This thesis is dedicated to my father. Thank you for encouraging me to embark on this academic journey even though you knew you would not be able to accompany me till the end. I would have loved to hear your input on this thesis and – as corny as it may sound – I hope you are proud of what I have achieved.*

# Contents

# 1 Introduction

## 1.1 Set Up

We work in a system with $N$ particles and focus on the phase-space $\mathbb{R}^{6N}$. To specify the interaction of these particles, we need a force $f^N : \mathbb{R}^3 \to \mathbb{R}^3$. Since forces with a quadratic scaling are of great interest in physics, they are the subject in this paper.

These forces, however, have a singularity that is hard to deal with and, for technical reasons, we therefore introduce a cut-off that depends on the number of particles $N$. We hereby modify the force in such a way that it becomes differentiable at least two-times.

While our results could also be proven for a slightly more general set of forces, for better readability and to avoid unnecessary variables, we restrict ourselves to pair interactions that satisfy the following properties:

$$f^N(q) = \begin{cases} \frac{q}{|q|} r_N(|q|) & \text{if} \quad |q| \leq c(N) \\ \frac{q}{|q|} \cdot \frac{1}{|q|^2} & \text{else} \end{cases} \tag{1.1}$$

where $c(N) = N^{-\frac{1}{3}} \ln(N)^{-\frac{1}{4}}$ and $r_N$ is used to create a two time differentiable cut-off. Furthermore, we need to be able to control how $r_N$ depends on $N$. So there needs to be a constant $C_r$ (independent of $N$) such that $|r_N|_\infty \leq C_r c(N)^{-2}$, $|r_N'|_\infty \leq C_r c(N)^{-3}$ and $|r_N''|_\infty \leq C_r c(N)^{-4}$. Furthermore, we want that $|f^{N+1}(q)| \leq C|f^N(q)|$ for all $q \in \mathbb{R}^6$ and a suitable constant $C$.

**Remark 1.1.** Such a force exists. For example, for $x \geq 0$ we can consider the polynomial $p(x) = -5x^6 + 6x^5$. We then consider

$$r_N(x) = p\left(\frac{x}{c(N)}\right) \cdot \frac{1}{x^2} = \frac{-5x^4}{c(N)^6} + \frac{6x^3}{c(N)^5}.$$

Note that by definition $r_N(0) = 0$ and for $|q| = c(N)$ we have $r_N(|q|) = \frac{1}{|q|^2}$, making $f^N$ continuous in this case. We also have $r_N'(0) = 0$ as well as

$$r_N'(c(N)) = \frac{-20}{c(N)^3} + \frac{18}{c(N)^3} = \frac{-2}{c(N)^3},$$

making $f^N$ continuously differentiable. The second derivative does not match up for $|q| = c(N)$, but this is tolerable as we will later see.

As for the estimates, one can easily check that on the interval $[0, c(N)]$ we have $|r_N|_\infty \leq 6c(N)^{-2}$, $|r_N'|_\infty \leq 18c(N)^{-3}$ and $|r_N''|_\infty \leq 36c(N)^{-4}$.

**Remark 1.2.** Possible generalizations include a factor $a \in \mathbb{R}$ in front of the force. Furthermore, the results should still hold for any force that is less singular, i.e. that satisfies similar bonds for the force itself and its derivatives.

Our $N$ particles are i.i.d. with respect to a continuously differentiable probability density $k : \mathbb{R}^6 \to \mathbb{R}$. As we base our work on the results of Graß in [9], we only consider densities that satisfy

$$\begin{aligned} k(x) &\leq C_0 \frac{1}{(1+|x|)^{4+\delta}} \quad \forall x \in \mathbb{R}^6 \\ |\nabla k(x)| &\leq C_0 \frac{1}{(1+|x|)^{3+\delta}} \quad \forall x \in \mathbb{R}^6 \\ \int_{\mathbb{R}^6} |p|^2 k(q,p) \,\mathrm{d}(q,p) &\leq C_0, \end{aligned} \tag{1.2}$$

for some $C_0, \delta > 0$.

Our particles are $x_i = ({}^1x_i, {}^2x_i) = (q_i, p_i) \in \mathbb{R}^6$. We use that notation more generally, and for any $z \in \mathbb{R}^6$ we use ${}^1z$ and ${}^2z$ to denote the position and momentum of $z$, respectively.

The trajectories of the particles are given by differential equations

$$q_i' = p_i \tag{1.3}$$

$$p_i' = F^N(q_i) = \frac{1}{N} \sum_{j \neq i} f^N(q_i - q_j). \tag{1.4}$$

Note that we assume the mass of the particles to be equal to 1 and hence velocity and momentum coincide. This is the reason for the equality in the first line.

There are two problems we have to address if we claim that our result describes real physical phenomenons.

First, we have a cut-off. This is a deviation from the real physical system, however the cut-off shrinks as the number of particles grows. In fact, the typical distance between particles in our configuration is $N^{-\frac{1}{3}}$ and our cut-off shrinks slightly faster because of the $\ln(N)^{-\frac{1}{4}}$. This means that for $N \to \infty$ the typical interaction between particles will not even feel the cut-off. At the same time we do not want to oversell the result, as the typical system will still have interactions that are effected by the cut-off.

Second, the prefactor $\frac{1}{N}$ in 1.4 might be a bit confusing.

For this problem there is a solution, namely a change of variables. To get from the system considered in this paper, where we use $q$ and $p$ and the force with the prefactor $\frac{1}{N}$, back to a setting with a normal force (still with a cut-off), we simply use new coordinates. We call them $\hat{q} = \sqrt[3]{N} q$ and $\hat{p} = \sqrt[3]{N} p$. We also get a new force

$$\hat{p}' = \sqrt[3]{N} p' = \sqrt[3]{N} \cdot \frac{1}{N} f(q_i - q_j) = \sqrt[3]{N} \frac{1}{N} f^N \left( \frac{1}{\sqrt[3]{N}} (\hat{q}_i - \hat{q}_j) \right) = f^N(\hat{q}_i - \hat{q}_j).$$

Hence, after rescaling by $\sqrt[3]{N}$ (which again is the expected distance between particles in our setting), we obtain a – up to the cut-off – physical force. We will also try to motivate our rescaling.

One can easily see that the prefactor $\frac{1}{N}$ can be incredible helpful when estimating the force, as without the prefactor the force on any particle might go to infinity in the limit $N \to \infty$.

The real problem, of course, lies in the fact that we did not allow our density $k$ to depend on $N$. There are technical reasons for this, mainly that some stochastic arguments would be more cumbersome.

A density that does not depend on $N$, however, would lead to problems (if we would not use the rescaling), as such a density would guarantee the unboundedness of potential energy per particle as $N \to \infty$. Without our prefactor this would be impossible to deal with.

To conclude: Normally for $N \to \infty$ the volume of our "gas" must go to infinity, but in our setting the volume stays constant and instead the force depends on $N$.

These are the reasons why this rescaling is typically used when working with mean-field arguments and often called "mean-field scaling" in literature.

We also introduce notation for the trajectories.

We write $\phi^N_{t_1,t_2}(x_i)$ for the time evolution of $x_i$ from $t_1$ to $t_2$. If we simply want to reference the location of $x_i$ in phase-space at a given point of time $t$, we use the short notation $x_i(t) = \phi^N_{0,t}(x_i)$.

The leading order effect of the interaction is in our setting given by the mean-field of the force (see [9]). We, therefore, introduce the mean-field dynamics next. For this we consider the Vlasov equation

$$\frac{d}{\mathrm{d}t} k_t + p \cdot \nabla_q k_t + f^N * \left( \int_{\mathbb{R}^3} k_t(\cdot, p) \, \mathrm{d}^3 p \right) \cdot \nabla_p k_t = 0,$$

which gives rise to the time evolution of the density, namely $k^N_t : \mathbb{R}^6 \to \mathbb{R}$. Note that for $N \in \mathbb{N}$ our force is regularized and hence Lipschitz continuous. The solution theory for this ODE is well known, see e.g. the work of Pfaffelmoser [20] or of Lions and Perthame [18].

However, we are also interested in the solution for $N = \infty$, i.e. the solution for the force without cut-off. Here the theory is more complicated but luckily well understood.

Like Graß, we refer to the result by Horst [10], which provides unique global existence under our conditions on the initial density.

As our force only depends on the spacial density, it is helpful to introduce (for $N \in \mathbb{N}$ or $N = \infty$)

$$\tilde{k}^N_t(q) = \int_{\mathbb{R}^3} k^N_t(q, p) \, \mathrm{d}^3 p.$$

With our spacial density we can now introduce the mean-field trajectories for our particles which are given by the differential equations

$$\begin{aligned} q' &= p, \\ p' &= f * \tilde{k}^N_t(q). \end{aligned}$$

These ODEs provide us the mean-field flow denoted by $\varphi^N_{t_1,t_2}(z)$ for any $z \in \mathbb{R}^6$. As with the Newtonian flow, we introduce a short notation for the position of the mean-field trajectories of our $x_i$. We will simply write $\widetilde{x}_i(t) = \varphi^N_{0,t}(x_i)$.

This means that $\varphi^N_{\cdot,s}(x)$ shall solve the ODEs above with the starting condition $\varphi^N_{s,s}(x) = x$. This implies that $\varphi^N_{t,s}(x) = \varphi^N_{t,r}(\varphi^N_{r,s}(x))$. Moreover, this evolution is by design compatible with the time evolution of the density, i.e.

$$k^N_t(x) = k_0(\varphi^N_{(0,t)}(x)).$$

This equality follows, as for $t = 0$ it is trivial and calculating the derivative in $t$ on both sides yields the same result. This makes the map $\varphi_{t,s} : \mathbb{R}^6 \to \mathbb{R}^6$ a Lebesgue-measure preserving diffeomorphism, which will become useful when we start computing integrals.

We also note that for this to be well defined, we need that in our time frame $[0, T]$ the spacial density stays bounded, i.e.

$$\sup_{N\in\mathbb{N}} \sup_{0\leq s\leq T} |k_t^N|_\infty \leq C(T, k_0).$$

Under our conditions on the $k_0$ this is proven in [10].

## 1.2 Previous results

In this section we give a brief historical overview of results yielding a "derivation" of Vlasov equation under different constraints on the density and interaction force.

The first mathematical derivation of Vlasov equation was possible in systems with Lipschitz continuous forces. Already in 1974 Neunzert and Wick [19] were able to prove the result in the Lipschitz setting, although the proof in the publication by Braun and Hepp [2] or in the book of Spohn [22] may be better known.

They were able to prove that if a particle distribution is close to the density $k_0$, then the particle distribution will stay close to the solution of the Vlasov equation $k_t$ for later times.

It is not surprising that with a Lipschitz force the mean-field approach is very fruitful. The main problem for the mean-field approach is very untypical interactions that still need to be considered, because of their strong impact, for example, collisions.

It took some time until Hauray and Jabin (see [14] and [13]) were able to extend the result to singular interactions. Namely, they were able to consider forces satisfying $|f(q)| \leq \frac{1}{|q|^\alpha}$ and $|\nabla f(q)| \leq \frac{1}{|q|^{\alpha+1}}$ with $\alpha < 1$. Here you have to deal with the fact that in theory two particles can come so close that this "collision" interaction dominates the effect of all the other particles. Two interesting things can be seen here. First, Hauray and Jabin use the second order nature of the differential equation. If two particles come very close, but their relative momentum is large enough, the interaction time will be small and one still obtains a reasonable estimate. Second, the result becomes probabilistic. Particles that have "collisions" with small relative momentum would be a problem, but are very unlikely. Hence, the result they were able to prove takes the form: For almost all initial particle distributions, which approximate the starting density $k_0$, the solution of the Vlasov equation will be a good estimate for the real dynamics.

Another big step was the result by Boers and Pickl [1] who considered forces bounded by $\frac{1}{|q|^\alpha}$ with $\alpha < 2$ with a cut-off at $N^{-\frac{1}{3}}$. The cut-off at $N^{-\frac{1}{3}}$ is notable, as it is the average distance of a particle to its nearest neighbor. Later Lazarovici and Pickl [17] were able to consider the real Coulomb case, however, with a cut-off at $N^{-c}$ with $c < \frac{1}{3}$.

There are many more results that work under different assumptions or in a modified setting.

Li, Pickl, Xin and Qitao were able to obtain similar estimates working in a Vlasov-Maxwell setting with electro-magnetic interaction [4].

Feistl and Pickl were able to consider a $N$-particle system with $\delta$-like interaction. Here they were able to show that typically the Newtonian trajectories converge to the characteristics of the corresponding Vlasov-Dirac-Benney system [6].

Serfaty was able to avoid a cut-off all together in [21] by assuming that the initial system is monokinetic.

Jabin and Wang were able to prove propagation of chaos in [15] using relative entropy.

Kießling considered jellium-like systems and was able to derive the Vlasov equations in this setting [16].

Finally, there also has been the approach of combining the pair interaction with a Brownian motion in the context of the Vlasov-Fokker-Planck equation [11][3].

All these approaches introduce new and interesting ideas, but our work is mostly based on the PHD-thesis by Phillip Graß. In his paper Graß was able to shrink the cut-off in the Coulomb case to $N^{-\frac{7}{18}+\varepsilon}$ [9] which is notably smaller than $N^{-\frac{1}{3}}$. The main new idea was to divide particles into two categories: The "good" particles that keep a reasonable distance in phase-space to all other particles; and the "bad" particles that have a relatively strong interaction with at least one partner.

Our work uses many of the same ideas and straight up borrows some of the results, so we highly recommend reading or at least glancing through his work.

Recently, Feistl was able to strengthen Graß's result in [5] by distinguishing not only good and bad particles but considering three groups of particles. With this she was able to shrink the cut-off further to $N^{-\frac{5}{12}+\sigma}$. As this result was released after most of this thesis was written, we build our arguments on the basis of Graß's paper instead. We are, however, confident that a similar approach could strengthen our result as well.

In the same paper Feistl also was able to use a probabilistic mean-field approach to prove that a small displacement of one particle at time zero entails a small effect for the dynamics of the whole system, i.e. the distance between the true dynamic and the disturbed dynamic is small for later times.

Depending on the readers familiarity with this topic, we recommend the lecture notes of Jabin [12] or the course by Golse [7] on this topic, as they give a insightful introduction to the framework of this thesis.

## 1.3 Motivation and heuristics

While the ideas in this paper are closely related to all the research above, the questions we are interested in are most comparable to Feistl's work at the end of [5].

Namely: What effect has a small disturbance of one particle on the whole system?

Feistl was able to consider forces of the form $f(q) = \frac{q}{|q|^{\lambda+1}}$ with a cut-off at $N^{-\beta}$ with $(\lambda+1)\beta < 1$ and $\lambda \leq 2$. We will limit our study to the case of $\lambda = 2$.

In this setting Feistl's result needs a cut-off that is larger than $N^{-\frac{1}{3}}$ or her estimates become significantly weaker.

She then considers what happens to the system if we move the starting position of one particle $x_0$ by a small $\delta \in \mathbb{R}^6$.

With her cut-off she was able to prove that with high probability the difference between the new trajectory of the disturbed particle and the original trajectory stays of order $\delta$ and that the effect on the other particles is of a smaller order. More precisely, there is some zero sequence $a_N$ such that the disturbance for the other particles is typically bounded by $a_N|\delta|$. In this paper we improve on her results.

We are able to show, that the mean-field trajectories are typically not only a good estimate for the real trajectories but also can be used to estimate the effect of moving the starting position of one particle by a small $\delta \in \mathbb{R}^6$. For this we will later introduce the disturbed system $X^\delta$, where we only changed the starting position of $x_0$ to $x_0 + \delta$.

We are able to show that the difference between the original Newtonian trajectory, i.e. $\phi_t(x_0)$ in $X$, and the trajectory of the disturbed particle, i.e. $\phi_t(x+\delta)$ in $X^\delta$, can

typically be estimated by the corresponding difference in mean-field trajectories, i.e. by $\varphi_t(x) - \varphi_t(x+\delta)$.

This difference of mean-field trajectories has order $\delta$, and we will prove that this gives in fact the leading order of $\delta$ in the difference of the corresponding Newtonian trajectories. Feistl's result proves that $|\phi_t(x) - \phi_t(x+\delta)| \leq C(T)|\delta|$ for all $t \in [0,T]$. We will show that

$$\Big|\varphi_t(x) - \phi_t(x+\delta) - \Big(\varphi_t(x) - \varphi_t(x+\delta)\Big)\Big| \leq N^{-\frac{1}{18}+4\sigma}|\delta|.$$

Because the mean-field trajectories are Lipschitz, this implies $|\varphi_t(x) - \phi_t(x+\delta)| \leq C(T)|\delta|$.

We will also strengthen the result for the remaining particles. Feistl was already able to prove that they are less effected using the zero sequence $a_N$. We are able to make this precise. The trajectories of these other particles typically only deviate by up to $N^{-\frac{1}{18}+2\sigma}$, where $\sigma > 0$ can be chosen arbitrarily small. If a given particle has no collisions and stays relatively far from the disturbed particle $x_0$, this can be improved to $N^{-\frac{1}{3}+2\sigma}$.

A final small improvement is achieved with regards to the cut-off. While the allowed cut-off changes only marginally from $> N^{-\frac{1}{3}}$ to $N^{-\frac{1}{3}}\ln(N)^{-\frac{1}{4}}$, this tiny improvement has some nice consequences. As we have discussed previously, the typical distance of particles in our set-up has order $N^{-\frac{1}{3}}$; so crossing this threshold has merit.

These results come at a cost, however. The result in [5] makes no assumptions over the moved particle. Following the ideas from Graß in [9], we consider two classes of particles. Bad particles which at some point come close to other particles in phase-space, and the other so called good particles. We only consider the case where the disturbed particle is good. Moving the starting position of a bad particle, will heavily effect its collision partners and makes the argument considerably more complicated. We still believe that the approach in this paper can be used as a blue print, and with some effort one should be able to modify the argument to include the case of bad particles. We will come back to this at the end of the paper.

While this restriction might seem as a setback compared to Feistl's result, it actually is the natural next step. In her arguments, she has to assume the worst case for all particles, i.e. all particles being bad, because she did not distinguish between those classes of particles. This, of course, comes at a price regarding the estimates. Introducing the notion of good and bad particles allows us to get a better understanding of the system to obtain stronger estimates and to pinpoint where and why certain results cannot be improved upon further. In this context the restriction that the moved particle has to be good should be understood as the natural first (and somewhat simpler) step in the effort to understand the more general situation without these kind of limitations.

# 2 Definitions and mathematical toolbox

## 2.1 The force and its derivatives

In this chapter we give a short overview over definitions and the most important lemmas used in our proof. We omit all proofs here and refer the interested reader to the appendix for a more detailed version of this chapter where (almost) all the proofs are given. As a consequence of this approach, some results are stated twice: Once here in the overview and a second time in the appendix.

**Definition 2.1.** We define $g^N : \mathbb{R}^3 \to \mathbb{R}^+$ by

$$q \mapsto \begin{cases} C_{Df} c(N)^{-3} & \text{if } |q| \leq 2c(N), \\ C_{Df} |q|^{-3} & \text{else.} \end{cases}$$

This function can be used like the derivative of $f^N$ in many cases as the following lemma shows.

**Lemma 2.2.** *Let $f^N$ be our force from 1.1. The following estimates hold for all $q, q_0, q_1 \in \mathbb{R}^3$:*

$$\left|Df^N(q)\right| \leq g(q)$$
$$\left|f^N(q_1) - f^N(q_0)\right| \leq g^N(\min(q_0, q_1))|q_1 - q_0|$$

*Further, if $\delta \in \mathbb{R}^6$ satisfies $|\delta| \leq c(N)$ or $|\delta| < \frac{1}{2}|q|$, it holds that*

$$\left|f^N(q) - f^N(q+\delta)\right| \leq g^N(q)|\delta|.$$

**Definition 2.3.** For $\mathbf{c} \geq 1$ we define $h^N_{\mathbf{c}} : \mathbb{R}^3 \to \mathbb{R}^+$ by

$$q \mapsto \begin{cases} c_h c(N)^{-4} & \text{if } |q| \leq 2\,\mathbf{c} \cdot c(N), \\ (2\,\mathbf{c})^4 c_h |q|^{-4} & \text{else.} \end{cases}$$

The $\mathbf{c}$ will be chosen later and it will not depend on the number of particles $N$ but only on the time frame $T$ and our density $k$.

The following Lemma shows how $h_{\mathbf{c}}$ can be used and why the constant $\mathbf{c}$ in the cutoff is needed. It allows us to give good estimates as long as the disturbance $\delta$ satisfies $|\delta| \leq \mathbf{c}\, c(N)$.

**Lemma 2.4.** *Let $f^N$ be our force from 1.1. The following estimates hold for all $q \in \mathbb{R}^3$ for fixed $\mathbf{c} \geq 1$.*

$$\left|DDf^N(q)\right| \leq h^N_{\mathbf{c}}(q) \text{ for } |q| \neq c(N)$$
$$\left|Df^N(q_0) - Df^N(q_1)\right| \leq h^N_{\mathbf{c}}(\min(q_0, q_1))|q_0 - q_1|.$$

*Further, if $\delta \in \mathbb{R}^6$ satisfies $|\delta| \leq \mathbf{c}\, c(N)$ or $|\delta| < \frac{1}{2}|q|$, it holds that*

$$\left|Df^N(q) - Df^N(q+\delta)\right| \leq h^N_{\mathbf{c}}(q)|\delta|.$$

The following lemma allows us to estimate the mean-field version of $Df$ and $h$.

**Lemma 2.5.** *Let $q(s)$ be any point in $\mathbb{R}^3$ (here $s \in [0,T]$ is some fixed time). Then the following three inequalities hold:*

$$\int_{\mathbb{R}^3} \left| Df^N(q(s)-y) \right| \tilde{k}_s^N(y)\,\mathrm{d}y \leq C\ln(N), \tag{a}$$

$$\left| \int_{\mathbb{R}^3} Df^N(q(s)-y)\tilde{k}_s(y)\,\mathrm{d}y \right| \leq C_{Df}, \tag{b}$$

$$\int_{\mathbb{R}^3} h_{\mathbf{c}}^N(q(s)-y)\tilde{k}_s(y)\,\mathrm{d}y \leq C_h c(N)^{-1}, \tag{c}$$

*where $C_h$ **does** depend on the constant $\mathbf{c}$ chosen in $h$.*

## 2.2 The notion of good and bad particles

The following result states that the mean-field trajectories behave Lipschitz.

**Lemma 2.6.** *For all $T>0$ there is a constant $C>0$ such that for all $x,y \in \mathbb{R}^6$, $N \in \mathbb{N}$ and $t_0, t_1 \in [0,T]$ it holds that*

$$|\varphi_{t_0,t_1}(x) - \varphi_{t_0,t_1}(y)| \leq |x-y| e^{C|t_1-t_0|}.$$

The next lemma will be used many times, as it is well suited to estimated the integrals we will be dealing with later.

**Lemma 2.7.** *Let $1<\alpha\leq 3$ and consider a function $g^N : \mathbb{R}^3 \to \mathbb{R}^n$ such that*

$$|g^N(q)| \leq \begin{cases} C c_N^{-\alpha} & \text{if} \quad |q| \leq c_N \\ \frac{C}{|q|^\alpha} & \text{if} \quad |q| > c_N \end{cases}.$$

*Then the following estimates hold:*

*a) Let $y,z \in \mathbb{R}^6$, $t_{min}$ be a point in time where*

$$\begin{aligned} &\min |{}^1\widetilde{z}(s) - {}^1\widetilde{y}(s)| = |{}^1\widetilde{z}(t_{min}) - {}^1\widetilde{y}(t_{min})| =: \Delta q > 0 \\ &\text{and} \qquad |{}^2\widetilde{z}(t_{min}) - {}^2\widetilde{y}(t_{min})| =: \Delta p > 0. \end{aligned}$$

*Then there exists a constant $C>0$ (independent of $y,z$ and $N$) such that*

$$\int_0^T \left| g\left( {}^1\widetilde{z}(s) - {}^1\widetilde{y}(s) \right) \right| \mathrm{d}s \leq C \min\left( \frac{1}{\Delta q^\alpha}, \frac{1}{c_N^{\alpha-1}\Delta p}, \frac{1}{\Delta q^{\alpha-1}\Delta p} \right).$$

*b) Let $T>0, z_1, z_2 \in \mathbb{R}^6$ be given and consider differentiable ${}^1y_1, {}^1y_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$. We denote the derivatives by ${}^2y_1, {}^2y_2$ and write $y_k = ({}^1y_k, {}^2y_k)$. Assume that for some $\sigma > 0$*

$$N^\sigma |{}^1\widetilde{z_1}(t_{min}) - {}^1\widetilde{z_2}(t_{min})| \leq |{}^2\widetilde{z_1}(t_{min}) - {}^2\widetilde{z_2}(t_{min})| = \Delta p \tag{A1}$$

*and*

$$\sup_{0\leq s\leq T} |\widetilde{z_1}(s) - y_1(s)| \leq N^{-\sigma}\Delta p \qquad \wedge \qquad \sup_{0\leq s\leq T} |\widetilde{z_2}(s) - y_2(s)| \leq N^{-\sigma}\Delta p. \tag{A2}$$

*Then there exists some $N_0$ such that for all $N \geq N_0$ we have*

$$\int_0^T \left| g\left( {}^1y_1(s) - {}^1y_2(s) \right) \right| \mathrm{d}s \leq C \min\left( \frac{1}{c_N^{\alpha-1}\Delta p}, \frac{1}{\min_{0\leq s\leq T} |y_1(s)-y_2(s)|^{\alpha-1}\Delta p} \right).$$

We now give a rigorous definition of good and bad particles.

**Definition 2.8.** Let $R, V > 0$. We say a particle $z \in \mathbb{R}^6$ has a "bad" collision with a particle $y \in \mathbb{R}^6$ in the time frame $[t_0, t_1]$ if

$$\exists t \in [t_0, t_1] : |^1\widetilde{z}(t) - {}^1\widetilde{y}(t)| \leq R \quad \wedge \quad |^2\widetilde{z}(t) - {}^2\widetilde{y}(t)| \leq V.$$

Most often we will work on the time interval $[0, T]$, and in this case we will often omit the interval. If the two particles do not have a bad collision, we write $z \in G^N_{R,V}(y)$. We call a particle $x_i$ good if all other $x_j \in G^N_{R,V}(x_i)$. Otherwise it is called bad. For a configuration $X \in \mathbb{R}^{6N}$ the set of good and bad particles is denoted by $\mathcal{M}^N_g(X)$ and $\mathcal{M}^N_b(X)$, respectively.

For the vast majority of this paper we will work in the setting $R = 6N^{-\frac{2}{9}-\sigma}, V = N^{-\frac{2}{9}}$. Whenever we talk about good and bad particles, it is meant with respect to this choice of $R$ and $V$, unless we specify it to be something else.

We also define the related sets $M^{N,(t_1,t_2)}_{(r,R)(v,V)}(y) \subseteq \mathbb{R}^6$, where $0 \leq r < R, 0 \leq v < V$ by

$$z \in M^{N,(t_1,t_2)}_{(r,R)(v,V)}(y) \subseteq \mathbb{R}^6 \iff \exists t \in [t_1, t_2] :$$
$$r \leq \min_{s \in [t_0,t_1]} |^1\widetilde{z}(s) - {}^1\widetilde{y}(s)| = |^1\widetilde{z}(t) - {}^1\widetilde{y}(t)| \leq R \quad \wedge \quad v \leq |^2\widetilde{z}(t) - {}^2\widetilde{y}(t)| \leq V.$$

We will also write

$$M^N_{R,V}(y) = M^N_{(0,R),(0,V)}(y).$$

These sets have a focus on the physical distance, since the distance in space dictates the points we are interested in. Only for the points of minimal spacial distance we even consider the relative momentum.

**Remark 2.9.** Both sets, i.e. $G^N_{R,V}(y)$ and $M^N_{R,V}(y)$, measure how close particles come to each other or, more precisely, whether they satisfy a certain condition on their distance. One might be curious why we need both definitions. The first definition is more simple and therefore arguably more natural. However, in some cases we want lower bounds on the relative position and momentum. It is hard to introduce such bounds in a meaningful way without changing the definition fundamentally. This is where the second definition is helpful, as the introduction of the minimal relative position allows us to introduce the lower bound in a sensible way.

This definition, however, becomes less stable, as the point of minimal distance to another particle might jump drastically even if we only move one particle a tiny bit.

So, depending on the circumstances, both notions of distance are useful.

The sets $M^{N,(t_1,t_2)}_{(r,R)(v,V)}$ will mainly be used to define the following partition of $\mathbb{R}^{6N}$.

**Definition 2.10.** For each $r_0, v_0, \varepsilon > 0$ and $y \in \mathbb{R}^6$ we introduce a cover of $\mathbb{R}^6$:

(i) $M^N_{(0,r_0),(0,v_0)}(y)$

(ii) $M^N_{(0,r_0),(N^{l\varepsilon}v_0,N^{(l+1)\varepsilon}v_0)}(y), \quad 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\varepsilon \ln(N)} \right\rfloor$

(iii) $M^N_{(0,r_0),(1,\infty)}(y)$

(iv) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(0,v_0)}(y), \quad 0 \le k \le \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor$

(v) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(N^{l\varepsilon}v_0,N^{(l+1)\varepsilon}v_0)}(y), \quad 0 \le k \le \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor, \quad 0 \le l \le \left\lfloor \frac{\ln(\frac{1}{v_0})}{\varepsilon \ln(N)} \right\rfloor$

(vi) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(1,\infty)}(y), \quad 0 \le k \le \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor$

(vii) $M^N_{(N^{-\varepsilon},\infty),(0,\infty)}(y)$

Here $\lfloor \cdot \rfloor$ is the floor operator.

Note that this is indeed a – not necessarily disjoint – cover of $\mathbb{R}^6$ as seen in 2.1.

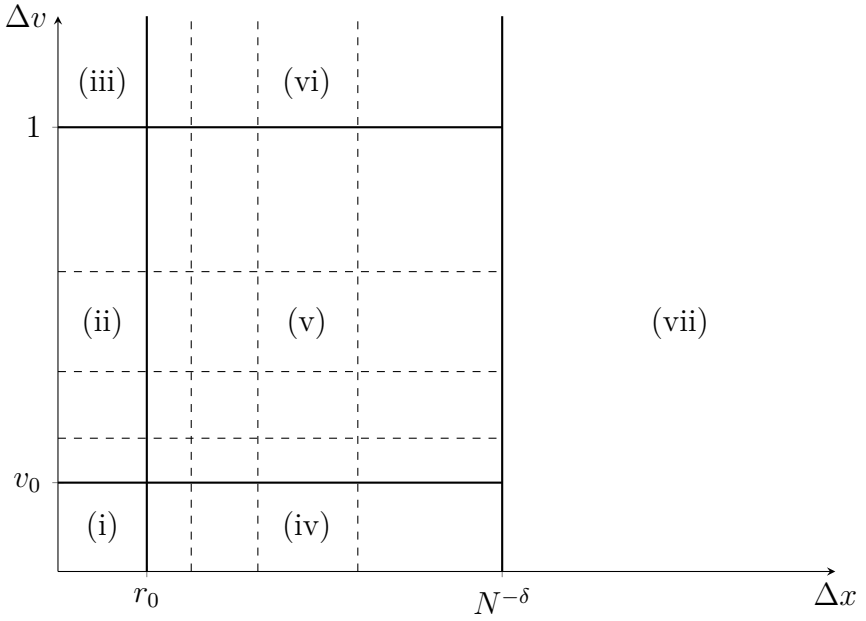


Figure 2.1: Collision classes

Here we see the seven classes depending on the relative position of a particle to $y$. The dotted lines are used to indicate the subdivision we use in (ii) and (iv)-(vi). For smaller $\varepsilon$ this subdivision becomes finer and can be considered as a discrete density.

The number of pieces used in the cover grows, as $r_0, v_0$ and $\varepsilon$ get smaller. In our applications $r_0$ and $v_0$ will depend on $N$. Note that this is okay as long as they are bounded from below by some power of $N$, i.e. $r_0, v_0 \ge N^{-\alpha}$ with $\alpha > 0$. One checks that, if $\varepsilon$ is independent of $N$, the number of $k$ and $l$ only depends on the $\alpha$ in the bound.

The following lemma gives some nice probabilistic estimates.

**Lemma 2.11.** *There exists a constant $C > 0$ such that for all $\Delta q, \Delta p > 0$, $N \in \mathbb{N}$, $y \in \mathbb{R}^6$ and $[t_0, t_1] \subseteq [0, T]$ it holds that*

*i)* $\mathbb{P}\left(x \in \mathbb{R}^6 : x \in \left(G^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y)\right)^C\right) \le C(\Delta q^2 \Delta p^4 \Delta t + \Delta q^3 \max(\Delta q, \Delta p)^3)$

*ii)* $\mathbb{P}\left(x \in \mathbb{R}^6 : x \in M^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y)\right) \le C(\Delta q^2 \Delta p^4 \Delta t + \Delta q^3 \max(\Delta q, \Delta p)^3)$

*iii)* $\mathbb{P}\left(x \in \mathbb{R}^6 : \min_{0 \le s \le T} |{}^1\widetilde{x}(s) - {}^1\widetilde{y}(s)| \le \Delta q\right) \le C\Delta q^2.$

These estimates allow us to obtain bounds on the number of bad particles. The estimates are taken with regard to $X = (x_1, \dots, x_N) \in \mathbb{R}^{6N}$.

**Corollary 2.12.** *Let $0 < r \leq R, 0 < v \leq V$ and $y \in \mathbb{R}^6$. Then it holds for every $\gamma > 0$ that*

$$\mathbb{P}\Bigg( \sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) > \ln(N)^{\frac{1}{4}} \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\ \vee \, |\, \mathcal{M}^N_b(X)| > \ln(N)^{\frac{1}{4}} N^{\frac{2}{3}} \Bigg) \leq C_\gamma N^{-\gamma}.$$

This, in turn, gives us estimates when considering the effect of bad particles.

**Lemma 2.13.** *Let $0 < r \leq R, 0 < v \leq V$ and $y \in \mathbb{R}^6$. Let $X \in \mathbb{R}^{6N}$ be a configuration of $N$ particles $x_i$ such that*

$$\sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) \leq \ln(N)^{\frac{1}{4}} \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \wedge |\, \mathcal{M}^N_b(X)| \leq \ln(N)^{\frac{1}{4}} N^{\frac{2}{3}}.$$

*Further, let $\omega^N : \mathbb{R}^3 \to \mathbb{R}^n$ be a family of functions such that*

$$|\omega^N(q)| \leq \begin{cases} Cc(N)^{-\alpha} & \text{if} \quad |q| \leq c(N) \\ \frac{C}{|q|^\alpha} & \text{if} \quad |q| > c(N) \end{cases}.$$

*Then*

$$\left| \frac{1}{N} \sum_{j \text{ bad}} \int_{t_0}^{t_1} \omega^N(\tilde{q}_i - \tilde{q}_j) \; \mathbf{1}_{M_{(6N^{-\frac{2}{9}-\sigma},\infty)(0,\infty)}(x_i)}(x_j) \, \mathrm{d}t \right| \leq \begin{cases} CN^{-\frac{1}{3}+3\sigma} & \text{if } \alpha = 2 \\ CN^{-\frac{1}{3}+4\sigma} & \text{if } \alpha = 3 \\ CN^{-\frac{1}{9}+5\sigma} & \text{if } \alpha = 4 \end{cases}$$

*and*

$$\left| \frac{1}{N} \sum_{j \text{ bad}} \int_{t_0}^{t_1} \omega^N(\tilde{q}_i - \tilde{q}_j) \mathbf{1}_{G^N(x_i)}(x_j) \, \mathrm{d}t \right| \leq \begin{cases} CN^{-\frac{1}{3}+4\sigma} & \text{if } \alpha = 2 \\ CN^{-\frac{1}{9}+5\sigma} & \text{if } \alpha = 3 \\ CN^{\frac{2}{9}+6\sigma} & \text{if } \alpha = 4 \end{cases}.$$

When estimating the effect of good particles, we apply a version of the law of large numbers to obtain:

**Corollary 2.14.** *Let $\gamma > 0$ and $y \in \mathbb{R}^6$. There exists a $C_\gamma > 0$ (not depending on $y$) such that for large enough $N$ the following estimates hold*

*a)*

$$\mathbb{P}\Bigg( \exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j=1}^{N} \int_{t_1}^{t_2} f^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j) \, \mathrm{d}t \\ - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f^N({}^1\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z) \, \mathrm{d}t \cdot k(z) \, \mathrm{d}z \Bigg| > N^{-\frac{4}{9}+\sigma} \Bigg) \leq C_\gamma N^{-\gamma}$$

*b)*

$$\mathbb{P}\left(\exists t_1,t_2\in[0,T]:\left|\frac{1}{N}\sum_{j=1}^{N}\int_{t_1}^{t_2}Df^N({}^1\widetilde{y}-\tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t\right.\right.$$
$$\left.\left.-\int_{\mathbb{R}^6}\int_{t_1}^{t_2}Df^N({}^1\widetilde{y}-{}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\cdot k(z)\,\mathrm{d}z\right|>N^{-\frac{1}{9}+\sigma}\right)\leq C_\gamma N^{-\gamma}$$

*c)*

$$\mathbb{P}\left(\exists t_1,t_2\in[0,T]:\left|\frac{1}{N}\sum_{j=1}^{N}\int_{t_1}^{t_2}g^N({}^1\widetilde{y}-\tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t\right.\right.$$
$$\left.\left.-\int_{\mathbb{R}^6}\int_{t_1}^{t_2}g^N({}^1\widetilde{y}-{}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\cdot k(z)\,\mathrm{d}z\right|>N^{-\frac{1}{9}+\sigma}\right)\leq C_\gamma N^{-\gamma}$$

*d)*

$$\mathbb{P}\left(\exists t_1,t_2\in[0,T]:\left|\frac{1}{N}\sum_{j=1}^{N}\int_{t_1}^{t_2}h^N_{\mathbf{c}}({}^1\widetilde{y}-\tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t\right.\right.$$
$$\left.\left.-\int_{\mathbb{R}^6}\int_{t_1}^{t_2}h^N_{\mathbf{c}}({}^1\widetilde{y}-{}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\cdot k(z)\,\mathrm{d}z\right|>N^{-\frac{1}{9}+\sigma}c(N)^{-1}\right)\leq C_\gamma N^{-\gamma}$$

**Remark 2.15.** Because of the special role of the moved particle we will sometimes consider $N-1$ particles instead of $N$. We often use this lemma in combination with 2.5 to estimate the sums over $f, g$ and $h$.

We also note that in the appendix we have a version that yields better estimates, as long as we restrict to nicer interactions, i.e. instead of $G^N$ we restrict to $\left(M^N_{N^{-\frac{1}{6}+\sigma},N^{-\frac{1}{6}+\sigma}}\right)^C$.

We will only need this version in one case, namely particles that stay reasonably far from $x_0$. In the next section we will introduce rings that allow us to characterize how far from $x_0$ a particle is.

## 2.3 Categorizing the distance to $\mathbf{x_0}$ using rings

It is not surprising that we can strengthen our result by considering the distance of particles from the disturbed particle $x_0$. For this, we will now define hypercylinders in phase-space around $x_0$. The degree to which particles feel the disturbance will be smaller the further out they are. We define $K=\frac{2}{9\sigma}-1$. Without loss of generality we assume that $K$ is integer, i.e. $\sigma=\frac{1}{18}\cdot\frac{1}{m}$ for some natural number $m$. The general case follows from monotonicity of our estimates. One also could use rounding functions instead, but restricting $\sigma$ in this way, makes the estimates cleaner and less convoluted.

For $k=-1,0,\dots K$ we consider

$$B^k=\{x\in\mathbb{R}^6\mid\exists t|{}^1\widetilde{x}-{}^1\widetilde{x}_0|\leq N^{-\frac{2}{9}+k\sigma}\text{ and }|{}^2\widetilde{x}-{}^2\widetilde{x}_0|\leq N^{-\frac{2}{9}+k\sigma}\}.$$

Note that $B^{-1}$ is actually empty, as $x_0$ is a good particle by assumption. It still is a useful notion, as our basic idea is to define the hypercylinder $R^k$ as $B^k\setminus B^{k-1}$. However, we need to be careful, as bad interactions can carry the error over the borders of the rings. For this reason we define:

**Definition 2.16.** Let $k \in \{0, 1, \dots, K\}$.

$$x_j \in R^k \iff k = \min \Big\{ m \in \mathbb{N}_0 \,|\exists n \in \mathbb{N}_0 \, \exists x_{j_0}, \dots x_{j_n}$$
$$\text{with } x_{j_i} \in G(x_{j_{i+1}})^C \text{ and } x_{j_0} = x_j \text{ and } x_{j_n} \in B^m \setminus B^{m-1} \Big\}.$$

Here the use of the letter $R$ stems from the fact, that we will simply call these hyper-cylinders rings from now on. While this might less precise mathematically, we believe that the image of rings is actually better suited for our later arguments.

In our definition a particle is considered to be in $R^k$ if either itself is in the ring or if there is a chain of bad interactions connecting the particle to the ring. This can be thought of as an adjustment of the naive definition of rings, as can be seen in Figure 2.2.

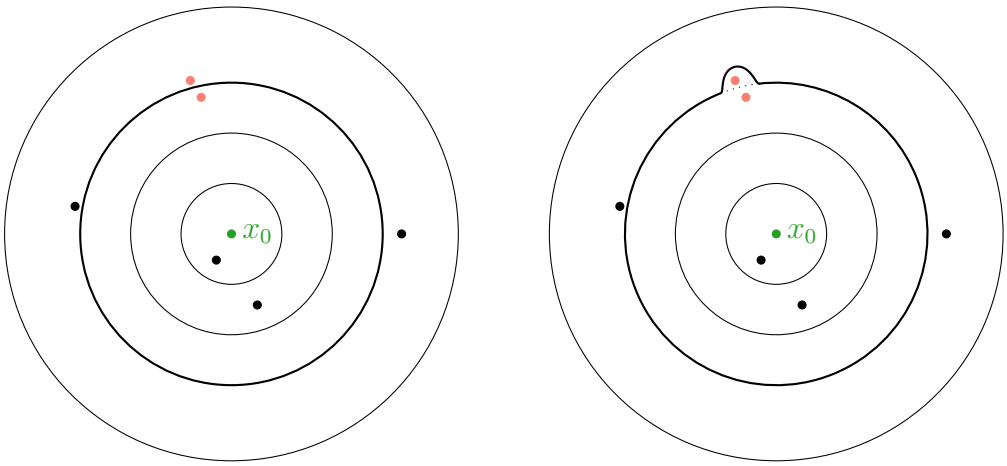


Figure 2.2: Adjusting rings to account for bad particles

Since typically there are only very few bad particles and bad particles stay very close to each other, this is more of a nuisance than a real problem. Also note that by 2.6 we will see that particles whose mean-field trajectory at some point comes close to $\tilde{x}_0$ (as in our definition) stay close to $\tilde{x}_0$ for the whole duration.

We will formulate and prove both these statements rigorously in the beginning of chapter 4 in the two lemmas 4.5 and 4.6.

We also define $R^{K+1} := R^\infty := \mathbb{R}^6 \setminus \bigcup R^k$. Note that for $y \in R^{K+1}$ we have that $|\tilde{y} - \tilde{x_0}| \geq N^{-\sigma}$. In many ways $R^{K+1}$ behaves exactly as one would expect. One has to be a little careful, however, as for all other $k$ we have an often useful bound on the volume of the ring, which is not the case for $R^\infty$.

Later we will use the notion of particles near $x_0$. These are particles that lie in $R^k$ with $k \leq \frac{1}{18\sigma} + 2$. The other particles are called far from $x_0$. This definition will be recalled whenever we use it and is only stated here for completion.

The notion of rings allows us to differentiate our estimates depending on the distance to the moved particle $x_0$. In our main theorem 4.2 the estimate for the good particles closest to $x_0$ is of order $N^{-\frac{1}{9}+3\sigma}$, while the estimate for those particles that are reasonably far from $x_0$ is of order $N^{-\frac{1}{3}+2\sigma}$ instead.

As most particles will have distance $> N^{-\sigma}$ to $x_0$ as $N$ goes to infinity, these stronger estimates can be applied to almost all particles for large enough $N$.

# 3 A derivation of Vlasov(-Poisson) equation as the mean-field limit of particle systems with regularized interaction

In this chapter we prove a version of Graß's result in [9] that is better suited for our needs.

Graß was able to prove that for a typical system the distance between a particle and its mean-field trajectory stays closer than $N^{-\frac{2}{9}}$.

If one takes a closer look at his paper, it becomes clear that he actually has proven the following stronger result. For good particles the distance typically stays closer than $CN^{-\frac{7}{18}+\frac{\sigma}{2}}$, and for bad particles it stays closer than $CN^{-\frac{2}{9}-\frac{3}{2}\sigma}$. This can be seen on page 51 and page 61 of [9], respectively.

In Graß's paper there is no special particle. To make his result more suitable for our setting, it is helpful to use a slightly different version of his result. It states that for a typical configuration $X^{\geq 1}$ the Newtonian trajectories $x_i(s)$ and the mean-field trajectories $\tilde{x}_i(s)$ in the system $X = X^{\geq 1} \cup \{x_0\}$ stay close to each other.

**Theorem 3.1.** *Let $\sigma > 0$ and let $f^N : \mathbb{R}^3 \to \mathbb{R}^3$ be the force from our set up and $k$ be a density satisfying the conditions we introduced in section 1.1. Let $T > 0$ and $\gamma > 0$. We find a constant $C_\gamma$ such that for any $x_0 \in \mathbb{R}^6$ we have*

$$\mathbb{P}\left( \sup_{x_i \in X \text{ good}} \sup_{0 \leq s \leq T} |x_i(s) - \tilde{x}_i(s)| > N^{-\frac{4}{9}+3\sigma} \;\middle|\; x_0 \text{ is good particle in } X^{\geq 1} \cup \{x_0\} \right) < C_\gamma N^{-\gamma},$$

$$\mathbb{P}\left( \sup_{x_i \in X \text{ bad}} \sup_{0 \leq s \leq T} |x_i(s) - \tilde{x}_i(s)| > N^{-\frac{1}{3}+\sigma} \;\middle|\; x_0 \text{ is good particle in } X^{\geq 1} \cup \{x_0\} \right) < C_\gamma N^{-\gamma}.$$

*Here all probabilities are taken with respect to $X^{\geq 1} \in \mathbb{R}^{6(N-1)}$. Importantly $x_0$, unlike all the other $x_i$, is not a random variable. It is, however, one of the good particles considered in the first estimate.*

Our version differs from Graß's theorem in two points.

First, we have a larger cut-off and can leverage this for better estimates. Second, Graß considers the whole system including $x_0$ to be random variables, while we emphasis the special role of $x_0$ and rigorously show that it is no problem for our probabilistic arguments if one particle is chosen instead of randomly distributed.

The idea is to control the distance between mean-field and Newtonian trajectories using

the following three values:

$$\begin{aligned}\epsilon_0(t) &= \sup_{0\le s\le t} |x_0(s) - \tilde{x}_0(s)| \\ \epsilon_g(t) &= \sup_{0\le s\le t} \sup_{i\neq 0 \text{ good}} |x_i(s) - \tilde{x}_i(s)| \\ \epsilon_b(t) &= \sup_{0\le s\le t} \sup_{i \text{ bad}} |x_i(s) - \tilde{x}_i(s)|.\end{aligned}$$

By abuse of notation we will also use $\epsilon_0(t) = x_0 - \tilde{x}_0(t)$.

For this we introduce the following stopping time $\tau$:

$$\tau = \sup\{t \in [0,T] \mid \epsilon_0(t) \le N^{-\frac{4}{9}+3\sigma}, \epsilon_g(t) \le N^{-\frac{4}{9}+\frac{5\sigma}{2}}, \epsilon_b(t) \le N^{-\frac{1}{3}+\sigma}\}.$$

We now want to show that the stopping time typically does not trigger, i.e. that $\tau = T$. For this we consider $t_0 \le t_1 \le \tau$ and prove that, if the estimates held so far, they still hold at $t_1$. This then implies that the stopping time (typically) never triggers.

We have three different kind of particles. The new particle $x_0$, the other good particles and finally the bad particles. While the arguments are similar in all cases, each type of particle has different problems to take care of. For example, when estimating bad particles, one has to take special care when estimating the effect of their bad interactions (i.e. the interactions with the particles that make the particle a bad particle).

For this reason we now consider the three cases one by one.

## 3.1 First case: $\mathbf{x_i}$ is the newly added particle $\mathbf{x_0}$

The central idea is always to use a Grönwall style argument, and hence we are interested in the force difference between $x_0$ and $\tilde{x}_0$. We estimate

$$\begin{aligned}&\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0} f^N(q_0 - q_j)\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{\mathbb{R}^6} f^N(\tilde{q}_0 - \tilde{y}) k_t^N(\tilde{y})\,\mathrm{d}y\,\mathrm{d}t \\ \le &\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0} \left(f^N(q_0 - q_j) - f^N(\tilde{q}_0 - \tilde{q}_j)\right)\mathrm{d}t\right| \qquad (1a) \\ &+ \left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0} f^N(\tilde{q}_0 - \tilde{q}_j)\,\mathrm{d}t - \int_{t_0}^{t_1} f^N(\tilde{q}_0 - \tilde{y}) k_t^N(\tilde{y}) \mathbf{1}_{G^N(x_0)}(y)\,\mathrm{d}t\right| \qquad (1b) \\ &+ \left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6} f^N(\tilde{q}_0 - \tilde{y}) k^N(\tilde{y}) \mathbf{1}_{G^N(x_0)}(y)\,\mathrm{d}y\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{\mathbb{R}^6} f^N(\tilde{q}_0 - \tilde{y}) k_t^N(\tilde{y})\,\mathrm{d}y\,\mathrm{d}t\right|. \qquad (1c)\end{aligned}$$

Est. 1a)

$$\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0} \left(f^N(q_0 - q_j) - f^N(\tilde{q}_0 - \tilde{q}_j)\right)\mathrm{d}t\right|$$

For this we consider the effect of the good and bad interacting particles separately. First, we consider the good $x_j$.

**Good prts.** For the good $x_j$ we can introduce $g^N$ by 2.2, because we know that $t_1 \le \tau$ and hence

$$|q_0 - q_j - (\tilde{q}_0 - \tilde{q}_j)| \le \epsilon_0 + \epsilon_g \le N^{-\frac{4}{9}+3\sigma} + N^{-\frac{4}{9}+\frac{5}{2}\sigma} \le c(N).$$

It then remains to estimate

$$\int_{t_0}^{t_1} \frac{1}{N} \sum_{x_j \text{ good}} g^N(\tilde{q}_0 - \tilde{q}_j)(N^{-\frac{4}{9}+\frac{5}{2}\sigma} + \epsilon_0)\,\mathrm{d}t.$$

To estimate this, we want to use our law of large numbers. For this we need to remove a set of bad starting configurations

$$\begin{aligned} &X^{\geq 1} \in \mathcal{A}_{1,a} \subset \mathbb{R}^{6(N-1)} \qquad (\mathcal{A}_1)\\ &\quad \iff \exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j\neq 0} \int_{t_0}^{t_1} |g^N(\tilde{q}_0 - \tilde{q}_j)| \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}t \\ &- \int_{\mathbb{R}^6} \int_{t_0}^{t_1} |g^N(\tilde{q}_0 - {}^1\tilde{y})| \mathbf{1}_{G^N(x_0)}(y)\,\mathrm{d}t\; k_0^N(y)\,\mathrm{d}y \Bigg| > 4N^{-\frac{1}{9}+\sigma} \end{aligned}$$

and

$$\begin{aligned} &X^{\geq 1} \in \mathcal{A}_{1,b} \subset \mathbb{R}^{6(N-1)}\\ &\quad \iff \exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j\neq 0} \int_{t_0}^{t_1} |f^N(\tilde{q}_0 - \tilde{q}_j)| \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}t \\ &- \int_{\mathbb{R}^6} \int_{t_0}^{t_1} |f^N(\tilde{q}_0 - {}^1\tilde{y})| \mathbf{1}_{G^N(x_0)}(y)\,\mathrm{d}t\; k_0^N(y)\,\mathrm{d}y \Bigg| > N^{-\frac{4}{9}+\sigma}. \end{aligned}$$

Defining $\mathcal{A}_1$ as the union of these sets, we can apply 2.14 to estimate $\mathbb{P}(X^{\geq 1} \in \mathcal{A}_1) \leq C_\gamma N^{-\gamma}$. Note that this is a property of $X^{\geq 1}$ and that the constant does not depend on $x_0$. From now on we work on the complement of $\mathcal{A}_1$, and therefore we can use 6.24 with $\beta_N = N^{-\frac{4}{9}+\frac{5}{2}\sigma} + \epsilon_0$ to estimate

$$\begin{aligned} &\int_{t_0}^{t_1} \frac{1}{N} \sum_{x_j \text{ good}} g^N(\tilde{q}_0 - \tilde{q}_j)(N^{-\frac{4}{9}+\frac{5}{2}\sigma} + \epsilon_0)\,\mathrm{d}t\\ \leq & CN^{-\frac{1}{9}+\sigma}(N^{-\frac{4}{9}+\frac{5}{2}\sigma} + \epsilon_0) + \int_{t_0}^{t_1} C\ln(N)(N^{-\frac{4}{9}+\frac{5}{2}\sigma} + \epsilon_0)\,\mathrm{d}t\\ \leq & CN^{-\frac{4}{9}+\frac{8}{3}\sigma} + \int_{t_0}^{t_1} C\ln(N)\epsilon_0\,\mathrm{d}t. \end{aligned}$$

**Bad prts.** For the effect from the bad particles $x_j$ we have to put in a bit more effort. We want to estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \left( f^N(q_0 - q_j) - f^N(\tilde{q}_0 - \tilde{q}_j) \right) \mathrm{d}t \right|.$$

For this we split the sum further using 2.10 with $y = x_0$ and $r_0 = 6N^{-\frac{2}{9}-\sigma}$, $v_0 = N^{-\frac{2}{9}}$ and $\varepsilon = \sigma$.

We start with the classes where the minimal distance between $\tilde{q}_0$ and $\tilde{q}_j$ is larger than $6N^{-\frac{2}{9}-\sigma}$.

Using that $t_1 \leq \tau$, we can estimate

$$|q_0 - q_j - (\tilde{q}_0 - \tilde{q}_j)| \leq |q_0 - \tilde{q}_0| + |q_j - \tilde{q}_j| \leq 3N^{-\frac{2}{9}-\sigma}.$$

Because $|\tilde{q}_0 - \tilde{q}_j| \geq 6N^{-\frac{2}{9}-\sigma}$, we can introduce $g$ by 2.2, and it remains to estimate

$$CN^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0 \text{ bad}} g(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_0)}(x_j) \, \mathrm{d}t.$$

We want to use Lemma 2.13. Therefore, we have to assume that there are few bad particles. To this end we use Lemma 2.12, which states that the probability for a large number of bad particles is in fact small. Note that this is a probabilistic result, so we have to remove a set of bad configurations again.

$$\begin{aligned} X^{\geq 1} \in \mathcal{A}_2 \subseteq \mathbb{R}^{6(N-1)} \qquad & (\mathcal{A}_2) \\ \iff \Bigg( & \sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(x_0)}(x_j) > \ln(N)^{\frac{1}{4}} \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\ & \vee |\, \mathcal{M}_b^N(X)| > \ln(N)^{\frac{1}{4}} N^{\frac{2}{3}} \Bigg). \end{aligned}$$

Using 2.12, we can estimate $\mathbb{P}(X^{\geq 1} \in \mathcal{A}_2) \leq C_\gamma N^{-\gamma}$.

One has to be a little careful here. The definition of being bad does depend on $N$ in two ways. First, for larger $N$, particles need to come closer to be considered bad. Second, the mean-field trajectory does depend on $N$. Importantly, both these things do not depend on $x_0$ and, because we assumed that $x_0$ would be a good particle in the new setting, $x_0$ does not corrupt any particle. There is still one small inconvenience, namely that 2.12 needs $N$ i.i.d. particles. This, however, can be solved by adding an imaginary $N$th i.i.d. particle. This, of course, could only increase the number of bad particles.

We already take the union over all the $r, R, v, V$ that come up in the partition here, as we will need it later. Now we can apply 2.13 to estimate

$$\begin{aligned} & CN^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0 \text{ bad}} g^N(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_0)}(x_j) \, \mathrm{d}t \\ \leq & N^{-\frac{2}{9}-\sigma} \cdot CN^{-\frac{1}{3}+3\sigma} \leq \underline{CN^{-\frac{5}{9}+2\sigma}}. \end{aligned}$$

We are left with the particles that come closer than $6N^{-\frac{2}{9}-\sigma}$, which are divided into the following collision classes (see 2.10) with $r_0 = 6N^{-\frac{2}{9}-\sigma}, v_0 = N^{-\frac{2}{9}}$:

1. $M^N_{(0,r_0),(0,v_0)}(x_0)$
2. $M^N_{(0,r_0),(N^{l\delta}v_0,N^{(l+1)\delta}v_0)}(x_0), 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\delta \ln(N)} \right\rfloor$
3. $M^N_{(0,r_0),(1,\infty)}(x_0)$

Here we cannot introduce $g$. However, because $x_0$ is a good particle, the relative momentum at their closest point has to be larger than $N^{-\frac{2}{9}}$. This means that the first class is in fact empty (and hence trivial) in this case.

In the following we will simply write $x_j \in M(x_0)$ as a placeholder to later plug in one of the remaining classes. When we use $v, V$ or $R$, it is always meant with respect to the class, i.e. we always have $R = 6N^{-\frac{2}{9}-\sigma}$ and, for example, in the third class $v = 1$ and $V = \infty$.

With this being said, we estimate $f^N$ by 2.7 to obtain

$$\begin{aligned}
&C\int_{t_0}^{t_1}\frac{1}{N}\sum_{\substack{j\neq 0 \text{ bad}\\ x_j\in M(x_0)}}\left|f^N(q_0-q_j)-f^N(\tilde{q}_0-\tilde{q}_j)\right|\mathrm{d}t\\
\leq& C\int_{t_0}^{t_1}\frac{1}{N}\sum_{\substack{j\neq 0 \text{ bad}\\ x_j\in M(x_0)}}\left|f^N(q_0-q_j)\right|+\left|f^N(\tilde{q}_0-\tilde{q}_j)\right|\mathrm{d}t\\
\leq& C\frac{1}{c(N)v}\frac{1}{N}\sum_{\substack{j\neq 0 \text{ bad}\\ x_j\in M(x_0)}}1.
\end{aligned}$$

Note that for the estimate of $|f^N(q_0-q_j)|$ we need the second part of 2.7. We apply this with $y_1=x_0, y_2=x_j$ and $z_1=x_0, z_2=x_j$. To verify the conditions, we use $|{}^1\widetilde{x}_0(t_{\min})-{}^1\widetilde{x}_j(t_{\min})|\leq 6N^{-\frac{2}{9}-\sigma}$ and $|{}^2\widetilde{x}_0(t_{\min})-{}^2\widetilde{x}_j(t_{\min})|\geq N^{-\frac{2}{9}}$. This yields

$$N^{\frac{\sigma}{2}}|{}^1\widetilde{x}_0(t_{\min})-{}^1\widetilde{x}_j(t_{\min})|\leq N^{-\frac{2}{9}}\leq|{}^2\widetilde{x}_0(t_{\min})-{}^2\widetilde{x}_j(t_{\min})|=\Delta p$$

as well as

$$\sup|\widetilde{x}_0(t)-x_0(t)|\leq N^{-\frac{2}{9}-\sigma}\leq N^{-\frac{\sigma}{2}}\Delta p$$

and

$$\sup|\widetilde{x}_j(t)-x_j(t)|\leq N^{-\frac{2}{9}-\sigma}\leq N^{-\frac{\sigma}{2}}\Delta p.$$

Moving on with our estimate, we use $R=6N^{-\frac{2}{9}-\sigma}$, $v\geq N^{-\frac{2}{9}}$ as well as $c(N)\geq N^{-\frac{1}{3}-\frac{\sigma}{2}}$. We also need that the number of bad particles can be estimated by $a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil$ as we work on the complement of $\mathcal{A}_2$.

$$\begin{aligned}
&C\frac{1}{c(N)v}\frac{1}{N}a_n\sum_{\substack{j\neq 0 \text{ bad}\\ x_j\in M(x_0)}}1\\
\leq& C\frac{1}{c(N)v}\frac{1}{N}a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil\\
\leq& C\frac{1}{c(N)v}\frac{1}{N}a_N N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4+C\frac{1}{c(N)^2v}\frac{1}{N}a_N\\
\leq& CN^{-\frac{4}{9}+\sigma}\frac{\min(V,1)}{v}+CN^{-\frac{4}{9}+\sigma}\leq \underline{CN^{-\frac{4}{9}+2\sigma}}.
\end{aligned}$$

Here we used that in the remaining classes either $v=1$ or $V=N^\sigma v$.

Combining these estimates, we are done with the first summand and have proven:

$$\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0}f(q_0-q_j)-f^N(\tilde{q}_0-\tilde{q}_j)\right|\leq \underline{\underline{N^{-\frac{4}{9}+\frac{8}{3}\sigma}+\int_{t_0}^{t_1}\ln(N)\cdot{}^1\epsilon_0}}.$$

Est. 1b) Next, we consider the second summand, i.e.

$$\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0}f^N(\tilde{q}_0-\tilde{q}_j)-\int_{t_1}^{t_2}\int_{\mathbb{R}^6}f^N(\tilde{q}_0-{}^1\tilde{y})k(\tilde{y})\mathbf{1}_{G^N(x_0)}(y)\,\mathrm{d}y\right|.$$

This is a simple application of our law of large numbers, i.e. 2.14. We already took care of the associated bad configurations in $\mathcal{A}_1$ and obtain

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0} f^N(\tilde{q}_0 - \tilde{q}_j) - \int_{t_1}^{t_2} \int_{\mathbb{R}^6} f^N(\tilde{q}_0 - {}^1\tilde{y}) k_0^N(y) \mathbf{1}_{G^N(x_0)}(y) \, \mathrm{d}y \right| \leq \underline{\underline{N^{-\frac{4}{9}+\sigma}.}}$$

Again we note that this does not depend on $x_0$.

Est. 1c) Finally, the last summand can be estimated by

$$\begin{aligned}
&\left| \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_0 - \tilde{y}) k_0^N(y) \mathbf{1}_{G^N(x_0)}(y) \, \mathrm{d}y \, \mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_0 - \tilde{y}) k_0^N(y) \, \mathrm{d}y \, \mathrm{d}t \right| \\
=&\left| \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_0 - \tilde{y}) k_0^N(y) \mathbf{1}_{G^N(x_0)^C}(y) \, \mathrm{d}y \, \mathrm{d}t \right| \\
\overset{x_j \text{ iid}}{\leq}& C(T) |f|_\infty \mathbb{P}(x_1 \notin G^N(x_0)) \\
\leq& C(T) N^{\frac{2}{3}+\sigma} N^{-\frac{4}{3}-2\sigma} \\
\leq& \underline{\underline{N^{-\frac{2}{3}}.}}
\end{aligned}$$

Grönwall Combining all estimates, we obtain

$$\begin{aligned}
&\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0} f^N(q_0 - q_j) \, \mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_0 - \tilde{y}) k(\tilde{y}) \, \mathrm{d}y \, \mathrm{d}t \right| \\
\leq& N^{-\frac{4}{9}+\frac{8}{3}\sigma} + \int_{t_0}^{t_1} C \ln(N)^1 \epsilon_0 \, \mathrm{d}t,
\end{aligned}$$

which implies for any $t_n \leq t \leq \tau$

$$\begin{aligned}
{}^1\epsilon_0(t) &\leq {}^1\epsilon_0(t_n) + \int_{t_n}^{t} {}^2\epsilon_0(s) \mathrm{d}s \\
{}^2\epsilon_0(t) &\leq {}^2\epsilon_0(t_n) + C t^* \ln(N) \cdot {}^1\epsilon_0(t) + C(T, N),
\end{aligned}$$

where $C(T, N) = C(T) N^{-\frac{7}{18}+\frac{3}{5}\sigma}$. This has the perfect form to apply a Grönwall style argument. As we will need the following version of Grönwall multiple times, we state it as a lemma.

**Lemma 3.2.** *Let $v_1, v_2 : \mathbb{R} \to \mathbb{R}_+$ be two functions, $C(T, N)$ be a function depending on the time $T$ and the positive integer $N$, and $C$ be a constant. Let $[0, T]$ be divided in equidistant intervals $[t_n, t_{n+1}]$ with distance $t^*$. Assume $v_1(t_0) = 0$ and*

$$v_1(t) \leq v_1(t_n) + \int_{t_n}^{t} v_2(s) \mathrm{d}s \tag{3.1}$$

$$v_2(t) \leq v_2(t_n) + C t^* \ln(N) v_1(t) + C(T, N) \tag{3.2}$$

*for any $t \in [t_n, t_{n+1}]$. Then for $N$ large enough, we have*

$$v_2(T) \leq C(T, N) N^{\varepsilon},$$

*where the question of how large $N$ has to be depends only on $T, C$ and $\varepsilon \in \mathbb{R}_+$.*

Before we get to the proof, we apply the result to $\epsilon_0$. Using $v_1 = {}^1\epsilon_0, v_2 = {}^2\epsilon_0$ as well as $C(T,N) = C(T)N^{-\frac{7}{18}+\frac{8}{3}\sigma}$, one easily checks that the conditions of the lemma are met. Hence, with $\varepsilon = \frac{\sigma}{6}$ we obtain ${}^2\epsilon_0(\tau) \leq C(T)N^{-\frac{4}{9}+\frac{17}{6}\sigma}$. This then implies that

$$ {}^1\epsilon_0(\tau) \leq C(T) \cdot {}^2\epsilon_0 \leq C(T)N^{-\frac{4}{9}+\frac{17}{6}\sigma}.$$

Combining this, we obtain that for large enough $N$ (depending on $C(T), \sigma$) we have

$$\epsilon_0(\tau) \leq N^{-\frac{4}{9}+3\sigma}.$$

Hence, the stopping-time was not triggered by $x_0$.

Before we move on to the other good particles, we quickly prove the lemma.

*Proof of Lemma.* With a small induction 3.1 and $v_1(t_0) = 0$ implies

$$v_1(t) \leq \sum_{k=1}^{n} t^* \sup_{0\leq s\leq t_k} v_2(s) + \int_{t_n}^{t} v_2(s)ds$$

for any $t \in [t_n, t_{n+1}]$. Using this and 3.2, we can calculate for $t \in [t_n, t_{n+1}]$

$$\begin{aligned} v_2(t) &\leq v_2(t_n) + Ct^* \ln(N)v_1(t) + C(T,N) \\ &\leq v_2(t_n) + Ct^* \ln(N) \left( \sum_{k=1}^{n} t^* \sup_{0\leq s\leq t_k} v_2(s) + \int_{t_n}^{t} v_2(s)ds \right) + C(T,N) \\ &\leq \left(1 + (t^*)^2 C \ln(N)\right) \sum_{k=1}^{n} \sup_{0\leq s\leq t_k} v_2(s) + C(T,N) + Ct^* \ln(N) \int_{t_n}^{t} v_2(s)\mathrm{d}s. \end{aligned}$$

Grönwall yields for $t \in [t_n, t_{n+1}]$

$$v_2(t) \leq \left( C(T,N) + \left(1 + (t^*)^2 C \ln(N)\right) \sum_{k=1}^{n} \sup_{0\leq s\leq t_k} v_2(s) \right) e^{(t^*)^2 C \ln(N)}.$$

We can replace the left side by the supremum over all $t \in [0, t_{n+1}]$.
If we choose $t^* = \frac{1}{\sqrt{C\ln(N)}}$, this becomes

$$\sup_{0\leq t\leq t_{n+1}} v_2(t) \leq \left( C(T,N) + 2\sum_{k=1}^{n} \sup_{0\leq s\leq t_k} v_2(s) \right) e. \tag{3.3}$$

We will show by induction over $n$ that

$$\sup_{0\leq t\leq t_{n+1}} v_2(t) \leq C(T,N)e(2e+1)^n.$$

The base case $n = 0$ follows directly from 3.3. Assume it holds for all $k \in \{0, \ldots, n-1\}$. We again can use 3.3 to estimate

$$\begin{aligned} \sup_{0\leq t\leq t_{n+1}} v_2(t) &\leq \left( C(T,N) + 2\sum_{k=1}^{n} \sup_{0\leq s\leq t_k} v_2(s) \right) e \\ &\leq \left( C(T,N) + 2\sum_{k=1}^{n} C(T,N)e(2e+1)^{k-1} \right) e \\ &= \left( C(T,N) + 2C(T,N)e\frac{(2e+1)^n - 1}{(2e+1) - 1} \right) e \\ &= C(T,N)e(2e+1)^n. \end{aligned}$$

With our choice of $t^*$ we need $\lceil T\sqrt{C\ln(N)}\rceil$ many points to cover $[0,T]$ where the last point is $T$. Hence,

$$\begin{aligned} v_2(T) &\leq \sup_{0\leq t\leq T} v_2(t) \leq C(T,N)e(2e+1)^{T\sqrt{C\ln(N)}+1} \\ &= C(T,N)e(2e+1)(c(T))^{\sqrt{\ln(N)}} \\ &\leq C(T,N)e(2e+1)N^{\varepsilon} \end{aligned}$$

with $c(T) = (2e+1)^{T\sqrt{C}}$ and using $(c(T))^{\sqrt{\ln(N)}} = N^{\frac{\ln(c(T))\sqrt{\ln(N)}}{\ln(N)}} \leq N^{\frac{\varepsilon}{2}}$ for $N$ large enough. Hence, again for $N$ large enough, we obtain

$$v_2(T) \leq C(T,N)N^{\varepsilon}.$$

□

## 3.2 Second case: $\mathbf{x_i}$ is a good particle

Next, we want to estimate $\epsilon_g$. For this we consider a good particle $x_i$. This works almost analogously to the case $x_i = x_0$, as we assumed $x_0$ to be a good particle in the new system $X^{\geq 1} \cup \{x_0\}$. However, we need to be a little careful, as the effect of $x_0$ on $x_i$ cannot be as easily estimated by our law of large numbers, as $x_0$ is not i.i.d. like the other particles. Because this is only one particle, this can be handled without too much effort.

For $x_i$ good we have for any $t_0, t_1 \in [0,\tau]$:

$$\begin{aligned} &\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i} f^N(q_i-q_j)\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{\mathbb{R}^6} f^N(\tilde{q}_i-\tilde{y})k_0^N(y)\,\mathrm{d}y\,\mathrm{d}t\right| \\ \leq &\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i}\Big(f^N(q_i-q_j) - f^N(\tilde{q}_i-\tilde{q}_j)\Big)\,\mathrm{d}t\right| \qquad (2a) \\ &+\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i} f^N(\tilde{q}_i-\tilde{q}_j)\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{\mathbb{R}^6} f^N(\tilde{q}_i-\tilde{y})k(y)\mathbf{1}_{G^N(x_i)}(y)\,\mathrm{d}y\,\mathrm{d}t\right| \qquad (2b) \\ &+\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6} f^N(\tilde{q}_i-\tilde{y})k(y)\mathbf{1}_{G^N(x_i)}(y)\,\mathrm{d}y\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{\mathbb{R}^6} f^N(\tilde{q}_i-\tilde{y})k_0(y)\,\mathrm{d}y\,\mathrm{d}t\right|. \qquad (2c) \end{aligned}$$

Est. 2a) We start with the first summand:

$$\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i}\Big(f^N(q_i-q_j) - f^N(\tilde{q}_i-\tilde{q}_j)\Big)\,\mathrm{d}t\right|.$$

For this, we consider the effect of the good and bad interacting particles separately. First we consider the effect of the good $x_j$.

**Good prts.** As the stopping time has not been triggered yet, in this case we know that

$$|q_i - q_j - (\tilde{q}_i - \tilde{q}_j)| \leq \epsilon_g + \max(\epsilon_g + \epsilon_0) \leq 2N^{-\frac{4}{9}+3\sigma} \leq c(N).$$

Hence, we can introduce $g^N$ by 2.2:

$$\begin{aligned}&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i \text{ good}} f^N(q_i-q_j)-f^N(\tilde{q}_i-\tilde{q}_j)\,\mathrm{d}t\right|\\ \leq&\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i,0 \text{ good}} g^N(\tilde{q}_i-\tilde{q}_j)\cdot 2\epsilon_g\,\mathrm{d}t+\int_{t_0}^{t_1}\frac{1}{N}g^N(\tilde{q}_i-\tilde{q}_0)\cdot(\epsilon_0+\epsilon_g)\,\mathrm{d}t.\end{aligned}$$

To further estimate we need to use our law of large numbers. Hence, we had to split off the effect of $x_0$ as it is not i.i.d..

To estimate the effect of $x_0$, we use 2.7 and the fact that as $q_i$ is a good particle we know $\max(|\tilde{q}_i-\tilde{q}_0|,|\tilde{p}_i-\tilde{p}_0|)\geq N^{-\frac{2}{9}}$. This yields

$$\int_{t_0}^{t_1}\frac{1}{N}g^N(\tilde{q}_i-\tilde{q}_0)\cdot(\epsilon_g+\epsilon_0)\,\mathrm{d}t\leq\frac{C(T)}{N}\frac{N^{\frac{2}{9}}}{c(N)^2}(N^{-\frac{4}{9}+3\sigma}+N^{-\frac{4}{9}+\frac{5}{2}\sigma})\leq\underline{C(T)N^{-\frac{5}{9}+4\sigma}}.$$

For the remaining particles we can use our law of large numbers after removing a set of bad starting configurations, where the estimates might not hold. This is the first time we need to be careful about the fact that we have removed $x_0$, and hence we need to apply 6.23 here. To this end we define

$$\begin{aligned}&X^{\geq 1}\in\mathcal{A}_{3i}\subseteq\mathbb{R}^{6(N-1)}\\ :\Longleftrightarrow\ &\exists t_1,t_2\in[0,T]:\left|\frac{1}{N}\sum_{j=1}^{N-1}\int_{t_1}^{t_2}g^N({}^1\widetilde{x}_i-\tilde{q}_j)\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}t\right.\\ &\qquad\left.-\int_{\mathbb{R}^6}\int_{t_1}^{t_2}g^N({}^1\widetilde{x}_i-{}^1\widetilde{z})\mathbf{1}_{G^N(x_i)}(z)\,\mathrm{d}t\cdot k(z)\,\mathrm{d}z\right|>CN^{-\frac{1}{9}+\sigma}.\end{aligned}$$

Defining $\mathcal{A}_3$ as the union over these sets, we can estimate $\mathbb{P}(X^{\geq 1}\in\mathcal{A}_3)\leq C_\gamma N^{-\gamma}$ with 6.23. Note that, as the probability drops faster than any power of $N$, it is no problem that we need to apply this estimate up to $N$ times for all the (good) $x_i$.

With this we can estimate using 6.24 with $\beta_N=\epsilon_g$.

$$\begin{aligned}&\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i,0 \text{ good}} g^N(\tilde{q}_i-\tilde{q}_j)\cdot 2\epsilon_g\\ \leq&\underline{C(T)N^{-\frac{1}{9}+\sigma}\epsilon_g(t_1)+\int_{t_0}^{t_1}C\ln(N)\epsilon_g(t)\,\mathrm{d}t}\end{aligned}$$

**Bad prts.** For the bad $j$ we do not have to worry about $x_0$, so this works exactly the same as the corresponding argument when estimating $\epsilon_0$. Hence, we will keep this section short and only repeat the main ideas. We have to estimate

$$\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i \text{ bad}}\left(f^N(q_i-q_j)-f^N(\tilde{q}_i-\tilde{q}_j)\right)\mathrm{d}t\right|.$$

We split the sum using 2.10 with $r_0=6N^{-\frac{2}{9}-\sigma}$, $v_0=N^{-\frac{2}{9}}$ and $\varepsilon=\sigma$.

If $|\tilde{q}_i-\tilde{q}_j|>6N^{-\frac{2}{9}-\sigma}$, we use that the stopping time has not triggered and hence we know

$$|q_i-q_j-(\tilde{q}_i-\tilde{q}_j)|\leq|q_i-\tilde{q}_i|+|q_j-\tilde{q}_j|\leq 3N^{-\frac{2}{9}-\sigma}.$$

Hence, we can introduce $g$ by 2.2 and it remains to estimate

$$C(T)N^{-\frac{2}{9}-\sigma}\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i \text{ bad}} g(\tilde{q}_i-\tilde{q}_j)\mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_i)}(x_j)\,\mathrm{d}t.$$

We want to use Lemma 2.13, for which we need to control the number of bad particles. For this we define

$$\begin{aligned} X^{\geq 1}\in\mathcal{A}_{4i}\subseteq\mathbb{R}^{6(N-1)} \qquad (\mathcal{A}_4)\\ \iff \Bigg(\sum_{\text{j bad}}\mathbf{1}_{M^N_{(r,R),(v,V)}(x_i)}(x_j) > \ln(N)^{\frac{1}{4}}\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil\\ \vee\,|\,\mathcal{M}^N_b(X)| > \ln(N)^{\frac{1}{4}}N^{\frac{2}{3}}\Bigg). \end{aligned}$$

Again, we have $\mathbb{P}(X^{\geq 1}\in\mathcal{A}_4)\leq C_\gamma N^{-\gamma}$ for $\mathcal{A}_4=\bigcup\mathcal{A}_{4i}$. This is a property of $X^{\geq 1}$. Recall that we need to be careful here, as 2.13 needs $N$ i.i.d. particles, which can, for example, be solved by adding an hypothetical $N$-th i.i.d. particle and noting that this could only increase the number of bad particles.

Now we can apply 2.13 to estimate

$$\begin{aligned} &N^{-\frac{2}{9}-\sigma}\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i \text{ bad}} g^N(\tilde{q}_i-\tilde{q}_j)\mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_i)}(x_j)\,\mathrm{d}t\\ \leq& N^{-\frac{2}{9}-\sigma}\cdot CN^{-\frac{1}{3}+4\sigma}\leq \underline{CN^{-\frac{5}{9}+3\sigma}}. \end{aligned}$$

We are left with the particles that come closer than $6N^{-\frac{2}{9}-\sigma}$, which are divided into the following collision classes (see 2.10) with $r_0=6N^{-\frac{2}{9}-\sigma}, v_0=N^{-\frac{2}{9}}$:

1. $M^N_{(0,r_0),(0,v_0)}(x_i)$
2. $M^N_{(0,r_0),(N^{l\delta}v_0,N^{(l+1)\delta}v_0)}(x_i), 0\leq l\leq\left\lfloor\frac{\ln(\frac{1}{v_0})}{\delta\ln(N)}\right\rfloor$
3. $M^N_{(0,r_0),(1,\infty)}(x_i)$

As $x_i$ is a good particle, the relative momentum at their closest point has to be larger than $N^{-\frac{2}{9}}$, and hence we can rule out the first collision class.

With our established notation we estimate $f^N$ by 2.7 to obtain

$$\begin{aligned} &C\int_{t_0}^{t_1}\frac{1}{N}\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}}\left|f^N(q_i-q_j)-f^N(\tilde{q}_i-\tilde{q}_j)\right|\mathrm{d}t\\ \leq& C\int_{t_0}^{t_1}\frac{1}{N}\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}}\left|f^N(q_i-q_j)\right|+\left|f^N(\tilde{q}_i-\tilde{q}_j)\right|\mathrm{d}t\\ \leq& C\frac{1}{c(N)v}\frac{1}{N}\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}}1. \end{aligned}$$

Note that for the estimate of $|f^N(q_i-q_j)|$ we need the second part of 2.7. The required estimates for $x_i$ work just as with $x_0$.

Moving on with our estimate, we use $R = 6N^{-\frac{2}{9}-\sigma}$, $v \geq N^{-\frac{2}{9}}$ as well as $c(N) \geq N^{-\frac{1}{3}-\frac{\sigma}{2}}$. Combining this with the upper bound for the number of bad particles in this collision class obtained in $\mathcal{A}_4$, we estimate using $a_N = \ln(N)^{\frac{1}{4}}$

$$\begin{aligned}
&C\frac{1}{c(N)v}\frac{1}{N}a_n \sum_{\substack{j\neq i \text{ bad} \\ x_j \in M(x_i)}} 1 \\
&\leq C\frac{1}{c(N)v}\frac{1}{N}a_N \lceil N^{\frac{2}{3}}R^2 \min(\max(V,R),1)^4 \rceil \\
&\leq C\frac{1}{c(N)v}\frac{1}{N}a_N N^{\frac{2}{3}}R^2 \min(\max(V,R),1)^4 + C\frac{1}{c(N)v}\frac{1}{N}a_N \\
&\leq CN^{-\frac{4}{9}+\sigma}\frac{\min(V,1)}{v} + CN^{-\frac{4}{9}+\sigma} \leq \underline{CN^{-\frac{4}{9}+2\sigma}}.
\end{aligned}$$

Here we used that in the remaining classes either $v = 1$ or $V = N^\sigma v$.

Combining these estimates, we are done with the first summand and have proven:

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i} f(q_i - q_j) - f^N(\tilde{q}_i - \tilde{q}_j) \right| \leq \underline{\underline{CN^{-\frac{4}{9}+2\sigma} + \int_{t_0}^{t_1} \ln(N) \cdot {}^1\epsilon_g.}}$$

Est. 2b) We need to estimate:

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i} f^N(\tilde{q}_i - \tilde{q}_j)\,\mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k(y) \mathbf{1}_{G^N(x_i)}(y)\,\mathrm{d}y\,\mathrm{d}t \right|.$$

Compared to the previous segment, this is relatively easy. We split off the effect of $x_0$ and then use the appropriate version of our law of large numbers.

The effect of $x_0$ can be estimated by 2.7, as we know that either the minimal distance between $\tilde{x}_0$ and $\tilde{x}_i$ is greater than $6N^{-\frac{2}{9}+\sigma}$, or their relative momentum at this point is greater than $N^{-\frac{2}{9}}$.

$$\left| \int_{t_0}^{t_1} \frac{1}{N} f^N(\tilde{q}_i - \tilde{q}_0)\,\mathrm{d}t \right| \leq \frac{C}{N}\frac{N^{\frac{2}{9}}}{c(N)} \leq \underline{N^{-\frac{4}{9}+\sigma}.}$$

Now it remains to estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i,0} f^N(\tilde{q}_i - \tilde{q}_j)\,\mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k(y) \mathbf{1}_{G^N(x_i)}(y)\,\mathrm{d}y\,\mathrm{d}t \right|.$$

For this we need to remove a set of bad starting configurations, i.e.

$$\begin{aligned}
&X^{\geq 1} \in \mathcal{A}_{5i} \\
&\quad \iff \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N-1} \int_{t_1}^{t_2} f^N({}^1\widetilde{x}_i - \tilde{q}_j) \mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}t \right. \\
&\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f^N({}^1\widetilde{x}_i - {}^1\widetilde{z}) \mathbf{1}_{G^N(x_i)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > CN^{-\frac{4}{9}+\sigma}.
\end{aligned}$$

Setting $\mathcal{A}_5$ as the union, we can estimate the probability by $C_\gamma N^{-\gamma}$ using 6.23.
Working on the complement, we can directly estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i,0} f^N(\tilde{q}_i - \tilde{q}_j) \, \mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k(y) \mathbf{1}_{G^N(x_i)}(y) \, \mathrm{d}y \, \mathrm{d}t \right| \leq \underline{C N^{-\frac{4}{9}+\sigma}}.$$

Est. 2c) Finally, for the last summand we simply have to estimate

$$\begin{aligned} &\left| \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k(y) \mathbf{1}_{G^N(x_i)}(y) \, \mathrm{d}y \, \mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k_0(y) \, \mathrm{d}y \, \mathrm{d}t \right| \\ =& \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k(y) \mathbf{1}_{(G^N(x_i))^C}(y) \, \mathrm{d}y \, \mathrm{d}t. \end{aligned}$$

For this we note that

$$\mathbb{P}\left( y \in \left( G^N(x_i) \right)^C \right) \leq C(T) \left( N^{-\frac{2}{9}-\sigma} \right)^2 \left( N^{-\frac{2}{9}} \right)^4 = C(T) N^{-\frac{12}{9}-2\sigma}$$

by 2.11.

Further, we can estimate $f^N(\tilde{q}_i - \tilde{y})$ by its maximum value, i.e. $|f^N|_\infty \leq Cc(N)^2 \leq CN^{\frac{2}{3}+\sigma}$

Combining these two estimates, we obtain

$$\begin{aligned} &\int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k(y) \mathbf{1}_{(G^N(x_i))^C}(y) \, \mathrm{d}y \, \mathrm{d}t \\ \leq& C(T) N^{\frac{2}{3}} \int_{(G^N(x_i))^C} k(y) \, \mathrm{d}y \\ \leq& C(T) N^{\frac{2}{3}+\sigma-\frac{4}{3}-2\sigma} = \underline{\underline{C(T) N^{-\frac{2}{3}-\sigma}}}. \end{aligned}$$

Grönwall Combining these estimates, we have proven that

$$\begin{aligned} &\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} f^N(q_i - q_j) \, \mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\tilde{q}_i - \tilde{y}) k_0(y) \, \mathrm{d}y \, \mathrm{d}t \right| \\ \leq& C(T) N^{-\frac{4}{9}+2\sigma} + \int_{t_0}^{t_1} C \ln(N) \cdot {}^1\epsilon_g(t) \, \mathrm{d}t. \end{aligned}$$

As this estimate holds for all the good particles $x_i$, this implies for any $t_n \leq t \leq \tau$

$$\begin{aligned} {}^1\epsilon_g(t) &\leq {}^1\epsilon_g(t_n) + \int_{t_n}^{t} {}^2\epsilon_0(s) \mathrm{d}s \\ {}^2\epsilon_g(t) &\leq {}^2\epsilon_g(t_n) + C t^* \ln(N) \cdot {}^1\epsilon_0(t) + C(T, N), \end{aligned}$$

where $C(T, N) = C(T) N^{-\frac{4}{9}+2\sigma}$.

Hence, applying our Grönwall Lemma 3.2, yields

$$\epsilon_g(\tau) \leq N^{-\frac{4}{9}+3\sigma}.$$

This proves that the stopping time cannot be triggered by a good particle.

## 3.3 Second case: $x_i$ is a bad particle

We have weaker bounds for bad particles that make estimates harder.

We circumvent this problem using the following Ansatz from Grass. We define a cloud of auxiliary particles around our bad particle, i.e. we define

$$J_N = \{-\lceil N^{\frac{1}{9}} \rceil, \dots, -1, 0, 1, \dots \lceil N^{\frac{1}{9}} \rceil\}^6.$$

Then, for $(k_1, \dots, k_6) \in J_N$ we define $x_{i,k_1,\dots,k_6} = x_i + \sum_{j=1}^6 k_j N^{-\frac{4}{9}+2\sigma} e_j$, where $e_j$ is the $j$-th basis vector in $\mathbb{R}^6$.

We want to show that for any $t_0 \leq \tau$ there is an auxiliary particle $x_{i,k_1,\dots,k_6}$ that satisfies

$$\sup_{s\in[t_0,t+N^{-\frac{1}{18}}]} |^1 x_i - {}^1\widetilde{x}_{i,k_1,\dots,k_6}| \leq N^{-\frac{4}{9}+3\sigma}, \qquad \sup_{s\in[t_0,t+N^{-\frac{1}{18}}]} |^2 x_i - {}^2\widetilde{x}_{i,k_1,\dots,k_6}| \leq N^{-\frac{7}{18}+\frac{3}{4}\sigma}.$$

First, we note that

$$|\varphi^N_{0,t_0}(\phi^N_{t_0,0}(x_i)) - x_i| \leq C(T)|x_i(t_0) - \tilde{x}_i(t_0)| \leq \epsilon_b = N^{-\frac{1}{3}+\sigma},$$

where we used 2.6 in the first step, and the fact that $t \leq \tau$ in the second estimate.

This distance is smaller than the auxiliary cloud, and hence we find an auxiliary particle $x_{i,k_1,\dots,k_6}$ that satisfies:

$$|\varphi^N_{0,t_0}(\phi^N_{t,0}(x_i)) - x_{i,k_1,\dots,k_6}| \leq CN^{-\frac{4}{9}+2\sigma}.$$

Applying 2.6 again, we obtain

$$|x_i(t_0) - \tilde{x}_{i,k_1,\dots,k_6}(t_0)| \leq C(T)N^{-\frac{4}{9}+2\sigma}.$$

Hence, for large enough $N$ we at least have a small time interval such that

$$\sup_{s\in[t_0,t_1]} |^1 x_i(s) - {}^1\tilde{x}_{i,k_1,\dots,k_6}(s)| \leq N^{-\frac{4}{9}+3\sigma},$$

as well as

$$\sup_{s\in[t_0,t_1]} |^2 x_i(s) - {}^2\tilde{x}_{i,k_1,\dots,k_6}(s)| \leq N^{-\frac{7}{18}+\frac{3}{4}\sigma}.$$

The idea now is to estimate $\sup_{t_0\leq s\leq t} |x_i(s) - \tilde{x}_i(s)|$ using the fact that it is bounded by

$$\sup_{t_0\leq s\leq t} |x_i(s) - \tilde{x}_{i,k_1,\dots,k_6}(s)| + \sup_{t_0\leq s\leq t} |\tilde{x}_{i,k_1,\dots,k_6}(s) - \tilde{x}_i(s)|.$$

This is helpful, as both summands are easy to handle. The first estimate works very similar to the estimates for a good particle, because we know that the spacial distance is bounded by $N^{-\frac{4}{9}}$. The second part is well behaved, as we compare two mean-field trajectories. For ease of notation we write $\hat{x}_i(s) = \widetilde{x}_{i,k_1,\dots,k_6}$ for the auxiliary particle that satisfies the above condition on the interval $s \in [t_k, t_{k+1}]$. So $\hat{x}_i$ is the mean-field trajectory of an auxiliary particle.

We start by estimating the second summand, i.e. the mean-field difference. For this we simply apply 2.6 again to obtain:

$$\begin{aligned}
&|\tilde{x}_{i,k_1,\dots,k_6}(t) - \tilde{x}_i(t)| \\
&\leq e^{C(t-t_0)}|\tilde{x}_{i,k_1,\dots,k_6}(t_0) - \tilde{x}_i(t_0)| \\
&\leq e^{C(t-t_0)}\left(|\tilde{x}_{i,k_1,\dots,k_6}(t_0) - x_i(t_0)| + |x_i(t_0) - \tilde{x}_i(t_0)|\right) \\
&\leq e^{C(t-t_0)}\left(N^{-\frac{4}{9}+3\sigma} + |x_i(t_0) - \tilde{x}_i(t_0)|\right),
\end{aligned}$$

where we used the choice of $\hat{x}_i$ in the last estimate.

It remains to estimate $|x_i - \hat{x}_i|$.
The idea is to use the same estimates we have already used for the good particles. We simply replace $\tilde{x}_i$ by $\hat{x}_i$.

There is one additional problem we need to take care of, namely the bad interactions. So far we only have worked with good particles, so even the interaction with other bad particles was reasonable. In this situation the bad particle has at least one bad interaction. These interactions are handled in an extra step in this estimate.

For $x_i$ bad we have for any $t_0, t_1 \in [0, \tau]$:

$$\begin{aligned}
&\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} f^N(q_i - q_j) \,\mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k_0(y) \,\mathrm{d}y \,\mathrm{d}t \right| \\
\leq &\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} f^N(q_i - q_j) \mathbf{1}_{(G^N(\hat{x}_i))^C}(x_j) \,\mathrm{d}t \right| && \text{(3a)} \\
&+ \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} \left( f^N(q_i - q_j) - f^N(\hat{q}_i - \tilde{q}_j) \right) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \,\mathrm{d}t \right| && \text{(3b)} \\
&+ \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} f^N(\hat{q}_i - \tilde{q}_j) \,\mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k_0^N(y) \mathbf{1}_{G^N(\hat{x}_i)}(y) \,\mathrm{d}y \,\mathrm{d}t \right| && \text{(3c)} \\
&+ \left| \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k_0^N(y) \mathbf{1}_{G^N(\hat{x}_i)}(y) \,\mathrm{d}y \,\mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k_0^N(y) \,\mathrm{d}y \,\mathrm{d}t \right| && \text{(3d)}
\end{aligned}$$

Est. 3a) We have to estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} f^N(q_i - q_j) \mathbf{1}_{(G^N(\hat{x}_i))^C}(x_j) \,\mathrm{d}t \right|.$$

This estimate was not necessary when working with the good particles, as it estimates the effect of bad interactions. Again we need to remove some undesirable starting configurations. This time we define

$$\begin{aligned}
&X^{\geq 1} \in \mathcal{A}_{6,i,k_1,\ldots,k_6} && (\mathcal{A}_6) \\
&\iff \sum_{j \neq i} \mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma}, N^{-\frac{2}{9}})}(x_{i,k_1,\ldots,k_6})}(x_j) \geq \ln(N)^{\frac{1}{4}}.
\end{aligned}$$

By 2.11 we know that $\mathbb{P}\left(y \in \mathbb{R}^6 \mid y \notin G^N(z)\right) \leq CN^{-\frac{4}{3}-2\sigma}$, for all $z \in \mathbb{R}^6$. Hence, for $a_N = \lceil (\ln N)^{\frac{1}{4}} \rceil$ we can estimate

$$\mathbb{P}(X^{\geq 1} \in \mathcal{A}_{6,i,k_1,\ldots,k_6}) \leq \binom{N}{a_N} (CN^{-\frac{4}{3}-2\sigma})^{a_N} \leq \frac{N^{a_N}}{(Ca_N)^{a_N}} (CN^{-\frac{4}{3}-2\sigma})^{a_N} \leq N^{a_N(-\frac{1}{3}-\sigma)}.$$

Defining $\mathcal{A}_6$ as the union over all $i \in \{1, \ldots, N\}$ and $(k_1, \ldots, k_6) \in J_N$, we have

$$\mathbb{P}(X^{\geq 1} \in \mathcal{A}_6) \leq N(CN^{\frac{1}{9}})^6 N^{a_N(-\frac{1}{3}-\sigma)} = N^{-a_N\sigma} CN^{\frac{5}{3}} \cdot N^{-\frac{1}{3}a_N} \leq N^{-a_N \frac{1}{3}}.$$

Clearly, this decreases faster than any power of $N$. Working on the complement of $\mathcal{A}_6$, we can estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} f^N(q_i - q_j) \mathbf{1}_{(G^N(\hat{x}_i))^C} \, \mathrm{d}t \right| \leq \Delta_t C(T) \cdot \frac{1}{N} |f^N|_\infty \cdot \ln(N)^{\frac{1}{4}} \leq \underline{\underline{\Delta_t C(T) \cdot N^{-\frac{1}{3}+\frac{\sigma}{2}}}}.$$

Est. 3b) We have to estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} \left( f^N(q_i - q_j) - f^N(\hat{q}_i - \tilde{q}_j) \right) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \, \mathrm{d}t \right|.$$

For this, again we consider the effect of good interactions and bad interactions separately.

**Good prts.** As the stopping time has not been triggered yet, and by our choice of $\hat{x}_i$, in this case we know that

$$|q_i - q_j - (\hat{q}_i - \tilde{q}_j)| \leq \max(\epsilon_g, \epsilon_0) + N^{-\frac{4}{9}+\frac{5}{2}\sigma} \leq 2N^{-\frac{4}{9}+3\sigma} \leq c(N).$$

Hence, we can introduce $g^N$ by 2.2 to obtain

$$\begin{aligned} &\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i \text{ good}} f^N(q_i - q_j) - f^N(\hat{q}_i - \tilde{q}_j) \, \mathrm{d}t \right| \\ \leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i \text{ good}} g^N(\hat{q}_i - \tilde{q}_j) \cdot 2N^{-\frac{4}{9}+3\sigma}. \end{aligned}$$

To further estimate, we need to use our law of large numbers. Before this is possible, we need to split off the effect of $x_0$ as it is not i.i.d.. For this we use 2.7 and the fact that $x_0$ is a good particle to $\hat{x}_i$ and hence still has a reasonable distance to $\hat{x}_i$, i.e. $\max(|\hat{q}_i - \tilde{q}_0|, |\hat{p}_i - \tilde{p}_0|) \geq N^{-\frac{2}{9}}$. This yields

$$\int_{t_0}^{t_1} \frac{1}{N} g^N(\hat{q}_i - \tilde{q}_0) \cdot 2N^{-\frac{4}{9}+3\sigma} \leq \frac{C}{N} \frac{N^{\frac{2}{9}}}{c(N)^2} N^{-\frac{4}{9}+3\sigma} \leq \underline{CN^{-\frac{1}{9}+\sigma} N^{-\frac{4}{9}+3\sigma}.}$$

For the remaining particles we can use our law of large numbers after we remove a set of bad starting configurations where the estimates might not hold. As we have removed $x_0$, we need to apply 6.23 here. We define

$$\begin{aligned} &X^{\geq 1} \in \mathcal{A}_{7i,(k_1,\ldots,k_6)} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{A}_7) \\ \iff &\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N-1} \int_{t_1}^{t_2} g^N(\hat{q}_i - \tilde{q}_j) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \, \mathrm{d}t \right. \\ &\qquad \left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} g^N(\hat{q}_i - {}^1\widetilde{z}) \mathbf{1}_{G^N(\hat{x}_i)}(z) \, \mathrm{d}t \cdot k(z) \, \mathrm{d}z \right| > CN^{-\frac{1}{9}+\sigma}. \end{aligned}$$

Setting $\mathcal{A}_7$ as the union over these sets, we can estimate $\mathbb{P}\left(X^{\geq 1} \in \mathcal{A}_7\right) \leq C_\gamma N^{-\gamma}$ with Corollary 6.23. Note that as the probability drops faster than any power of $N$,

it is no problem, that we need to apply this estimate up to $N$ times for all the $x_i$ and then again $\approx (N^{\frac{1}{9}})^6$ times for all the auxiliary particles in the clouds around them.

With this we can estimate using Lemma 6.24

$$\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i, 0 \text{ good}} g^N(\tilde{q}_i - \tilde{q}_j) \cdot 2N^{-\frac{4}{9}+\frac{5}{2}\sigma} \leq \underline{N^{-\frac{4}{9}+3\sigma}}.$$

Next, we deal with the interaction with other bad particles.

**Bad prts.** We have seen these arguments more or less twice now, so we keep it short. The only difference is that we have $\hat{x}_i$ instead of $\tilde{x}_i$ and that, this time, we in theory would have to worry about the bad interaction partners. But we have already estimated their effect in (3a). So it now remains to estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i \text{ bad}} \left( f^N(q_i - q_j) - f^N(\tilde{q}_i - \tilde{q}_j) \right) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \, \mathrm{d}t \right|.$$

We split the sum using 2.10 with $r_0 = 6N^{-\frac{2}{9}-\sigma}$, $v_0 = N^{-\frac{2}{9}}$ and $\varepsilon = \sigma$.

If $|\hat{q}_i - \tilde{q}_j| > 6N^{-\frac{2}{9}-\sigma}$, we use that

$$|q_i - q_j - (\hat{q}_i - \tilde{q}_j)| \leq |q_i - \hat{q}_i| + |q_j - \tilde{q}_j| \leq 3N^{-\frac{2}{9}-\sigma}.$$

Hence, we can introduce $g$ by 2.2 and it remains to estimate

$$N^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i \text{ bad}} g(\hat{q}_i - \tilde{q}_j) \mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(\hat{x}_i)}(x_j) \, \mathrm{d}t.$$

We want to use Lemma 2.13, i.e. we need to assume that there are few bad particles. For this we define

$$\begin{aligned} & X \in \mathcal{A}_{8i,(k_1,\dots,k_6)} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{A}_8) \\ \iff & \Bigg( \sum_{\text{j bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(\hat{x}_i)}(x_j) > \ln(N)^{\frac{1}{4}} \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\ & \quad \vee |\mathcal{M}_b^N(X)| > \ln(N)^{\frac{1}{4}} N^{\frac{2}{3}} \Bigg). \end{aligned}$$

We use Corollary 2.12 to estimate the probability of $\mathcal{A}_8$, which is defined as the union over all these sets. Note that we have to be careful, as we need to apply the estimate not only for all the bad particles, but all their auxiliary particles as well. But as the total number of particles is bounded by some power of $N$, we still obtain $\mathbb{P}(X^{\geq 1} \in \mathcal{A}_8) \leq C_\gamma N^{-\gamma}$. As we have seen two times before, this is indeed a property of $X^{\geq 1}$. Now we can apply Lemma 2.13 to estimate

$$\begin{aligned} & N^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i \text{ bad}} g^N(\tilde{q}_i - \tilde{q}_j) \mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(\hat{x}_i)}(x_j) \, \mathrm{d}t \\ \leq & N^{-\frac{2}{9}-\sigma} \cdot C N^{-\frac{1}{3}+4\sigma} \leq \underline{C N^{-\frac{5}{9}+3\sigma}}. \end{aligned}$$

We are left with the particles that come closer than $6N^{-\frac{2}{9}-\sigma}$, which are divided into the following collision classes (see 2.10) with $r_0 = 6N^{-\frac{2}{9}-\sigma}, v_0 = N^{-\frac{2}{9}}$:

1. $M^N_{(0,r_0),(0,v_0)}(\hat{x}_i)$
2. $M^N_{(0,r_0),(N^{l\delta}v_0,N^{(l+1)\delta}v_0)}(\hat{x}_i), 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\delta \ln(N)} \right\rfloor$
3. $M^N_{(0,r_0),(1,\infty)}(\hat{x}_i)$

As we only consider particles in $G^N(\hat{x}_i)$, we know that the relative momentum at their closest point has to be larger than $N^{-\frac{2}{9}}$, and hence we can rule out the first collision class.

With our established notation we estimate $f^N$ by 2.7 to obtain

$$\begin{aligned} & C\int_{t_0}^{t_1} \frac{1}{N} \sum_{\substack{j\neq 0 \text{ bad} \\ x_j \in M(x_0)}} \left| f^N(q_i - q_j) - f^N(\hat{q}_i - \tilde{q}_j) \right| \mathrm{d}t \\ \leq & C\int_{t_0}^{t_1} \frac{1}{N} \sum_{\substack{j\neq i \text{ bad} \\ x_j \in M(x_i)}} \left| f^N(q_i - q_j) \right| + \left| f^N(\hat{q}_i - \tilde{q}_j) \right| \mathrm{d}t \\ \leq & C\frac{1}{c(N)v}\frac{1}{N} \sum_{\substack{j\neq i \text{ bad} \\ x_j \in M(x_i)}} 1. \end{aligned}$$

Note that for the the estimate of $|f^N(q_i - q_j)|$ we need the second part of 2.7. The required estimates for $x_i$ work just as with $x_0$.

Moving on with our estimate, we use $R = 6N^{-\frac{2}{9}-\sigma}$, $v \geq N^{-\frac{2}{9}}$ as well as $c(N) \geq N^{-\frac{1}{3}-\frac{\sigma}{2}}$. Combining this with the upper bound for the number of bad particles in this collision class obtained in $\mathcal{A}_8$, we estimate

$$\begin{aligned} C\frac{1}{c(N)v}\frac{1}{N} \sum_{\substack{j\neq i \text{ bad} \\ x_j \in M(x_i)}} 1 &\leq C\frac{1}{c(N)v}\frac{1}{N} a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\ &\leq C\frac{1}{c(N)v}\frac{1}{N} a_N N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 + C\frac{1}{c(N)v}\frac{1}{N} a_N \\ &\leq CN^{-\frac{4}{9}+\sigma}\frac{\min(V,1)}{v} + CN^{-\frac{4}{9}+\sigma} \\ &\leq \underline{CN^{-\frac{4}{9}+2\sigma}}. \end{aligned}$$

Here we used that in the remaining classes either $v = 1$ or $V = N^\sigma v$.

Est. 3c) Here we have to estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i} f^N(\hat{q}_i - \tilde{q}_j) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \,\mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k(y) \mathbf{1}_{G^N(\hat{x}_i)}(y) \,\mathrm{d}y \,\mathrm{d}t \right|.$$

This is a simple application of our law of large numbers after considering the effect of $x_0$ separately. For $x_0$ we use Lemma 2.7, which can be applied as we have taken care of the bad interactions already, to obtain

$$\left| \int_{t_0}^{t_1} \frac{1}{N} f^N(\hat{q}_i - \tilde{q}_0) \mathbf{1}_{G^N(\hat{x}_i)}(x_0) \,\mathrm{d}t \right| \leq \frac{C(T)}{N}\frac{N^{\frac{2}{9}}}{c(N)} \leq \underline{C(T)N^{-\frac{4}{9}+\sigma}}.$$

It now remains to estimate

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i,0} f^N(\hat{q}_i - \tilde{q}_j) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \, \mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k_0^N(y) \mathbf{1}_{G^N(\hat{x}_i)}(y) \, \mathrm{d}y \, \mathrm{d}t \right|.$$

For this we need to remove another set of bad starting configurations

$$\begin{aligned} X^{\geq 1} \in \mathcal{A}_{9i,(k_1,\dots,k_6)} \subseteq \mathbb{R}^{6(N-1)} \qquad & (\mathcal{A}_9) \\ \iff \exists t_1, t_2 \in [0,T] : & \left| \frac{1}{N} \sum_{j=1}^{N-1} \int_{t_1}^{t_2} f^N({}^1\hat{x}_i - \tilde{q}_j) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \, \mathrm{d}t \right. \\ & \left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f^N({}^1\hat{x}_i - {}^1\tilde{z}) \mathbf{1}_{G^N(\hat{x}_i)}(z) \, \mathrm{d}t \cdot k(z) \, \mathrm{d}z \right| > C N^{-\frac{4}{9}+\sigma}. \end{aligned}$$

Again, we have to take the large amount of auxiliary particles $\hat{x}_i$ into account. As we have seen before, this is no problem. The number of times we have to apply Lemma 6.23 is bounded by some power of $N$, and hence we can modify $\gamma$ to still obtain that the probability of $\mathcal{A}_9 = \bigcup \mathcal{A}_{9i,(k_1,\dots,k_6)}$ can be estimated by $\mathbb{P}(X^{\geq 1} \in \mathcal{A}_9) \leq C_\gamma N^{-\gamma}$ for any $\gamma$. Working on the complement, we can directly estimate by

$$\begin{aligned} & \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i,0} f^N(\hat{q}_i - \tilde{q}_j) \mathbf{1}_{G^N(\hat{x}_i)}(x_j) \, \mathrm{d}t \right. \\ & \left. - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k_0^N(y) \mathbf{1}_{G^N(\hat{x}_i)}(y) \, \mathrm{d}y \, \mathrm{d}t \right| \leq \underline{\underline{C N^{-\frac{4}{9}+\sigma}.}} \end{aligned}$$

Est. 3d)

$$\left| \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k(y) \mathbf{1}_{G^N(\hat{x}_i)}(y) \, \mathrm{d}y \, \mathrm{d}t - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k_0^N(y) \, \mathrm{d}y \, \mathrm{d}t \right|$$

This works the same as our arguments for the good particles in the last section.

We note that $\mathbb{P}\left(y \in (G^N(\hat{x}_i))^C\right) \leq C(T)(N^{-\frac{2}{9}-\sigma})^2 (N^{-\frac{2}{9}})^4 = C(T) N^{-\frac{12}{9}-2\sigma}$ by 2.11.

Further, we can estimate $f^N(\hat{q}_i - \tilde{y})$ by its maximum value, i.e. $|f^N|_\infty \leq Cc(N)^2 \leq C(T) N^{\frac{2}{3}+\sigma}$.

Combining these two estimates, we obtain

$$\begin{aligned} & \int_{t_0}^{t_1} \int_{\mathbb{R}^6} f^N(\hat{q}_i - \tilde{y}) k(y) \mathbf{1}_{(G^N(\hat{x}_i))^C}(y) \, \mathrm{d}y \, \mathrm{d}t \\ \leq & C(T) N^{\frac{2}{3}+\sigma} \int_{(G^N(\hat{x}_i))^C} k(y) \, \mathrm{d}y \\ \leq & C(T) N^{\frac{2}{3}+\sigma-\frac{4}{3}-2\sigma} = \underline{\underline{C(T) N^{-\frac{2}{3}-\sigma}.}} \end{aligned}$$

Combining all these estimates, we have proven (for large enough $N$):

$$\begin{aligned}&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}f^N(q_i-q_j)\,\mathrm{d}t-\int_{t_0}^{t_1}\int_{\mathbb{R}^6}f^N(\tilde{q}_i-\tilde{y})k_0(y)\,\mathrm{d}y\,\mathrm{d}t\right|\\ \leq&(t_1-t_0)N^{-\frac{1}{3}+\frac{\sigma}{2}}+C(T)N^{-\frac{4}{9}+3\sigma}.\end{aligned}$$

$\Delta t = N^{-\frac{1}{18}}$ Now we can prove that $t_2 = t_1 + N^{-\frac{1}{18}}$ is a valid choice.

Recall that we have to prove that for this choice

$$\sup_{s\in[t_0,t_1]}|{}^1x_i(s)-{}^1\tilde{x}_{i,k_1,\dots,k_6}(s)|\leq N^{-\frac{4}{9}+3\sigma},$$

as well as

$$\sup_{s\in[t_0,t_1]}|{}^2x_i(s)-{}^2\tilde{x}_{i,k_1,\dots,k_6}(s)|\leq N^{-\frac{7}{18}+\frac{3}{4}\sigma}.$$

First, we note that for $t_1 - t_0 = N^{-\frac{1}{18}}$ we have

$$\begin{aligned}&|{}^2x_i(s)-{}^2\tilde{x}_{i,k_1,\dots,k_6}(s)|\\ \leq&|{}^2x_i(t_0)-{}^2\tilde{x}_{i,k_1,\dots,k_6}(t_0)|+\left|\int_{t_0}^{s}\frac{1}{N}\sum_{j\neq i}f^N(q_i-q_j)\,\mathrm{d}t-\int_{t_0}^{s}\int_{\mathbb{R}^6}f^N(\tilde{q}_i-\tilde{y})k_0(y)\,\mathrm{d}y\,\mathrm{d}t\right|\\ \leq&N^{-\frac{4}{9}+2\sigma}+(t_1-t_0)N^{-\frac{1}{3}+\frac{\sigma}{2}}+C(T)N^{-\frac{4}{9}+3\sigma}\leq N^{-\frac{7}{18}+\frac{3}{4}\sigma}.\end{aligned}$$

To obtain a strong estimate for $|{}^1x_i(s) - {}^1\tilde{x}_{i,k_1,\dots,k_6}(s)|$, we need the following result by Graß:

**Lemma 3.3.** *There exists $C > 0$ such that for all $\mathbb{N} \in \mathbb{N}$, $t, t_0 \in [0,T]$ where $|t - t_0| \leq 1$ and $x, y \in \mathbb{R}_6$ it holds that*

$$\begin{aligned}&\left|{}^1\varphi_{t,t_0}(x)-{}^1\varphi_{t,t_0}(y)-({}^1x-{}^1y)-({}^2x-{}^2y)(t-t_0)\right|\\ \leq&C(t-t_0)^2\Big(|{}^1x-{}^1y|+|{}^2x-{}^2y||t-t_0|\Big).\end{aligned}$$

*Proof.* This is Lemma 2.1.3. in [9]. □

Using that

$$|x_i(t_0)-\tilde{x}_{i,k_1,\dots,k_6}(t_0)|\leq N^{-\frac{4}{9}+2\sigma}$$

and that $|t_1 - t_0| \ll 1$, this estimate yields

$$\begin{aligned}&|{}^1x_i(s)-{}^1\tilde{x}_{i,k_1,\dots,k_6}(s)|\\ \leq&|{}^1x_i(t_0)-{}^1\tilde{x}_{i,k_1,\dots,k_6}(t_0)|+(t_1-t_0)N^{-\frac{7}{18}+\frac{3}{4}\sigma}+N^{-\frac{4}{9}+2\sigma}\\ \leq&N^{-\frac{4}{9}+2\sigma}+N^{-\frac{1}{18}}N^{-\frac{7}{18}+\frac{3}{4}\sigma}+C(T)N^{-\frac{4}{9}+2\sigma}\leq N^{-\frac{4}{9}+\frac{5}{2}\sigma}.\end{aligned}$$

This also proves that

$$\sup_{t_1\leq s\leq t_2}|x_i(s)-\hat{x}_i(s)|\leq C(T)N^{-\frac{7}{18}+\frac{3}{4}\sigma}. \tag{3.4}$$

Finally, this allows us to prove our actual goal, namely that $|x_i - \tilde{x}_i|$ stays small.

Recall that this term is bounded by

$$|x_i - \tilde{x}_i| \leq |x_i - \hat{x}_i| + |\hat{x}_i - \tilde{x}_i|.$$

Combining the estimate 3.4 with the estimate

$$|\hat{x}_i(s) - \tilde{x}_i(s)| \leq e^{C(s-t_0)} \left( N^{-\frac{4}{9}+3\sigma} + |x_i(t_0) - \tilde{x}_i(t_0)| \right)$$

from the beginning of this section, we obtain

$$\begin{aligned} &\sup_{t_0 \leq s \leq t_1} |x_i - \tilde{x}_i| \leq e^{CN^{-\frac{1}{18}}} \left( N^{-\frac{4}{9}+3\sigma} + |^1x_i(t_0) - {}^1\tilde{x}_i(t_0)| \right) + C(T)N^{-\frac{7}{18}+\frac{3}{4}\sigma} \\ \leq &e^{CN^{-\frac{1}{18}}} |^1x_i(t_0) - {}^1\tilde{x}_i(t_0)| + C(T)N^{-\frac{7}{18}+\frac{3}{4}\sigma}. \end{aligned}$$

Choosing equidistant $t_k$ with $t_k = k \cdot N^{-\frac{1}{18}}$, we obtain the series of inequalities

$$\sup_{t_n \leq s \leq t_{n+1}} \left| x_i - \tilde{x}_i \right| \leq e^{CN^{-\frac{1}{18}}} |^1x_i(t_n) - {}^1\tilde{x}_i(t_n)| + C(T)N^{-\frac{7}{18}+\frac{3}{4}\sigma}.$$

With a small induction one can conclude that

$$\sup_{0 \leq s \leq t_n} |x_i(s) - \tilde{x}_i(s)| \leq CN^{-\frac{7}{18}+\frac{3}{4}\sigma} \cdot \sum_{k=0}^{n-1} e^{2CN^{-\frac{1}{18}}k}.$$

Using that $TN^{\frac{1}{18}}$ constitutes an upper bound for $n$ and that $1 - e^{CN^{-\frac{1}{18}}} \leq CN^{-\frac{1}{18}}$ for large enough $N$, we get

$$\sup_{0 \leq s \leq \tau} |x_i(s) - \tilde{x}_i(s)| \leq C(T)N^{-\frac{6}{18}+\frac{3}{4}\sigma} \leq N^{-\frac{1}{3}+\sigma}.$$

This then implies that the stopping time cannot be triggered by a bad particle.

This concludes the proof of our version of Graß's result in [9].

# 4 Proof of the main theorem

After all this preparation we just need some notation before we are now able to state and prove our main theorem.

**Definition 4.1.** For $X = X^{\geq 1} \cup \{x_0\} \in \mathbb{R}^{6N}$ and $\delta \in \mathbb{R}^6$ we define the disturbed configuration $X^\delta = X^{\geq 1} \cup \{x_0 + \delta\} \in \mathbb{R}^{6N}$. This means, that $x_0^\delta(0) = x_0(0) + \delta$ and $x_i^\delta(0) = x_i(0)$. We write $\widetilde{x}_0^\delta$ for the mean-field evolution of $x_0^\delta$. As we only disturb one particle, it is reasonable to simply apply the same mean-field evolution. This also implies that for the other particles $(x_i,\ i \geq 1)$, where the starting position does not change, the mean-field trajectory does not change either, i.e. we have $\widetilde{x}_i^\delta = \widetilde{x}_i$.

We also introduce

$$\delta_i(t) = x_i^\delta(t) - x_i(t), \quad \delta_g(t) = \sup_{x_i \neq 0 \text{ good}} \sup_{0 \leq s \leq t} |\delta_i(s)| \quad \text{and} \quad \delta_b(t) = \sup_{X_i \text{bad}} \sup_{0 \leq s \leq t} |\delta_i(s)|,$$

as well as deltas for the particles in the $k$-th ring:

$$\delta_g^k(t) = \sup_{x_i \in R^k \text{ good}} \sup_{0 \leq s \leq t} |\delta_i(s)| \qquad \text{and} \qquad \delta_b^k(t) = \sup_{x_i \in R^k \text{ bad}} \sup_{0 \leq s \leq t} |\delta_i(s)|.$$

We use ${}^1\delta_i$ and ${}^2\delta_i$ to denote the position and momentum component, respectively. Analogously ${}^1\delta_g$ and ${}^1\delta_b$ are defined as the suprema over their respective ${}^1\delta_i$.

Lastly, we write $\widetilde{\delta}_0 = \widetilde{x}_0^\delta - \tilde{x}_0$.

By slight abuse of notation $\delta_0(t) = x_0^\delta(t) - x_0(t)$ will sometimes also be used for $\delta_0(t) = \sup_{0 \leq s \leq t} |\delta_0(s)|$. The same is true for $\tilde{\delta}_0$.

With this notation we can finally state our main result. It states that for a typical configuration $X^{\geq 1}$ the trajectories in the original system $X = X^{\geq 1} \cup \{x_0\}$ and the trajectories in the disturbed system $X^\delta = X^{\geq 1} \cup \{x_0 + \delta\}$ stay close to each other.

**Theorem 4.2.** *Let $\sigma > 0$ and $f^N$ be the force from our Set up and $k$ be a density satisfying the conditions we introduced in section 1.1. Let $T > 0$ and $\gamma > 0$. Then we find a $C_\gamma$ such that for all $x_0 \in \mathbb{R}^6$, all $\delta \leq c(N)$ and all $k \in [0, \frac{2}{9\sigma}]$ we have the following estimates for the conditional probabilities:*

$$\mathbb{P}\left( \sup_{0 \leq s \leq T} |\delta_0 - \tilde{\delta}_0| > N^{-\frac{1}{18}+4\sigma} \cdot |\delta| \quad \middle|\ x_0 \text{ is good} \right) < C_\gamma N^{-\gamma},$$

$$\mathbb{P}\left( \sup_{x_i \in R^k \text{ good}} \sup_{0 \leq s \leq T} |x_i^\delta - x_i| > N^{-\frac{1}{9}+2\sigma-k\sigma} \cdot |\delta| \quad \middle|\ x_0 \text{ is good} \right) < C_\gamma N^{-\gamma},$$

$$\mathbb{P}\left( \sup_{x_i \in R^k \text{ bad}} \sup_{0 \leq s \leq T} |x_i^\delta - x_i| > N^{-\frac{1}{18}+2\sigma-k\sigma} \cdot |\delta| \quad \middle|\ x_0 \text{is good} \right) < C_\gamma N^{-\gamma}.$$

*Here all probabilities are taken with respect to $X^{\geq 1} \in \mathbb{R}^{6(N-1)}$. Importantly $x_0$, unlike all the other $x_i$, is not a random variable.*

*By "$x_0$ is good", we mean that $x_0$ becomes a good particle in $X^{\geq 1} \cup \{x_0\}$ with respect to our definition 2.8.*

The theorem states that, with a very high probability, for all particles that were not disturbed the trajectory stays more or less the same, i.e. the change is of a significantly smaller order than the $\delta$. For the disturbed particle itself it gives a good estimate on how the trajectory changes. We can simply investigate how the mean-field trajectory changes when we move the particle by $\delta$, and this will give us a good estimate on how the real trajectory changes. As the change of the mean-field trajectory is easily estimated by Corollary 2.6, our result implies an estimate on $|\delta_0|$. More concretely, we find a constant $C_1(T)$ (independent of $\delta$, $X$ and $N$) such that

$$\sup_{0\leq s\leq T} |\widetilde{x}_0^\delta - \widetilde{x}_0| \leq C_1(T)|\widetilde{x}_0^\delta(0) - \widetilde{x}_0(0)| = C_1(T)|\delta|.$$

Hence, our theorem implies that in almost all cases $|\delta_0| \leq C_1(T)|\delta| + N^{-\frac{1}{18}+4\sigma}|\delta|$. Our result is much more interesting, as we were able to extract the leading order of $\delta$ in $\delta_0$.

As an alternative formulation of our result, one could consider a configuration to be stable if all good particles can be moved a bit without disturbing the system too much. This alternate formulation follows directly from the one we stated above, i.e. from Theorem 4.2. It is in fact slightly weaker as all the particles – including the disturbed particle – have to be i.i.d.. At its core, it is still a version of our result that, depending on the situation, might even be easier to work with.

**Theorem 4.3.** *Let $f^N : \mathbb{R}^3 \to \mathbb{R}^3$ be the force from our Set up and $k$ be a density satisfying the conditions we introduced in 1.1. Then for all $T > 0$ and all $\gamma > 0$ we find a constant $C_\gamma$ such that for all $\delta \leq c(N)$ and $k \in [0, \frac{2}{9\sigma}]$*

$$\mathbb{P}\left( \sup_{0\leq s\leq T} |\delta_0 - \tilde{\delta}_0| > N^{-\frac{1}{18}+4\sigma}|\delta| \quad \middle| \; x_0 \text{ is a good particle in } X \right) < C_\gamma N^{-\gamma}$$

$$\mathbb{P}\left( \sup_{x_i\in R^k} \sup_{good} \sup_{0\leq s\leq T} |x_i^\delta - x_i| > N^{-\frac{1}{9}+2\sigma-k\sigma}|\delta| \quad \middle| \; x_0 \text{ is a good particle in } X \right) < C_\gamma N^{-\gamma}$$

$$\mathbb{P}\left( \sup_{x_i\in R^k} \sup_{bad} \sup_{0\leq s\leq T} |x_i^\delta - x_i| > N^{-\frac{1}{18}+2\sigma-k\sigma}|\delta| \quad \middle| \; x_0 \text{ is a good particle in } X \right) < C_\gamma N^{-\gamma}.$$

*This time the probabilities are taken with respect to $X \in \mathbb{R}^{6N}$, and hence $x_0$ is a random variable.*

*This can be strengthened further. Instead of only considering $x_0$ in the probabilities, we can estimate the probability that there exists a good $x_i \in X$, which – when moved – disturbs the system too much.*

*Proof.* We assume $x_0$ is a good particle in $X$. Then

$$\begin{aligned}
&\mathbb{P}\left( \sup_{0\leq s\leq T} |\delta_0 - \tilde{\delta}_0| > N^{-\frac{1}{18}+4\sigma}\delta \right) \\
&\leq \int_{y\in\mathbb{R}^6} \mathbb{P}\left( \sup_{0\leq s\leq T} |\delta_0 - \tilde{\delta}_0| > N^{-\frac{1}{18}+4\sigma}\delta \quad | \; x_0 = y \right) \cdot k_0(y)\,\mathrm{d}y \\
&\leq \int_{y\in\mathbb{R}^6} C_\gamma N^{-\gamma} \cdot k_0(y)\,\mathrm{d}y = C_\gamma N^{-\gamma}.
\end{aligned}$$

Here we simply used Theorem 4.2, as the conditional probability has exactly the right structure, i.e. it considers the random variable $X^{\geq 1}$ together with a fixed $x_0$. Of course, we make heavy use of the fact that $X^{\geq 1}$ and $x_0$ are i.i.d. and that the constant $C_\gamma$ does not depend on $x_0 = y$. The other two estimates follow analogously.

Finally, we consider the probability

$$\mathbb{P}\Big(\exists x \in X \mid x \text{ good particle that infringes one of the estimates above}\Big).$$

As all particles are initially i.i.d. and as the probability for each particle is bounded by $C_\gamma N^{-\gamma}$, it is clear that this probability is bounded by $NC_\gamma N^{-\gamma}$. We set $\tilde{C}_\gamma = C_{\gamma+1}$ and conclude that the probability for a configuration to have one or more good particles, that, when moved, disturb the system is bounded by $\widetilde{C}_\gamma N^{-\gamma}$.

□

**Proof** We will spend the entire remaining chapter proving this result.

First, we fix the constant in our $h_{\mathbf{c}}$ namely $\mathbf{c} = 2C_1(T) + 1$ which, as promised, does not depend on $N$. From now on we will omit the index and simply write $h$. Later in the proof it will become clear why this choice was made.

Now we define $\tau$ as the latest point in time $\leq T$, for which

$$|\delta_0 - \tilde{\delta}_0| \leq N^{-\frac{1}{18}+4\sigma}|\delta|;\ \ \delta_0(\tau) \leq 2C_1(\tau)|\delta|;\ \ \delta_g^k(\tau) \leq N^{-\frac{1}{9}+2\sigma-k\sigma}|\delta|;\ \ \delta_b^k(\tau) \leq N^{-\frac{1}{18}+2\sigma-k\sigma}|\delta|.$$

We want to show that this stopping time typically never triggers, i.e. that $\mathbb{P}(\tau \neq T) < C_\gamma N^{-\gamma}$.

To this end we want to estimate the difference in force between the particles in the original system and their counterparts in the disturbed system. The advantage of this approach is that we can assume that our stopping time has not been triggered so far, and hence we have strong estimates on our particles. Our final goal will be a Grönwall-style argument.

In the proof we will distinguish different cases. We will first consider the difference of force for the 'good' particles that were not moved, then the 'bad' particles and finally we will have to deal with the moved particle $x_0$ separately.

What we will prove is a stochastic result. Hence, it is okay that we will not consider every possible starting configuration and rather work with a subset of configurations that have desirable properties. For this, we will remove sets of bad starting configurations multiple times throughout this proof.

Before we start, we can define the first such set. By slightly altering the result of Graß, we were able to prove that for any fixed $x_0$ configurations $x_1, \ldots, x_N$ will typically be well behaved with respect to $x_0$ in the sense that if $x_0$ is a good particle in $X = x_0 \cup X^{\geq 1}$, then the real trajectories in $X$ stay close to their mean-field trajectories. To make this precise, we define

$$\begin{aligned} X^{\geq 1} \in \mathcal{B}_1 \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_1) \\ :\Longleftrightarrow (\exists_{i \text{ good}} \sup_{0\leq s\leq T} |x_i - \widetilde{x_i}| > N^{-\frac{7}{18}+\sigma} \vee \exists_{i \text{ bad}} \sup_{0\leq s\leq T} |x_i - \widetilde{x_i}| > N^{-\frac{2}{9}-\sigma}). \end{aligned}$$

In Theorem 3.1 we have proven that for any $\gamma > 0$ there is a $C_\gamma > 0$ such that $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_1) < C_\gamma N^{-\gamma}$. From now on we will work on the complement of $\mathcal{B}_1$, i.e. we will only consider configurations where the particles stay reasonable close to their mean-field trajectories.

**Remark 4.4.** Graß uses a slightly different definition of good and bad particles, i.e. his definition uses the $M_{R,V}$ we defined in 2.8. However, our definition is more strict as $G_{R,V}(y) \subseteq M_{R,V}(y)$, and hence more particles are considered "bad" in our definition. In conclusion, Graß's result is even stronger than what we use here.

We already need to restrict our configuration further to make sure that our partition into rings works properly. For this, we remove

$$X^{\geq 1} \in \mathcal{B}_{2a,i} \subseteq \mathbb{R}^{6(N-1)} \tag{$\mathcal{B}_{2a}$}$$
$$:\iff \left( \sum_{j \text{ bad}} \mathbf{1}_{M^N_{N^{-\frac{1}{9}}, N^{-\frac{1}{9}}}(x_i)}(x_j) > \ln(N) \right)$$

By Corollary 2.12 the probability $X \in B_{2a,i}$ is less than $C_\gamma N^{-\gamma}$ for all $i \in \{0, 1 \ldots, N-1\}$, so the same is true for $\mathcal{B}_{2a} := \bigcup \mathcal{B}_{2a,i}$, i.e. $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_{2a}) \leq C_\gamma N^{-\gamma}$, for any $\gamma$ and suitable $C_\gamma$. From now on we will also work on the complement of $\mathcal{B}_2$, i.e. we work on $(\mathbb{R}^{6(N-1)} \setminus \mathcal{B}_1) \setminus \mathcal{B}_{2a}$. We will remove more bad starting configurations while proving our theorem, and in the end we will work on $\mathbb{R}^{6(N-1)} \setminus \bigcup \mathcal{B}_k$.

Working on the complement of $\mathcal{B}_{2a}$ allows us to prove the following two important lemmas for our rings of particles around $x_0$.

**Lemma 4.5.** *Let $k = 0, \ldots, K+1$. If $y \in R^k$, then $\sup_t |\tilde{y} - \tilde{x}_0| \leq C N^{-\frac{2}{9}+k\sigma}$. Only for $k = 0$ we have $\sup_t |\tilde{y} - \tilde{x}_0| \leq C \ln(N) N^{-\frac{2}{9}}$.*

*Proof.* Recall that our definition of the $k$-th ring definition 2.16 did account for chains of bad interactions. Otherwise this lemma would follow directly from the definition of the rings and the fact that the mean-field trajectories are Lipschitz.

With our definition, however, we still know that, if $y$ is in $R^k$, there has to be a particle $z$ in $B^k \setminus B^{k-1}$ and a chain of bad particles connecting the two. As we work on the complement of $\mathcal{B}_2$, we know that in the $N^{-\frac{1}{9}}$ ball around $z$ there are at most $\ln(N)$ bad particles. By Lemma 2.6 and their definition we know that the mean-field trajectories of the bad particles stay in a $C(T)N^{-\frac{2}{9}}$ ball around their collision partners. That means that in the worst case the $\ln(N)$ bad particles around $z$ can build a chain with length of at most $C(T)\ln(N)N^{-\frac{2}{9}}$. As $\min |\tilde{z} - \tilde{x}_0| \leq N^{-\frac{2}{9}+k\sigma}$, this means that $|\tilde{y} - \tilde{x}_0| \leq N^{-\frac{2}{9}+k\sigma} + C\ln(N)N^{-\frac{2}{9}}$.

If $k \geq 1$, we can estimate $C\ln(N)N^{-\frac{2}{9}}$ by $N^{-\frac{2}{9}+k\sigma}$ yielding the desired result. If $k = 0$, we estimate $N^{-\frac{2}{9}+k\sigma}$ by $\ln(N)N^{-\frac{2}{9}}$ to obtain the stated estimate.

□

**Lemma 4.6.** *For $j < k-1$ let $y \in R^k$ and $z \in R^j$. In this case we have $y \in M_{\frac{1}{2}N^{-\frac{2}{9}+(k-1)\sigma}, \frac{1}{2}N^{-\frac{2}{9}+(k-1)\sigma}}(z)^C$.*

*Proof.* Assume to the contrary that there is a point in time where $|\tilde{y} - \tilde{z}| \leq \frac{1}{2}N^{-\frac{2}{9}+(k-1)\sigma}$. By the lemma above this means that $|\tilde{y} - \tilde{x}_0| \leq |\tilde{y} - \tilde{z}| + |\tilde{z} - \tilde{x}_0| \leq N^{-\frac{2}{9}+(k-1)\sigma}$, yielding a contradiction as $y \notin B^{k-1}$. □

We can now also use our version of the law of large numbers to prove that in nearly all cases there are very few particles near $x_0$, i.e. in the rings $R^k$ with $k \leq \frac{1}{18\sigma} + 2$. This will be helpful as, when we move $x_0$, particles that are this close to the disturbance are most heavily impacted by it.

**Lemma 4.7.** *We have*

$$\mathbb{P}\left(\#\{i = 1, \dots, N \mid x_i \text{ is near } x_0\} > CN^{12\sigma}\right) \leq C_\gamma N^{-\gamma}.$$

*Proof.* We consider the set of particles near $x_0$. Recall that near means that $x_i \in R^k$ with $k \leq \frac{1}{18\sigma} + 2$. First, note that by Lemma 4.5 we know that these particles $x_i$ will stay in their corresponding $CN^{-\frac{2}{9}+k\sigma}$ ball around $x_0$. As $k \leq \frac{1}{18\sigma} + 2$, this yields that they will stay in the $CN^{-\frac{3}{18}+2\sigma}$ ball around $x_0$. The rest is a simple law of large numbers argument (i.e. Lemma 6.17). For this proof we will write $M = M^N_{(CN^{-\frac{1}{6}+2\sigma}, CN^{-\frac{1}{6}+2\sigma})}(x_0)$. As our $z_k$ we will use $\mathbf{1}_M(x_k) - \mathbb{P}(x_1 \in M)$.

By construction and as the $x_i$ are i.i.d., we have $\mathbb{E}(z_k) = 0$. Next, we set $b(N) = N^{12\sigma-1}$, and by Lemma 2.11 we have

$$\mathbb{E}(z_k^2) \leq \mathbb{E}(\mathbf{1}_M(x_k)^2) = \mathbb{P}(x_1 \in M) \leq CN^{12\sigma-1}.$$

We also can estimate $|z_k| = |\mathbf{1}_M(x_k) - \mathbb{P}(x_1 \in M)| \leq 1 \leq \sqrt{Nb(N)}$.

Now our law of large numbers 6.17 yields

$$\mathbb{P}(|\bar{z}| \geq CN^{6\sigma-1} \ln(N)) \leq N^{-\gamma}.$$

As a simple consequence we obtain

$$\mathbb{P}(|M| \geq CN^{6\sigma} \ln(N) + CN^{12\sigma}) \leq N^{-\gamma}.$$

Hence, typically we have that the number of particles in this ball around $x_0$ is bounded by $CN^{12\sigma}$. □

This allows us to define

$$X^{\geq 1} \in \mathcal{B}_{2b} \subseteq \mathbb{R}^{6(N-1)} \tag{$\mathcal{B}_{2b}$}$$
$$:\Longleftrightarrow \left| \bigcup_{k=-1}^{\frac{1}{18}\sigma+2} R^k \right| > CN^{12\sigma}.$$

The previous lemma directly implies that we can estimate $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_{2b}) \leq C_\gamma N^{-\gamma}$. In the following arguments we will need to restrict our possible starting configurations a few more times, but for now we are ready to start considering the difference in force for a good particle.

## 4.1 First case: $x_i$ is a good particle

We want to show that the stopping time does not trigger because of a good particle. For this we will prove that $\delta_g(\tau) < N^{-\frac{1}{9}+2\sigma}|\delta|$ and $\delta_g^k(\tau) < N^{-\frac{1}{9}+2\sigma-k\sigma}|\delta|$. To achieve this, we will estimate the force difference between $x_i$ and $x_i^\delta$ and then use a Grönwall estimation. From now on we will write $k(i)$ to reference the ring our particle $x_i$ is in, i.e. $x_i \in R^{k(i)}$. The same holds for all the other particles, i.e. $k(j)$ references the ring with $x_j \in R^{k(j)}$.

### 4.1.1 Estimating the difference in force

Recall that the force on the original particle $x_i$ is given as

$$\frac{1}{N}\sum_{j\neq i} f^N(q_i(t)-q_j(t))$$

while the force on $x_i^\delta$ is given as

$$\frac{1}{N}\sum_{j\neq i} f^N(q_i^\delta(t)-q_j^\delta(t)).$$

After taking the difference and integrating, we can apply the triangle-inequality multiple times to obtain:

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\left(f^N(q_i^\delta(t)-q_j^\delta(t))-f^N(q_i(t)-q_j(t))\right)\mathrm{d}t\right|\\
\leq&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\left(f^N(q_i^\delta(t)-q_j^\delta(t))-f^N(q_i(t)-q_j(t))\right)\mathrm{d}t\right.\\
&\qquad\left.-\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}Df^N(\tilde{q}_i-\tilde{q}_j)(\delta_i-\delta_j)\,\mathrm{d}t\right| &\text{(1a)}\\
&+\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}Df^N(\tilde{q}_i-\tilde{q}_j)(\delta_i-\delta_j)\,\mathrm{d}t-\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}Df^N(\tilde{q}_i-\tilde{q}_j)\delta_i\,\mathrm{d}t\right| &\text{(1b)}\\
&+\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}Df^N(\tilde{q}_i-\tilde{q}_j)\delta_i\,\mathrm{d}t-\int_{t_0}^{t_1}\int_{\mathbb{R}^6}Df^N(\tilde{q}_i-y)k(y)\mathbf{1}_{G^N(x_i)}(y)\delta_i\,\mathrm{d}y\,\mathrm{d}t\right| &\text{(1c)}\\
&+\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6}Df^N(\tilde{q}_i-y)k(y)\mathbf{1}_{G^N(x_i)}(y)\delta_i\,\mathrm{d}y\,\mathrm{d}t-\int_{t_0}^{t_1}\overline{Df}^N(\tilde{q}_i)\delta_i\,\mathrm{d}t\right| &\text{(1d)}\\
&+\left|\int_{t_0}^{t_1}\overline{Df}^N(\tilde{q}_i)\delta_i\,\mathrm{d}t\right| &\text{(1e)}\\
\leq&C\int_{t_0}^{t_1}\ln(N)\cdot{}^1\delta_g^k\,\mathrm{d}t+C(T)N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta|.
\end{aligned}$$

The last estimate is not obvious at all but rather our goal for this section. We will estimate each of the summands (1a)-(1e). Often we will need to further split the sums, and consider the effect of good and bad particles separately. For better readability we have marked the sections on the left. The final estimate for each of the summands (1a)-(1e) is <u><u>double underlined</u></u>. Important interim estimates, that are combined to get to the final estimate, are simply <u>underlined</u>.

Est. 1a) We can use the mean-value-theorem on $f^N(q_i^\delta(t)-q_j^\delta(t))-f^N(q_i(t)-q_j(t))$ to obtain

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\Big(f(q_i^\delta(t)-q_j^\delta(t))-f(q_i(t)-q_j(t))\Big)\,\mathrm{d}t\right.\\
&\qquad\left.-\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}Df^N(\tilde q_i-\tilde q_j)(\delta_i-\delta_j)\,\mathrm{d}t\right|\\
\le&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\int_0^1\Big(Df^N\big(q_i{-}q_j+s(\delta_i{-}\delta_j)\big)-Df^N(\tilde q_i{-}\tilde q_j)\Big)\,(\delta_i(t){-}\delta_j(t))\,\mathrm{d}s\;\mathrm{d}t\right|\\
\le&\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\int_0^1\Big|Df^N\big(q_i-q_j+s(\delta_i-\delta_j)\big)-Df^N(\tilde q_i-\tilde q_j)\Big|\,\mathrm{d}s\,|\delta_i(t)-\delta_j(t)|\,\mathrm{d}t.
\end{aligned}$$

We split the sum into the effect of the moved particle $x_0$, the good particles and the bad particles.

**Moved prt.** Here we need to estimate

$$\int_{t_0}^{t_1}\frac{1}{N}\int_0^1\Big|Df^N\big(q_i-q_0+s(\delta_i(t)-\delta_0(t))\big)-Df^N(\tilde q_i-\tilde q_0)\Big|\,\mathrm{d}s\,|\delta_i(t)-\delta_0(t)|\,\mathrm{d}t.$$

As we assume that the stopping time has not been triggered so far and as we work on the compliment of $\mathcal{B}_1$, we can estimate

$$\begin{aligned}
|q_i-q_0+s(\delta_i(t)-\delta_0(t))-(\tilde q_i-\tilde q_0)|&\le|q_i-\tilde q_i|+|q_0-\tilde q_0|+2C_1(T)|\delta|+|\delta_g|\\
&\le(2C_1(T)+1)c(N).
\end{aligned}$$

This allows us to introduce $h$ by lemma 2.4. With this we obtain

$$\begin{aligned}
&\int_{t_0}^{t_1}\frac{1}{N}\int_0^1\Big|Df^N(q_i-q_0+s(\delta_i(t)-\delta_0(t)))-Df^N(\tilde q_i-\tilde q_0)\Big|\,\mathrm{d}s\,|\delta_i(t)-\delta_0(t)|\,\mathrm{d}t\\
\le&(\delta_0+\delta_g)\frac{1}{N}\int_{t_0}^{t_1}h^N(\tilde q_i-\tilde q_0)(2C_1(T)+1)c(N)\,\mathrm{d}t\\
\le&(2C_1(T)+1)^2\delta c(N)\frac{1}{N}\int_{t_0}^{t_1}h^N(\tilde q_i-\tilde q_0)\,\mathrm{d}t.
\end{aligned}$$

We estimate the integral over $h$ with Lemma 2.7:

$$\begin{aligned}
&(2C_1+1)^2\delta c(N)\frac{1}{N}\int_{t_0}^{t_1}h^N(\tilde q_i-\tilde q_0)\,\mathrm{d}t\\
\le&Cc(N)\delta\frac{1}{N}\frac{C}{c(N)^3\max(\Delta q,\Delta p)}\\
\le&C|\delta|\frac{N^{\frac{2}{9}+\sigma-k(i)\sigma}}{c(N)^2N}\\
\le&C|\delta|N^{\frac{8}{9}+\frac{3}{2}\sigma-1-k(i)\sigma}=\underline{C|\delta|N^{-\frac{1}{9}+\frac{3}{2}\sigma-k(i)\sigma}},
\end{aligned}$$

where we used that for good particles in $R^{k(i)}$ either $\Delta q\ge\frac{1}{2}N^{-\frac{2}{9}+k(i)\sigma-\sigma}$ or $\Delta p\ge\frac{1}{2}N^{-\frac{2}{9}+k(i)\sigma-\sigma}$ by 4.6.

Note that this is true even for the particles in the 0-th ring, as there are no bad particles around $x_0$.

**Good prts.** For the effect of the good particles we need to estimate

$$\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i \text{ good}} \int_0^1 \left| Df^N\big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\big) - Df^N(\tilde{q}_i - \tilde{q}_j) \right| \mathrm{d}s \, |\delta_i(t) - \delta_j(t)| \, \mathrm{d}t.$$

Again, we can use that the stopping time has not been triggered yet in combination with the fact that we work on the complement of $\mathcal{B}_1$ to estimate

$$|q_i - q_j + s(\delta_i(t) - \delta_j(t)) - (\tilde{q}_i - \tilde{q}_j)| \leq |q_i - \tilde{q}_i| + |q_j - \tilde{q}_j| + 2N^{-\frac{1}{9}+2\sigma}\delta \leq 3N^{-\frac{7}{18}+\sigma}.$$

Hence, we can introduce $h$ by lemma 2.4. With this, we obtain

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0,i \text{ good}} \\
&\qquad \cdot \int_0^1 \left| Df^N(q_i - q_j + s(\delta_i(t) - \delta_j(t))) - Df^N(\tilde{q}_i - \tilde{q}_j) \right| \mathrm{d}s \, |\delta_i(t) - \delta_j(t)| \, \mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0,i \text{ good}} |h^N(\tilde{q}_i - \tilde{q}_j)| 3N^{-\frac{7}{18}+\sigma} |\delta_i(t) - \delta_j(t)| \, \mathrm{d}t \\
\leq &C \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0,i \text{ good}} |Df^N(\tilde{q}_i - \tilde{q}_j)| N^{-\frac{1}{18}+2\sigma} (|\delta_i(t)| + |\delta_j(t)|) \, \mathrm{d}t,
\end{aligned}$$

where in the last step we used that $|h| \leq Cc(N)|Df|$.

To make the following estimates a little easier, we start and estimate the $\delta_i$ part, i.e.

$$N^{-\frac{1}{18}+2\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0,i \text{ good}} |Df^N(\tilde{q}_i - \tilde{q}_j)| |\delta_i(t)| \, \mathrm{d}t.$$

For this, we want to use our law of large numbers 2.14 and hence we need to remove a set of bad starting configurations, i.e. for $i \in \{1, \dots, N-1\}$

$$\begin{aligned}
&X^{\geq 1} \in \mathcal{B}_{3a,i} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_3)\\
&:\Longleftrightarrow \ \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j\neq 0,i} \int_{t_1}^{t_2} |Df^N(\tilde{q}_i - \tilde{q}_j)| \mathbf{1}_{G^N(x_i)}(x_j) \, \mathrm{d}s \right.\\
&\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} |Df^N(\tilde{q}_i - {}^1\tilde{y})| \mathbf{1}_{G^N(x_i)}(y) \, \mathrm{d}s k(y) \, \mathrm{d}y \right| > N^{-\frac{1}{9}+\sigma}
\end{aligned}$$

as well as for $R = V = N^{-\frac{1}{6}+\sigma}$

$$\begin{aligned}
&X^{\geq 1} \in \mathcal{B}_{3b,i} \subseteq \mathbb{R}^{6(N-1)}\\
&:\Longleftrightarrow \ \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j\neq 0,i} \int_{t_1}^{t_2} |Df^N(\tilde{q}_i - \tilde{q}_j)| \mathbf{1}_{M^N_{R,V}(x_i)^C}(x_j) \, \mathrm{d}s \right.\\
&\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} |Df^N(\tilde{q}_i - \tilde{y})| \mathbf{1}_{M^N_{R,V}(x_i)^C}(y) \, \mathrm{d}s k(y) \, \mathrm{d}y \right| > N^{-\frac{1}{6}+\sigma}.
\end{aligned}$$

Defining $\mathcal{B}_3$ as the union over these sets, we directly obtain $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_3) \leq C_\gamma N^{-\gamma}$ by applying 6.19 and 6.20 for $\mathcal{B}_{3a}$ and $\mathcal{B}_{3b}$, respectively. From now on we will assume that we work on the complement of $\mathcal{B}_3$.

With these tools ready, we can estimate by

$$\begin{aligned}
&N^{-\frac{1}{18}+2\sigma}\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0,i \text{ good}}\left|Df^N(\tilde{q}_i-\tilde{q}_j)\right||\delta_i(t)|\,\mathrm{d}t\\
\leq& CN^{-\frac{1}{18}+2\sigma}\delta_g^{k(i)}(t_1)\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0,i \text{ good}}\left|Df^N(\tilde{q}_i-\tilde{q}_j)\right|\mathrm{d}t\\
\leq& CN^{-\frac{1}{18}+2\sigma}\delta_g^{k(i)}(t_1)\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\left|Df^N(\tilde{q}_i-\tilde{q}_j)\right|\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}t\\
\overset{\mathcal{B}_3}{\leq}& CN^{-\frac{1}{18}+2\sigma}\delta_g^{k(i)}(t_1)\cdot\left(N^{-\frac{1}{9}+\sigma}+\int_{\mathbb{R}^6}\int_{t_0}^{t_1}\left|Df^N(\tilde{q}_i-{}^1\tilde{y})\right|\mathbf{1}_{G^N(x_i)}(y)\,\mathrm{d}s k_0^N(y)\,\mathrm{d}y\right)\\
\leq& \underline{C\delta_g^{k(i)}(t_1)N^{-\frac{1}{18}+3\sigma}},
\end{aligned}$$

where we estimated the integral by $C(T)\ln(N)$ using Lemma 2.5.

It now remains to estimate

$$N^{-\frac{1}{18}+2\sigma}\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0,i \text{ good}}\left|Df^N(\tilde{q}_i-\tilde{q}_j)\right|\cdot|\delta_j|\,\mathrm{d}t.$$

As we have to deal with a very similar estimate when we deal with 1b, we simply estimate by

$$\sum_{j\neq 0,i \text{ good}}\int_{t_0}^{t_1}\frac{1}{N}\left|Df^N(\tilde{q}_i-\tilde{q}_j)\right|\cdot|\delta_j|\,\mathrm{d}t.$$

We now split the sum one last time to consider the effect of different rings. We start with the rings $R^{k(j)}$ with $k(j)\geq k(i)-1$ and $k(j)\neq K+1$. We implement this restriction via an indicator function by defining $R^{\geqslant}=\bigcup_{k(i)-1}^{K}R^k$. This allows us to use that the stopping time has not been triggered yet, to estimate $\delta_j$ by $\delta_g^{k(j)}\leq N^{-\frac{1}{9}+2\sigma-(k(i)-1)\sigma}$.

As we work on the complement of $\mathcal{B}_3$, we can apply Corollary 2.14 to estimate

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0,i \text{ good}}Df^N(\tilde{q}_i-\tilde{q}_j)\mathbf{1}_{R^{\geqslant}}(x_j)\delta_j\,\mathrm{d}t\right|\\
\leq&\frac{1}{N}\sum_{j\neq 0,i \text{ good}}\int_{t_0}^{t_1}\left|Df^N(\tilde{q}_i-\tilde{q}_j)\right|\mathbf{1}_{R^{\geqslant}}(x_j)N^{-\frac{1}{9}+2\sigma-(k(i)-1)\sigma}|\delta|\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}t\\
\leq& CN^{-\frac{1}{9}+\sigma}N^{-\frac{1}{9}+2\sigma-(k(i)-1)\sigma}|\delta|\\
&+N^{-\frac{1}{9}+2\sigma-(k(i)-1)\sigma}|\delta|\cdot\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6}\left|Df({}^1\tilde{q}_i(t)-{}^1\tilde{z})\right|\mathbf{1}_{G^N(x_i)}(z)\mathbf{1}_{R^{\geqslant}}(z)k(z)\,\mathrm{d}z\,\mathrm{d}s\right|\\
\overset{(\star)}{\leq}& CN^{-\frac{2}{9}+4\sigma-k(i)\sigma}|\delta|+CN^{-\frac{1}{9}+3\sigma-k(i)\sigma}N^{-3\sigma}\ln(N)|\delta|\\
\leq&\underline{N^{-\frac{1}{9}-k(i)\sigma+\sigma}|\delta|}.
\end{aligned}$$

At $(\star)$ we estimated the integral by

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|Df({}^1\tilde{q}_i(t)-{}^1\tilde{z})\right|\mathbf{1}_{G^N(x_i)}(z)\mathbf{1}_{R^{\geqslant}}(z)k(z)\,\mathrm{d}({}^1z,{}^2z)\\
\leq&\int_{|{}^2z-{}^2x_0|\leq CN^{-\sigma}}\int_{|{}^1z-{}^1x_0|\leq CN^{-\sigma}}|Df({}^1\tilde{q}_i(t)-{}^1\tilde{z})|\mathbf{1}_{G^N(x_i)}(z)|k|_\infty\,\mathrm{d}({}^1z,{}^2z)\\
\leq&CN^{-3\sigma}\int_{|{}^1z-{}^1\tilde{q}_i(t)|\leq c(N)}Cc(N)^3|k|_\infty\,\mathrm{d}\,{}^1z\\
&+CN^{-3\sigma}\int_{c(N)\leq|{}^1z-{}^1\tilde{q}_i(t)|\leq 1}C\frac{1}{|{}^1z-{}^1\,\tilde{q}_i(t)|^3}|k|_\infty\,\mathrm{d}\,{}^1z\\
&+CN^{-3\sigma}\int_{|{}^1z-{}^1\tilde{q}_i(t)|\geq 1\text{ and }|{}^1z-{}^1x_0|\leq CN^{-\sigma}}C|k|_\infty\,\mathrm{d}\,{}^1z\\
\leq&CN^{-3\sigma}\ln(N).
\end{aligned}$$

Here we made heavy use of the fact that $k\neq K+1$ to get the $N^{-3\sigma}$ in this estimate. For $k=K+1(=\infty)$:

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\mathbf{1}_{x_j\in R^{K+1}}Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\right|\\
\leq&\frac{1}{N}\sum_{j\neq i,k(j)=K+1}\int_{t_0}^{t_1}\left|Df^N(\tilde{q}_i-\tilde{q}_j)\right|\delta_g^\infty\mathbf{1}_{G^N(x_i)}(x_j)\\
\leq&CN^{-\frac{1}{9}+\sigma}\delta_g^\infty\\
&+\left|\int_{t_1}^{t_2}\delta_g^\infty(s)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sup_{s\in[0,T]}\left|Df^N({}^1\tilde{q}_i(t)-{}^1\tilde{z})\right|\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}sk(z)\,\mathrm{d}z\right|\\
\overset{2.5}{\leq}&CN^{-\frac{1}{9}+\sigma}\cdot\delta_g^\infty+\int_{t_1}^{t_2}C\ln(N)\cdot\delta_g^\infty\,\mathrm{d}s\\
\leq&N^{-\frac{1}{3}}\delta+\int_{t_1}^{t_2}C\ln(N)\cdot\delta_g^\infty\,\mathrm{d}s.
\end{aligned}$$

Now it remains to estimate the effect of the good particles that are significantly closer than $x_i$, i.e. the ones with $x_j\in R^{k(j)}$ with $k(j)<k(i)-1$. We first consider the particles that still are reasonably far from $x_0$, i.e. the particles with $k(j)\geq\frac{1}{18\sigma}+2$. Again we will use an indicator function and define

$$R^{\leqslant}_{\text{far}}=\bigcup_{k=\frac{1}{18\sigma}+2}^{k(i)-2}R^k.$$

For all $x_j\in R^{\leqslant}_{\text{far}}$ we know $\delta_j\leq N^{-\frac{1}{9}+2\sigma-(\frac{1}{18\sigma}+2)\sigma}|\delta|=N^{-\frac{3}{18}}|\delta|$. Also note that the set of such particles is empty if $k(i)\leq\frac{1}{18\sigma}+3$. So by Lemma 4.6 we know that the mean-field distance between $x_i$ and $x_j$ has to stay at least of order $CN^{-\frac{2}{9}+\frac{1}{18}+2\sigma}\geq N^{-\frac{1}{6}+\sigma}$. This means that for $R=V=N^{-\frac{1}{6}+\sigma}$ we can add the indicator function for $M_{R,V}(x_i)^C$, which then allows us to obtain the stronger estimate when switching to the integral using 6.20.

In this case we need to consider each ring $R^k\subseteq R^{\leqslant}_{\text{far}}$ individually. However, as there is a finite amount of rings only depending on $\sigma$, this is no problem.

All in all, for the particles that are closer than $x_i$ but still are far from $x_0$ we can estimate for each $k(j)$ with $R^{k(j)} \subseteq R^{\leqslant}_{\text{far}}$

$$\begin{aligned}&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0,i} \mathbf{1}_{R^k}(x_j)\mathbf{1}_{M_{R,V}(x_i)^C}(x_j) Df^N(\tilde{q}_i - \tilde{q}_j)|\delta_j| \,\mathrm{d}t\right| \\ \leq &\frac{1}{N} \sum_{j\neq 0,i} \int_{t_0}^{t_1} \left|Df^N(\tilde{q}_i - \tilde{q}_j)\right| \delta_g^{k(j)} \mathbf{1}_{M_{R,V}(x_i)^C}(x_j)\mathbf{1}_{CB^{k(j)}}(x_j) \,\mathrm{d}t.\end{aligned}$$

Here we used that particles in the ring $R^{k(j)}$ have to stay in $C(T)B^{k(j)}$ for some constant by Lemma 4.5. This estimate is necessary, as we want to apply our law of large numbers and our definition of $R^{k(j)}$ depends on the complete configuration $x_0, \dots, x_N$.

With this we can now apply the law of large numbers in the form of 6.24 with $\beta_N = \delta_g^{k(j)}$. Note that we can use the stronger version here (see remark 2.15), since we have the suitable restriction on the particles. This yields

$$\begin{aligned}&\frac{1}{N} \sum_{j\neq 0,i} \int_{t_0}^{t_1} \left|Df^N(\tilde{q}_i - \tilde{q}_j)\right| \delta_g^{k(j)} \mathbf{1}_{M_{R,V}(x_i)^C}(x_j)\mathbf{1}_{CB^{k(j)}}(x_j) \,\mathrm{d}t \\ \leq &CN^{-\frac{3}{18}+\sigma}\delta_g^{k(j)}(t_1) \\ &+ \left|\int_{t_1}^{t_2} \delta_g^k(s) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |Df({}^1\widetilde{q}_i(t) - {}^1\widetilde{z})| \mathbf{1}_{M_{R,V}(x_i)^C}(z)\mathbf{1}_{CB^k}(z) \,\mathrm{d}s k(z) \,\mathrm{d}z\right| \\ \overset{(\star\star)}{\leq} &CN^{-\frac{3}{18}+\sigma} \cdot \delta_g^{k(j)}(t_1) + \int_{t_1}^{t_2} CN^{-\frac{6}{9}+3k(j)\sigma} \ln(N) \cdot \delta_g^{k(j)}(s) \,\mathrm{d}s \\ \leq &CN^{-\frac{3}{18}+\sigma} N^{-\frac{3}{18}}|\delta| + CN^{-\frac{6}{9}+3k(j)\sigma} \ln(N) N^{-\frac{2}{9}+2\sigma-k(j)\sigma}|\delta| \\ \leq &CN^{-\frac{1}{3}+\sigma}|\delta| + CN^{-\frac{6}{9}+2k(j)\sigma+k(i)\sigma-k(i)\sigma} \ln(N) N^{-\frac{2}{9}+2\sigma}|\delta| \\ \overset{k\leq k(i)-2}{\leq} &CN^{-\frac{1}{3}+\sigma}|\delta| + CN^{-\frac{6}{9}+3k(i)\sigma-2\sigma} \ln(N) N^{-\frac{2}{9}+2\sigma-k(i)\sigma}|\delta| \\ \overset{k(i)\leq \frac{2}{9\sigma}}{\leq} &CN^{-\frac{1}{3}+\sigma}|\delta| + CN^{-\sigma} \ln(N) N^{-\frac{2}{9}-k(i)\sigma}|\delta| \\ \leq &\underline{CN^{-\frac{1}{3}+\sigma}|\delta| + CN^{-\frac{2}{9}+1.5\sigma-k(i)\sigma}|\delta|}.\end{aligned}$$

In $(\star\star)$ we used a similar trick as before by estimating for each $k(j)$

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |Df({}^1\widetilde{q}_i(t) - {}^1\widetilde{z})| \mathbf{1}_{G^N(y)}(x_i)\mathbf{1}_{CB^{k(j)}}(z) k(z) \,\mathrm{d}({}^1z, {}^2z) \qquad (4.1) \\ \leq &\int_{|{}^2z - {}^2x_0|\leq CN^{-\frac{2}{9}+k(j)\sigma}} \int_{|{}^1z - {}^1x_0|\leq CN^{-\frac{2}{9}+k(j)\sigma}} |Df({}^1\widetilde{q}_i(t) - {}^1\widetilde{z})| \mathbf{1}_{G^N(y)}(x_i)|k|_\infty \,\mathrm{d}({}^1z, {}^2z) \\ \leq &N^{-\frac{6}{9}+3k(j)\sigma} \int_{|{}^1z - {}^1\tilde{q}_i(t)|\leq c(N)} Cc(N)^3|k|_\infty \,\mathrm{d}\,{}^1z \\ &+ N^{-\frac{6}{9}+3k(j)\sigma} \int_{c(N)\leq|{}^1z - {}^1\tilde{q}_i(t)|\leq 1} C\frac{1}{|{}^1z - {}^1\,\tilde{q}_i(t)|^3}|k|_\infty \,\mathrm{d}\,{}^1z \\ &+ N^{-\frac{6}{9}+3k(j)\sigma} \int_{|{}^1z - {}^1\tilde{q}_i(t)|\geq 1 \text{ and } |{}^1z - {}^1x_0|\leq N^{-\sigma}} C|k|_\infty \,\mathrm{d}\,{}^1z \\ \leq &CN^{-\frac{2}{3}+3k(j)\sigma} \ln(N).\end{aligned}$$

Finally, for the particles that are closer than $k(i)-1$ and are also near $x_0$, we estimate the integral over $Df$ with 2.7. This can be done leveraging that by Lemma 4.6 we have $\Delta q \geq N^{-\frac{2}{9}-\sigma+k(i)\sigma}$ or $\Delta p \geq N^{-\frac{2}{9}-\sigma+k(i)\sigma}$ . We can also estimate the number of summands by Lemma 4.7 as we work on the compliment of $\mathcal{B}_{2b}$.

$$\begin{aligned}
&\delta_g^k \frac{1}{N} \sum_{j<k(i)-1 \text{ near}} \int_{t_0}^{t_1} Df^N(\tilde{q}_i - \tilde{q}_j)\,\mathrm{d}t \\
\leq& N^{-\frac{1}{9}+2\sigma-k\sigma}|\delta| \frac{1}{N} \sum_{j<k(i)-1 \text{ near}} \frac{C}{c(N)^2 \max(\Delta q, \Delta p)} \\
\leq& N^{-\frac{1}{9}+2\sigma-k\sigma} C|\delta| N^{12\sigma} \frac{N^{\frac{2}{9}+\sigma-k(i)\sigma}}{c(N)^2 N} \\
\leq& C|\delta| N^{-\frac{1}{9}+16\sigma+\frac{8}{9}-1-k(i)\sigma} \leq \underline{C|\delta| N^{-\frac{1}{9}-k(i)\sigma}}.
\end{aligned}$$

All these estimates satisfy our goal, as they are bounded by $C(T)N^{-\frac{1}{9}-k(i)\sigma+1.5\sigma}|\delta| + C(T)N^{-\frac{1}{3}}$.

This concludes the effect of the other good particles. Next, we look at the effect that the bad particles have.

**Bad prts.** We have to work with weaker estimates when considering the bad particles. However, we know that there are few of them, which can be leveraged to our advantage.

This time we want to estimate

$$C \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i \text{ bad}} \int_0^1 \left| Df^N(q_i - q_j + s(\delta_i(t) - \delta_j(t))) - Df^N(\tilde{q}_i - \tilde{q}_j) \right| \mathrm{d}s\, |\delta_i(t) - \delta_j(t)|\,\mathrm{d}t.$$

We now split the sum further using definition 2.10 with $r_0 = 6N^{-\frac{2}{9}-\sigma}$, $v_0 = N^{-\frac{2}{9}}$ and $\varepsilon = \sigma$.

We start with the classes where the minimal distance between $\tilde{q}_i$ and $\tilde{q}_j$ is larger than $6N^{-\frac{2}{9}-\sigma}$.

For the bad particles we can estimate

$$|q_i - q_j + s(\delta_i(t) - \delta_j(t)) - (\tilde{q}_i - \tilde{q}_j)| \leq |q_i - \tilde{q}_i| + |q_j - \tilde{q}_j| + 2\max(\delta_g, \delta_b) \leq 3N^{-\frac{2}{9}-\sigma}.$$

This means that because $|\tilde{q}_i - \tilde{q}_j| \geq 6N^{-\frac{2}{9}-\sigma}$ we can introduce $h$ by Lemma 2.4, and it remains to estimate

$$C(\delta_b + \delta_g) N^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i \text{ bad}} h^N(\tilde{q}_i - \tilde{q}_j) \mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma}, \infty), (0,\infty)}(x_i)}(x_j)\,\mathrm{d}t.$$

We want to use Lemma 2.13 and hence need to make sure that there are few bad particles, which we have already seen in Lemma 2.12. Note that this was a probabilistic result, so we have to remove a set of bad configurations again. Namely, let

$$\begin{aligned}
X^{\geq 1} \in& \mathcal{B}_{4a,i,(r,R)(v,V)} \subseteq \mathbb{R}^{6(N-1)} \qquad \text{with } i \in \{0, \dots, N-1\} \qquad (\mathcal{B}_4) \\
:\Longleftrightarrow& \left( \sum_{\text{j bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(x_i)}(x_j) > a_N \left\lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \right\rceil \right. \\
& \left. \vee\, |\,\mathcal{M}_b^N(X)| > a_N N^{\frac{2}{3}} \right)
\end{aligned}$$

and

$$\begin{aligned} X^{\geq 1} \in & \mathcal{B}_{4b,i,j(r,R)(v,V)} \subseteq \mathbb{R}^{6(N-1)} \qquad \text{with } i \in \{1,\dots,N-1\} \\ :\Longleftrightarrow & \left( \sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j) \right. \\ & \left. > a_N \left\lceil N^{\frac{2}{3}} R_1^2 R_2^2 \min(\max(V_1,R_1),1)^4 \cdot \min(\max(V_2,R_2),1)^4 \right\rceil \right), \end{aligned}$$

where $a_N = \ln(N)^{\frac{1}{4}}$. We then define $\mathcal{B}_4$ as the union over these sets over all $i$ and all collision classes we are considering in definition 2.10. As the number of collision classes does not depend on $N$, by Lemma 2.12 (for $\mathcal{B}_{4a}$) and 6.26 (for $\mathcal{B}_{4b}$) we have that $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_4) \leq C_\gamma N^{-\gamma}$ for any $\gamma$ and suitable constant. From now on we work on the complement of $\mathcal{B}_4$, i.e. we assume that we can control the number of bad particles in each class. Note that we will only need $\mathcal{B}_{4b}$ later when considering the interaction of two bad particles.

Now we can apply Lemma 2.13 to estimate

$$\begin{aligned} & C(\delta_b + \delta_g) N^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i \text{ bad}} h^N(\tilde{q}_i - \tilde{q}_j) \mathbf{1}_{M^N_{(6N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_i)}(x_j) \, \mathrm{d}t \\ \leq & C N^{-\frac{1}{18}+2\sigma} \delta N^{-\frac{2}{9}-\sigma} \cdot C N^{-\frac{1}{9}+5\sigma} \leq \underline{N^{-\frac{1}{3}}\delta}. \end{aligned}$$

We are left with the particles that come closer than $6N^{-\frac{2}{9}-\sigma}$, which are divided into the following collision classes (see definition 2.10) with $r_0 = 6N^{-\frac{2}{9}-\sigma}, v_0 = N^{-\frac{2}{9}}$:

1. $M^N_{(0,r_0),(0,v_0)}(x_i)$
2. $M^N_{(0,r_0),(N^{l\sigma}v_0,N^{(l+1)\sigma}v_0)}(x_i), 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\sigma \ln(N)} \right\rfloor$
3. $M^N_{(0,r_0),(1,\infty)}(x_i)$

Here we cannot introduce $h$. However, because $x_i$ is a good particle the relative momentum at their closest point has to be larger than $N^{-\frac{2}{9}}$. This means that we do not need to care about the first of theses classes.

In the following we will simply write $x_j \in M(x_i)$ as a placeholder, to later plug in one of the remaining classes. When we use $v, V$ or $R$ its always meant with respect to the class.

With this being said, we estimate $Df$ by Lemma 2.7 to obtain

$$C\int_{t_0}^{t_1}\frac{1}{N}\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}} \cdot\int_0^1\left|Df^N\Big(q_i-q_j+s(\delta_i(t)-\delta_j(t))\Big)-Df^N\Big(\tilde{q}_i-\tilde{q}_j\Big)\right|\mathrm{d}s\,|\delta_i(t)-\delta_j(t)|\,\mathrm{d}t$$

$$\leq C\int_{t_0}^{t_1}\frac{1}{N}\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}} \cdot\int_0^1\left|Df^N\Big(q_i-q_j+s(\delta_i(t)-\delta_j(t))\Big)\right|+\left|Df^N\Big(\tilde{q}_i-\tilde{q}_j\Big)\right|\mathrm{d}s\,|\delta_i(t)-\delta_j(t)|\,\mathrm{d}t$$

$$\leq C\frac{1}{c(N)^2v}\frac{1}{N}a_n\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}}(|\delta_j|+|\delta_i|).$$

Note that for the estimate of $\left|Df^N\Big(q_i-q_j+s(\delta_i(t)-\delta_j(t))\Big)\right|$ we need the second part of Lemma 2.7. We apply this with $y_1=x_i+s\delta_i, y_2=x_j+s\delta_j$ and $z_1=x_i, z_2=x_j$.

To verify the conditions of the lemma, we use $|{}^1\widetilde{x}_i(t_{\min})-{}^1\widetilde{x}_j(t_{\min})|\leq 6N^{-\frac{2}{9}-\sigma}$ and $|{}^2\widetilde{x}_i(t_{\min})-{}^2\widetilde{x}_j(t_{\min})|\geq N^{-\frac{2}{9}}$. This yields

$$N^{\frac{\sigma}{2}}|{}^1\widetilde{x}_i(t_{\min})-{}^1\widetilde{x}_j(t_{\min})|\leq N^{-\frac{2}{9}}\leq|{}^2\widetilde{x}_i(t_{\min})-{}^2\widetilde{x}_j(t_{\min})|=\Delta p$$

as well as

$$\sup|\widetilde{x}_i(t)-x_i(t)-s\delta_i|\leq N^{-\frac{2}{9}-\sigma}+\delta_g\leq 2N^{-\frac{2}{9}-\sigma}\leq N^{-\frac{\sigma}{2}}\Delta p$$

as well as

$$\sup|\widetilde{x}_j(t)-x_j(t)-s\delta_j|\leq N^{-\frac{2}{9}-\sigma}+\delta_b\leq 2N^{-\frac{2}{9}-\sigma}\leq N^{-\frac{\sigma}{2}}\Delta p.$$

Moving on with our estimate, we use $R=6N^{-\frac{2}{9}-\sigma}$, $v\geq N^{-\frac{2}{9}}$ as well as $c(N)^2\geq N^{-\frac{2}{3}-\frac{\sigma}{2}}$ to estimate the $\delta_i$ part first. This is done easily, as the number of elements in the sum can be estimated by $a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil$ using Corollary 2.12. Thus

$$\begin{aligned}
&C\frac{1}{c(N)^2v}\frac{1}{N}a_n\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}}|\delta_i|\\
&\leq C\delta_i\frac{1}{c(N)^2v}\frac{1}{N}a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil\\
&\leq C\delta_g^{k(i)}\frac{1}{c(N)^2v}\frac{1}{N}a_NN^{\frac{2}{3}}R^2\min(\max(V,R),1)^4+C\delta_g^{k(i)}\frac{1}{c(N)^2v}\frac{1}{N}a_N\\
&\leq C\delta_g^{k(i)}N^{-\frac{1}{9}+\sigma}\frac{\min(V,1)}{v}+C\delta_g^{k(i)}N^{-\frac{1}{9}+\sigma}\leq \underline{CN^{-\frac{1}{9}+2\sigma}\delta_g^{k(i)}}.
\end{aligned}$$

Here we used that in remaining classes either $v=1$ or $V=N^\sigma v$.

For the $\delta_j$ we need to further split the sum. First, we consider the case of the far off particles, i.e. $k(j)\geq k(i)-1$. Note that here $k(j)=K+1$ is allowed. Again we used

2.12 and that all the $\delta_b^{k(j)}$ are bounded by $N^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta|$, as the stopping time has not been triggered yet.

$$\begin{aligned}
&C\frac{1}{c(N)^2 v}\frac{1}{N}\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}}|\delta_j|\\
\leq& CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}\delta\frac{1}{c(N)^2 v}\frac{1}{N}a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil\\
\leq& CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta|\frac{1}{c(N)^2 v}\frac{1}{N}a_N N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\\
&+CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta|\frac{1}{c(N)^2 v}\frac{1}{N}a_N\\
\leq& CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta|N^{-\frac{1}{9}+\sigma}\frac{\min(V,1)}{v}\\
&+CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta|N^{-\frac{1}{9}+\sigma}\\
\leq& \underline{N^{-\frac{1}{9}+\sigma-k(i)\sigma}\delta},
\end{aligned}$$

where again we used that $v=1$ or $V=N^\sigma v$.

Finally, if $k(j)<k(i)-1$, we know that $\Delta p\geq CN^{-\frac{2}{9}+k(i)\sigma-\sigma}$ by Lemma 4.6 and the fact that in the collision classes we are considering right now $\Delta q$ is small. We also know that the corresponding particles stay close to $x_0$. To leverage this, we use Lemma 6.26 with $R_2,V_2=CN^{-\frac{2}{9}+k(j)\sigma}$ to obtain a strong enough bound for the number of particles we have to consider.

$$\begin{aligned}
&C\frac{1}{c(N)^2 v}\frac{1}{N}\sum_{\substack{j\neq i \text{ bad}\\ x_j\in M(x_i)}}|\delta_j|\\
\leq\ & C\delta_b^{k(j)}\frac{1}{c(N)^2 v}\frac{1}{N}a_N\lceil N^{\frac{2}{3}}R_1^2\min(V_1,1)N^{6(-\frac{2}{9}+k(j)\sigma)}\rceil\\
\leq\ & C\delta_b^{k(j)}\frac{\min(1,V_1)}{v}N^{\frac{2}{3}-\frac{4}{9}-2\sigma+\frac{2}{3}+\sigma-1}N^{6(-\frac{2}{9}+k(j)\sigma)}+C\delta_b^{k(j)}\frac{1}{c(N)^2 v}\frac{1}{N}a_N\\
\leq\ & C|\delta|N^{-\frac{1}{18}+2\sigma-k(j)\sigma}N^{-\frac{1}{9}-\sigma}N^{-\frac{12}{9}+6k(j)\sigma}+C|\delta|N^{-\frac{1}{18}-k(j)\sigma+2\sigma}N^{-\frac{1}{9}-k(i)\sigma}\\
\leq\ & C|\delta|N^{-\frac{1}{18}+2\sigma}N^{-\frac{1}{9}-\sigma}N^{-\frac{12}{9}+5k(j)\sigma}+C|\delta|N^{-\frac{1}{18}-k(j)\sigma+2\sigma}N^{-\frac{1}{9}-k(i)\sigma}\\
\leq\ & \underline{N^{-\frac{1}{9}-k(i)\sigma+1.5\sigma}|\delta|},
\end{aligned}$$

where we used

$$N^{-\frac{12}{9}+5k\sigma}=N^{-\frac{12}{9}+5k\sigma+k(i)\sigma-k(i)\sigma}\leq N^{-\frac{12}{9}+6k(i)\sigma-k(i)\sigma}\leq N^{-k(i)\sigma}.$$

Combining all of this, we obtain

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}\left(f(q_i-q_j)-f(q_i^\delta-q_j^\delta)\right)-\frac{1}{N}\sum_{j\neq i}Df^N(\tilde{q}_i-\tilde{q}_j)(\delta_i(t)-\delta_j(t))\,\mathrm{d}t\right|\\
\leq& \underline{\underline{C(T)N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta|+C(T)N^{-\frac{1}{3}}|\delta|+\int_{t_1}^{t_2}C\ln(N)\cdot\delta_g^\infty\,\mathrm{d}s.}}
\end{aligned}$$

Est. 1b) Clearly, we have

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)(\delta_i-\delta_j)\,\mathrm{d}t - \int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_i\,\mathrm{d}t\right| \\
=&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\,\mathrm{d}t\right| \\
\leq&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0,i \text{ good}} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\,\mathrm{d}t\right| + \left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0,i \text{ bad}} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\,\mathrm{d}t\right| \\
&+\left|\int_{t_0}^{t_1} \frac{1}{N} Df^N(\tilde{q}_i-\tilde{q}_0)\delta_0\,\mathrm{d}t\right|
\end{aligned}$$

For the third summand we can apply Lemma 2.7 and estimate by

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N} Df^N(\tilde{q}_i-\tilde{q}_0)\delta_0\,\mathrm{d}t\right| \leq C\delta_0(t_1)\frac{1}{N}\frac{1}{c(N)^2\max(\Delta q,\Delta p)} \\
\leq& CN^{-\frac{1}{9}-(k(i)-1)\sigma+\frac{1}{2}\sigma}|\delta| \leq \underline{CN^{-\frac{1}{9}-k(i)\sigma+\frac{3}{2}\sigma}\delta},
\end{aligned}$$

where we used that as $x_i \in R^{k(i)}$ either $\Delta q \geq N^{-\frac{2}{9}+(k(i)-1)\sigma}$ or $\Delta p \geq N^{-\frac{2}{9}+(k(i)-1)\sigma}$.
The second summand can be split further. If $\Delta q > 6N^{-\frac{2}{9}-\sigma}$, we can use 2.13 to obtain

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i \text{ bad}} Df^N(\tilde{q}_i-\tilde{q}_j)\mathbf{1}_{M_{(6N^{-\frac{2}{9}-\sigma},\infty)(0,\infty)}}(x_i)(x_j)\delta_j\,\mathrm{d}t\right| \\
\leq& C(T)N^{-\frac{1}{3}+4\sigma}\delta_b \leq \underline{N^{-\frac{1}{3}}|\delta|}
\end{aligned}$$

We have just handled the case $\Delta q \leq 6N^{-\frac{2}{9}-\sigma}$ in the section above, and hence we can estimate by

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i \text{ bad}} Df^N(\tilde{q}_i-\tilde{q}_j)\mathbf{1}_{M_{(0,6N^{-\frac{2}{9}-\sigma})(0,\infty)}}(x_i)(x_j)\delta_j\,\mathrm{d}t\right| \\
\leq& \underline{C(T)N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta| + C(T)N^{-\frac{1}{3}}|\delta| + \int_{t_1}^{t_2} C\ln(N)\cdot\delta_g^\infty\,\mathrm{d}s.}
\end{aligned}$$

The first summand is the summand we have already mentioned in our estimates for 1a) and there we have proven that

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0,i \text{ good}} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\right| \\
&\leq \underline{N^{-\frac{1}{9}-k(i)\sigma+1.5\sigma}|\delta| + C(T)N^{-\frac{1}{3}}|\delta| + \int_{t_0}^{t_1} C\ln(N)\delta_g^\infty.}
\end{aligned}$$

Combining all these estimates, we get the same estimate as for 1a)

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)(\delta_i-\delta_j) - \int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_i\right| \\
&\leq \underline{\underline{CN^{-\frac{1}{3}}|\delta| + CN^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta| + C\ln(N)\int_{t_0}^{t_1}\delta_g^\infty(t)\,\mathrm{d}t.}}
\end{aligned}$$

Est. 1c) This is a simple application of our law of large numbers, i.e. we can apply Lemma 2.14 to estimate

$$\begin{aligned}&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i}Df^N(\tilde{q}_i-\tilde{q}_j)\delta_i\,\mathrm{d}t-\int_{\mathbb{R}^6}\int_{t_0}^{t_1}Df^N(\tilde{q}_i-{}^1\tilde{y})k(y)\mathbf{1}_{G^N(x_i)}(y)\delta_i\,\mathrm{d}t\,\mathrm{d}y\right|\\ \leq&\underline{\underline{CN^{-\frac{1}{9}+\sigma}\delta_g^{k(i)}(t_1)}},\end{aligned}$$

where we again used that we work on the complement of $\mathcal{B}_3$.

Est. 1d) Here we simply estimate $Df$ by $|Df|_\infty$ and then use Lemma 2.11 to estimate the probability of particles coming this close.

$$\begin{aligned}&\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6}Df^N(\tilde{q}_i-{}^1y)k(y)\mathbf{1}_{G^N(x_i)}(y)\delta_i(t)\,\mathrm{d}t-\int_{t_0}^{t_1}\overline{Df}(\tilde{q}_i)\delta_i(t)\,\mathrm{d}t\right|\\ =&\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6}Df^N(\tilde{q}_i-{}^1y)k(y)\mathbf{1}_{G^N(x_i)^C}(y)\delta_i(t)\,\mathrm{d}t\right|\\ \leq&C\delta_gN^{1+\sigma}\int_{\mathbb{R}^6}k(y)\mathbf{1}_{G^N(x_i)^C}(y)\,\mathrm{d}y\\ \leq&C\delta_gN^{1+\sigma}\,\mathbb{P}\left(y\in G^N(x_i)^C\right)\\ \leq&CN^{1+\sigma}\delta_g(N^{-\frac{2}{9}-\sigma})^2(N^{-\frac{2}{9}})^4\\ \leq&CN^{1-\frac{12}{9}+\sigma-2\sigma}\delta_g\\ =&CN^{-\frac{1}{3}-\sigma}\delta_g\\ \leq&\underline{\underline{CN^{-\frac{5}{9}+3\sigma}|\delta|}}.\end{aligned}$$

Est. 1e) Using Lemma 2.5, we have

$$\left|\int_{t_0}^{t_1}\overline{Df}(\tilde{q}_i)^1\delta_i\right|\mathrm{d}t\leq\int_{t_0}^{t_1}{}^1\delta_g^k\int_{\mathbb{R}^3}|Df^N(\tilde{q}_i-y)|\tilde{k}_t(y)\,\mathrm{d}t\leq\underline{\underline{\int_{t_0}^{t_1}C\ln(N)\cdot{}^1\delta_g^k(t)\,\mathrm{d}t}}.$$

Now that we have a strong estimate on the difference in force, we can apply Grönwall's inequality to conclude that the stopping time was not triggered by a good particle.

### 4.1.2 Grönwall estimate

Combining all the previous estimates, we conclude that if $x_i$ is a good particle (and not the moved particle $x_0$) for any $0 \leq t_0 < t_1 \leq \tau$, we have

$$\begin{aligned}&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} \left( f\Big(q_i(t) - q_j(t)\Big) - f\Big(q_i^\delta(t) - q_j^\delta(t)\Big)\right) \mathrm{d}t\right| \\ \leq & C \int_{t_0}^{t_1} \ln(N) \cdot ({}^1\delta_g^{k(i)} + {}^1\delta_g^\infty) + C(T) N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta| + C(T)N^{-\frac{1}{3}}|\delta|.\end{aligned}$$

Before we apply our Grönwall lemma, we can simplify this a bit. First note that if $k(i) = \infty$, we can simply add up ${}^1\delta_g^{k(i)}$ and ${}^1\delta_g^\infty$. Otherwise, we know $K + 1 = \frac{2}{9\sigma}$ and hence $\delta_g^{K+1} \leq N^{-\frac{1}{9}+2\sigma-(K+1)\sigma}|\delta| \leq N^{-\frac{1}{3}+\sigma}$, which for $k(i) \leq K$ is a good enough estimate. Thus, after changing the constants, we can leave out the ${}^1\delta_g^\infty$.

Similarly, we also note that as $k(i) \leq \frac{2}{9\sigma}$, we have $N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta| \geq N^{-\frac{1}{3}+1.5\sigma}|\delta|$. This proves that we can (after changing the constants) leave out the $N^{-\frac{1}{3}}|\delta|$ in the estimate.

This leaves us with the new estimate

$$\begin{aligned}&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i} \left( f\Big(q_i(t) - q_j(t)\Big) - f\Big(q_i^\delta(t) - q_j^\delta(t)\Big)\right) \mathrm{d}t\right| \\ \leq & C \int_{t_0}^{t_1} \ln(N) \cdot {}^1\delta_g^{k(i)} + C(T) N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta|.\end{aligned}$$

Now it is time for our Grönwall argument. We have already proven a useful version of Grönwall in the chapter on our modification of Graß's result. Hence, we can simply apply it again here.

We apply Lemma 3.2 to $\delta_g^{k(i)}$. Using $v_1 = {}^1\delta_g^{k(i)}, v_2 = {}^2\delta_g^{k(i)}$ as well as $C(T,N) = C(T)N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta|$, one easily checks that the conditions of the lemma are met. Hence, with $\epsilon = \frac{\sigma}{4}$ we obtain ${}^2\delta_g(\tau) \leq C(T)N^{-\frac{1}{9}+1.75\sigma-k(i)\sigma}|\delta|$. This then implies that

$${}^1\delta_g(\tau) \leq C(T) \cdot {}^2\delta_g \leq C(T)N^{-\frac{1}{9}+1.75\sigma-k(i)\sigma}|\delta|.$$

Combining this, we obtain that for large enough $N$ (depending on $C(T), \sigma$) we have

$$\delta_g(\tau) \leq N^{-\frac{1}{9}+2\sigma-k(i)\sigma}|\delta|.$$

Thus, the stopping-time was not triggered by a good particle.

## 4.2 Second case: $x_i$ is a bad particle

This works very similar to our estimates for the good particles. However, this time the mean-field trajectories might be a significantly worse estimate for these particles. We circumvent this problem using the following ansatz from Graß. We define a cloud of auxiliary particles around our bad particle, i.e. we define $J_N = \{-\lceil N^{\frac{1}{9}}\rceil, \dots, -1, 0, 1, \dots \lceil N^{\frac{1}{9}}\rceil\}^6$. Then for $(k_1, \dots, k_6) \in J_N$, we define $x_{i,k_1,\dots,k_6} = x_i + \sum_{j=1}^6 k_j N^{-\frac{4}{9}+2\sigma} e_j$ where $e_j$ is the $j$-th basis vector in $\mathbb{R}^6$.

In the proof in Theorem 3.1 we were able to show that for any $t_0 \leq \tau$ there is an auxiliary particle $x_{i,k_1,\dots,k_6}$ that satisfies

$$\sup_{s\in[t_0,t+N^{-\frac{1}{18}}]} |{}^1x_i - {}^1\widetilde{x}_{i,k_1,\dots,k_6}| \leq N^{-\frac{4}{9}+3\sigma}, \qquad \sup_{s\in[t_0,t+N^{-\frac{1}{18}}]} |{}^2x_i - {}^2\widetilde{x}_{i,k_1,\dots,k_6}| \leq N^{-\frac{7}{18}+\frac{3}{4}\sigma}.$$

With this in mind, we define $t_0 = 0$ and $t_{k+1} = t_k + N^{-\frac{1}{18}}$. For now we fix any $k$, and for ease of notation we write $\hat{x}_i(s) = \widetilde{X}_{i,k_1,\dots,k_6}$ for the auxiliary particle that satisfies the condition above on the interval $s \in [t_k, t_{k+1}]$. So $\hat{x}_i$ is the mean-field trajectory of an auxiliary particle.

Using $\hat{x}_i$ instead of $\tilde{x}_i$, we can salvage most of our estimates for the good particles. There is one small problem we have to deal with. Good particles (that have a reasonable distance to $\tilde{x}_i$) may come too close to $\hat{x}_i$. For this reason we use $\hat{G}^N(\hat{x}_i) = G^{N,(t_k,t_{k+1})}_{4N^{-\frac{2}{9}-\sigma},0.5N^{-\frac{2}{9}}}(\hat{x}_i)$ in our estimates. This is helpful as now, if $x_j \in G^N(x_i)$, we can conclude $x_j \in \hat{G}^N(\hat{x}_i)$. Assume $z \in G^N(x_i)$, i.e. for all $s \in [0, T]$ we have

$$|{}^1\widetilde{z} - {}^1\widetilde{x}_i| \leq 6N^{-\frac{2}{9}-\sigma} \implies |{}^2\widetilde{z} - {}^2\widetilde{x}_i| > N^{-\frac{2}{9}}.$$

This then implies that on $[t_k, t_{k+1}]$ and for large enough $N$

$$\begin{aligned}
&|{}^1\widetilde{z} - {}^1\hat{x}_i| \leq 4N^{-\frac{2}{9}-\sigma} \\
\implies &|{}^1\widetilde{z} - {}^1\widetilde{x}_i| \leq |{}^1\widetilde{z} - {}^1\hat{x}_i| + |x_i - \hat{x}_i| + |x_i - \tilde{x}_i| \leq 4N^{-\frac{2}{9}-\sigma} + N^{-\frac{4}{9}+3\sigma} + N^{-\frac{2}{9}-\sigma} \leq 6N^{-\frac{2}{9}-\sigma} \\
\implies &|{}^2\widetilde{z} - {}^2\widetilde{x}_i| > N^{-\frac{2}{9}} \\
\implies &|{}^2\widetilde{z} - {}^2\hat{x}_i^\delta| \geq |{}^2\widetilde{z} - {}^2\widetilde{x}_i| - |\tilde{x}_i - x_i| - |\hat{x}_i - x_i| > N^{-\frac{2}{9}} - N^{-\frac{2}{9}-\sigma} - N^{-\frac{7}{18}+\sigma} \geq \frac{1}{2}N^{-\frac{2}{9}}.
\end{aligned}$$

We also need to make sure that this new definition does not mess up our definition of rings. To this end we prove the following modified version of Lemma 4.6 and Lemma 4.5:

**Lemma 4.8.** *Let $k > j + 1$ and $x_i \in R^k$ and $z \in R^j$. Then*

$$\hat{x}_i \in M_{\frac{1}{3}N^{-\frac{2}{9}+(k-1)\sigma},\frac{1}{3}N^{-\frac{2}{9}+(k-1)\sigma}}(z)^C$$

*and*

$$\frac{1}{2}N^{-\frac{2}{9}+(k-1)\sigma} < \inf_{s\in[0,T]} |\hat{x}_i - \tilde{x}_0| \leq \sup_{s\in[0,T]} |\hat{x}_i - \tilde{x}_0| < CN^{-\frac{2}{9}+k\sigma}.$$

*If $k = 0$, the upper bound is $C\ln(N)N^{-\frac{2}{9}}$ instead.*

*Proof.* Recall from Lemma 4.6 that we have $\tilde{x}_i \in M_{\frac{1}{2}N^{-\frac{2}{9}+(k-1)\sigma}, \frac{1}{2}N^{-\frac{2}{9}+(k-1)\sigma}}(z)^C$. As the distance between $\tilde{x}_i$ and $\hat{x}_i$ in phase-space is bounded by $N^{-\frac{1}{3}+\sigma}$, which is of significantly smaller order in $N$, the first part follows. For the second estimate we note that

$$\frac{1}{2}N^{-\frac{2}{9}+(k-1)\sigma} < \inf_{s\in[0,T]} |\hat{x}_i - \tilde{x}_0|$$

has to be true. For $k = 0$, this is the fact that $x_0$ is assumed to be a good particle, i.e. $|\tilde{x}_i - \tilde{x}_0| \geq 6N^{-\frac{2}{9}-\sigma}$. Using that $\tilde{x}_i$ and $\hat{x}_i$ stay very close, this yields the estimate. For $k \geq 1$, this has to be true as otherwise (again using how close $\hat{x}_i$ is to $\tilde{x}_i$) $\tilde{x}_i$ would be in $R^{k-1}$ or even closer to $x_0$.

The upper bound follows directly from Lemma 4.5 where we have seen that $\sup_t |\tilde{x}_i - \tilde{x}_0| \leq CN^{-\frac{2}{9}+k\sigma}$, with the same caveat for $k = 0$. Again, we simply use the triangle inequality and the fact that $|\hat{x}_i - \tilde{x}_i| \leq N^{-\frac{1}{3}+\sigma}$ to obtain the required estimate (with a slightly bigger $C$). □

The reason the estimates will become worse for bad particles is that we have to do $N^{\frac{1}{18}}$ estimates to cover the time period $T$.

### 4.2.1 Estimating the difference in force

With all this done, we can estimate

$$\begin{aligned}
&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i}\Big(f\big(q_i^\delta(t)-q_j^\delta(t)\big)-f\big(q_i(t)-q_j(t)\big)\Big)\,\mathrm{d}t\right| \\
\leq&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i}\Big(f\big(q_i^\delta(t)-q_j^\delta(t)\big)-f\big(q_i(t)-q_j(t)\big)\Big)\mathbf{1}_{\hat{G}^N(\hat{x}_i)^C}(x_j)\,\mathrm{d}t\right| && \text{(2a)}\\
&+\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i}\Big(f\big(q_i-q_j\big)-f\big(q_i^\delta-q_j^\delta\big)-Df^N(\hat{q}_i-\tilde{q}_j)({}^1\delta_i-{}^1\delta_j)\Big)\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j)\,\mathrm{d}t\right| && \text{(2b)}\\
&+\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\hat{q}_i-\tilde{q}_j)({}^1\delta_i-{}^1\delta_j)\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j)\,\mathrm{d}t\right. \\
&\qquad\left.-\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\hat{q}_i-\tilde{q}_j)\delta_i\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j)\,\mathrm{d}t\right| && \text{(2c)}\\
&+\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\hat{q}_i-\tilde{q}_j)\delta_i\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j)\,\mathrm{d}t\right. \\
&\qquad\left.-\int_{t_k}^{t_{k+1}}\int_{\mathbb{R}^6} Df^N(\hat{q}_i-{}^1\tilde{y})k(y)\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(y)\delta_i\,\mathrm{d}t\right| && \text{(2d)}\\
&+\left|\int_{t_k}^{t_{k+1}}\int_{\mathbb{R}^6} Df^N(\hat{q}_i-{}^1\tilde{y})k(y)\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(y)\delta_i\,\mathrm{d}t-\int_{t_k}^{t_{k+1}}\overline{Df}(\hat{q}_i)\delta_i\,\mathrm{d}t\right| && \text{(2e)}\\
&+\left|\int_{t_k}^{t_{k+1}}\overline{Df}(\hat{q}_i)\delta_i\,\mathrm{d}t\right| && \text{(2f)}\\
\leq& C\int_{t_k}^{t_{k+1}} \ln(N)^1\delta_b\,\mathrm{d}t + C(T)N^{-\frac{1}{9}+\sigma}|\delta|.
\end{aligned}$$

The estimate in the last line is not trivial at all, and we will now estimate the terms (2a)-(2f) individually to reach the desired estimate.

Est. 2a) This estimate was not necessary when working with the good particles, as it estimates the effect of bad interactions. Again, we need to remove some undesirable starting configurations. This time we define

$$X^{\geq 1} \in \mathcal{B}_{5,i,k_1,\dots,k_6} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_5)$$
$$\iff \sum_{j\neq i} \mathbf{1}_{M^N_{6N^{-\frac{2}{9}-\sigma},N^{-\frac{2}{9}}}(x_{i,k_1,\dots,k_6})}(x_j) \geq \ln(N)^{\frac{1}{4}}.$$

By Lemma 2.11 we know that $\mathbb{P}(y \in \mathbb{R}^6 : y \notin G^N(z)) \leq CN^{-\frac{4}{3}-2\sigma}$ for all $z \in \mathbb{R}^6$. Hence, for $a_N = \lceil (\ln N)^{\frac{1}{4}} \rceil$ we can estimate

$$\mathbb{P}(X^{\geq 1} \in \mathcal{B}_{5,i,k_1,\dots,k_6}) \leq \binom{N}{a_N} (CN^{-\frac{4}{3}-2\sigma})^{a_N} \leq \frac{N^{a_N}}{(Ca_N)^{a_N}} (CN^{-\frac{4}{3}-2\sigma})^{a_N} \leq N^{a_N(-\frac{1}{3}-\sigma)}.$$

Here the last estimate holds for large enough $N$ as $a_N \to \infty$.

Defining $\mathcal{B}_5$ as the union over all $i \in \{1,\dots,N\}$ and $(k_1,\dots,k_6) \in J_N$, we have

$$\mathbb{P}(X^{\geq 1} \in \mathcal{B}_5) \leq N(CN^{\frac{1}{6}})^6 N^{a_N(-\frac{1}{3}-\sigma)} \leq N^{-a_N\frac{1}{3}}.$$

Clearly, this drops faster than any power of $N$.

Working on the complement of $\mathcal{B}_5$, we can use the mean value theorem to estimate

$$\begin{aligned}
&\left| \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i} \Big( f\big(q_i(t) - q_j(t)\big) - f\big(q_i^\delta(t) - q_j^\delta(t)\big) \Big) \mathbf{1}_{\hat{G}^N(\hat{x}_i)^C}(x_j)\, \mathrm{d}t \right| \\
\leq &\left| \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i} \int_0^1 Df^N\Big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\Big)\, \mathrm{d}s\, ({}^1\delta_i(t) - {}^1\delta_j(t)) \mathbf{1}_{\hat{G}^N(\hat{x}_i)^C}(x_j)\, \mathrm{d}t \right| \\
\leq &\left| \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i} \frac{C}{c(N)^3} \cdot {}^1\delta_b^k \mathbf{1}_{\hat{G}^N(\hat{x}_i)^C}(x_j)\, \mathrm{d}t \right| \\
\leq &\int_{t_k}^{t_{k+1}} \ln(N)^{\frac{3}{4}} \cdot {}^1\delta_b^{k(i)} \sum_{j\neq i} \mathbf{1}_{\hat{G}^N(\hat{x}_i)^C}(x_j)\, \mathrm{d}t \\
= &\int_{t_k}^{t_{k+1}} \ln(N)^{\frac{3}{4}} \cdot {}^1\delta_b^{k(i)} \sum_{j\neq i} \mathbf{1}_{M^N_{6N^{-\frac{2}{9}-\sigma},N^{-\frac{2}{9}}}(\hat{x}_i)}(x_j)\, \mathrm{d}t \\
\leq &\int_{t_k}^{t_{k+1}} \ln(N)^{\frac{3}{4}} \cdot {}^1\delta_b^{k(i)} \ln(N)^{\frac{1}{4}}\, \mathrm{d}t \\
\leq &\underline{\underline{C \int_{t_k}^{t_{k+1}} \ln(N) \cdot {}^1\delta_b^{k(i)}\, \mathrm{d}t.}}
\end{aligned}$$

Here we used that all the $x_j \in \hat{G}^N(\hat{x}_i)^C$ must have a bad interaction with $x_i$ by our definition of $\hat{G}^N$ and hence have to be in the same ring $R^k$.

We also point out that this is the estimate that prevents us from shrinking our cut-off. While we can handle the $\ln(N)$ in front of our $\delta_b^k$, any power of $N$ would be detrimental to our Grönwall estimate later on.

Est. 2b) In a first step we can use the mean-value-theorem to rewrite the term into

$$\begin{aligned}
&\Big| \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i} \mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j) \\
&\qquad \cdot \int_0^1 \Big( Df^N\big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\big) - Df^N(\hat{q}_i - \tilde{q}_j) \Big) \big(\delta_i(t) - \delta_j(t)\big) \,\mathrm{d}s \,\mathrm{d}t \Big| \\
\leq & C \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i} \mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j) \\
&\qquad \cdot \int_0^1 \Big| Df^N\big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\big) - Df^N(\hat{q}_i - \tilde{q}_j) \Big| \,\mathrm{d}s \,\big|\delta_i(t) - \delta_j(t)\big| \,\mathrm{d}t.
\end{aligned}$$

Again, we need to consider the effect of the good and bad particles separately.

**Good prts.** In this case we know that for all times in $[t_k, t_{k+1}]$

$$|q_i - q_j + s(\delta_i(t) - \delta_j(t)) - (\hat{q}_i - \tilde{q}_j)| \leq |q_i - \hat{q}_i| + |q_j - \tilde{q}_j| + |\delta_i| + |\delta_g| \leq (2C_1 + 1)c(N).$$

Hence, we can introduce $h$ by lemma 2.4. Again, it is helpful to split off the effect of $x_0$ and use that for $i \neq 0$ we can strengthen the estimate above to $\leq 3N^{-\frac{4}{9}+3\sigma}$. With this we obtain:

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i \text{ good}} \\
&\qquad \cdot \int_0^1 \Big| Df^N\big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\big) - Df^N(\hat{q}_i - \tilde{q}_j) \Big| \,\mathrm{d}s \,\big|\delta_i(t) - \delta_j(t)\big| \,\mathrm{d}t \\
\leq & \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq 0,i \text{ good}} h^N(\hat{q}_i - \tilde{q}_j) \cdot 3N^{-\frac{4}{9}+3\sigma} |\delta_i(t) - \delta_j(t)| \,\mathrm{d}t \\
&+ (\delta_0 + \delta_b) \frac{1}{N} \int_{t_k}^{t_{k+1}} h^N(\hat{q}_i - \tilde{q}_0)(2C_1 + 1)c(N) \,\mathrm{d}t \\
\leq & 3N^{-\frac{4}{9}+3\sigma} \frac{1}{N} \sum_{j\neq 0,i \text{ good}} \int_{t_k}^{t_{k+1}} h^N(\hat{q}_i - \tilde{q}_j) \cdot (\delta_g^{k(j)}(t) + |\delta_i|(t)) \,\mathrm{d}t \\
&+ (2C_1 + 1)^2 |\delta| c(N) \frac{1}{N} \int_{t_k}^{t_{k+1}} h^N(\hat{q}_i - \tilde{q}_0) \,\mathrm{d}t.
\end{aligned}$$

We start by estimating the effect from the moved particle $x_0$, i.e. the second summand. To this end, we estimate the integral over $h$ with Lemma 2.7:

$$\begin{aligned}
&(2C_1 + 1)^2 |\delta| c(N) \frac{1}{N} \int_{t_k}^{t_{k+1}} h^N(\hat{q}_i - \tilde{q}_0) \,\mathrm{d}t \\
\leq & C c(N) |\delta| \frac{1}{N} \frac{C}{c(N)^3 \max(\Delta q, \Delta p)} \\
\leq & C |\delta| \frac{N^{\frac{2}{9}+\sigma-k(i)\sigma}}{c(N)^2 N} \\
\leq & C |\delta| N^{\frac{8}{9}+\frac{3}{2}\sigma-1-k(i)\sigma} = \underline{C|\delta| N^{-\frac{1}{9}+\frac{3}{2}\sigma-k(i)\sigma}},
\end{aligned}$$

where we used that for particles in $R^{k(i)}$ either $\Delta q \geq \frac{1}{3} N^{-\frac{2}{9}+k(i)\sigma-\sigma}$ or $\Delta p \geq \frac{1}{3} N^{-\frac{2}{9}+k(i)\sigma-\sigma}$ by Lemma 4.8.

Note that this is true even for the particles in the 0-th ring as there are no bad particles around $x_0$.

It remains to find a suitable estimate for the first summand. Here we need to define another set of bad starting configurations by defining

$$\begin{aligned} & X^{\geq 1} \in \mathcal{B}_{6,i} \in \mathbb{R}^{6(N-1)} \text{ with } i \in \{1, \dots, N-1\} \qquad (\mathcal{B}_6) \\ \iff & \left| \frac{1}{N} \sum_{j \neq 0,i} \int_{t_k}^{t_{k+1}} |h^N(\hat{q}_i - \tilde{q}_j)| \mathbf{1}_{\mathcal{M}_G(X)}(x_j) \right. \\ & \left. - \int_y \int_{t_k}^{t_{k+1}} |h^N(\hat{q}_i - {}^1\tilde{y})| \, \mathrm{d}s \mathbf{1}_{\mathcal{M}_G(X)}(y) k(y) \, \mathrm{d}y \right| > 4N^{-\frac{1}{9}+\sigma} c(N)^{-1}. \end{aligned}$$

Using 2.14, we can estimate $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_6) \leq C_\gamma N^{-\gamma}$. From now on we will work on the complement of $\mathcal{B}_6$, and hence we apply Lemma 6.24. To make the following estimates easier, we start and estimate the $\delta_i$ part which we can estimate by $\delta_b^{k(i)}$ to obtain

$$\begin{aligned} & CN^{-\frac{4}{9}+3\sigma} |\delta_i(t_k)| \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j \neq i \text{ good}} |h^N(\hat{q}_i - \tilde{q}_j)| \, \mathrm{d}t \\ \leq & CN^{-\frac{4}{9}+3\sigma} \delta_b^{k(i)} \cdot C(T) c(N)^{-1} \\ \leq & \underline{C \delta_b^{k(i)}(t_1) N^{-\frac{1}{9}+4\sigma}}, \end{aligned}$$

where we used $c(N) = \ln(N)^{-\frac{1}{4}} N^{-\frac{1}{3}}$.

It now remains to estimate

$$3N^{-\frac{4}{9}+3\sigma} \sum_{j \neq 0,i \text{ good}} \int_{t_k}^{t_{k+1}} \frac{1}{N} |h^N(\hat{q}_i - \tilde{q}_j)| \cdot \delta_b^{k(i)}(t) \, \mathrm{d}t.$$

As we have to deal with a very similar estimate when we deal with 2b, we simply estimate (quite generously) by

$$\sum_{j \neq 0,i \text{ good}} \int_{t_k}^{t_{k+1}} \frac{1}{N} |Df^N(\hat{q}_i - \tilde{q}_j)| \cdot \delta_b^k(t) \, \mathrm{d}t.$$

To estimate this further, we want to use our law of large numbers 2.14. Again, we need to remove a set of bad starting configurations, i.e.

$$\begin{aligned} & X^{\geq 1} \in \mathcal{B}_{7,i} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_7) \\ & :\iff \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0,i} \int_{t_k}^{t_{k+1}} |Df^N(\hat{q}_i - \tilde{q}_j)| \mathbf{1}_{G^N(x_i)}(x_j) \, \mathrm{d}s \right. \\ & \left. - \int_{\mathbb{R}^6} \int_{t_k}^{t_{k+1}} |Df^N(\hat{q}_i - {}^1\tilde{y})| \mathbf{1}_{G^N(x_i)}(y) \, \mathrm{d}s k(y) \, \mathrm{d}y \right| > 4N^{-\frac{1}{9}+\sigma}. \end{aligned}$$

Defining $\mathcal{B}_7$ as the union over these sets, one can show that $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_7) \leq C_\gamma N^{-\gamma}$ by applying Lemma 6.20 and Corollary 2.14. After removing these configurations, we can estimate the effect of the $x_j$ with $k(j) \geq k(i) - 1$, $k(j) \neq K+1$. We consider each possible $k$ individually. For the particles in the corresponding ring we can introduce

an indicator function on the set $C(T)B^k$ for some constant by Lemma 4.8. This allows us to estimate:

$$\begin{aligned}
&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{\substack{j\neq i,0\\ j \text{ good}}} Df^N(\hat{q}_i - \tilde{q}_j)\mathbf{1}_{C(T)B^k}(x_j)\delta_j\right| \\
\leq &\frac{1}{N} \sum_{\substack{j\neq i,0\\ j \text{ good}}} \int_{t_k}^{t_{k+1}} \left|Df^N(\hat{q}_i - \tilde{q}_j)\right| N^{-\frac{1}{9}+2\sigma-(k(i)-1)\sigma}|\delta|\mathbf{1}_{G^N(x_i)}(x_j)\cdot\mathbf{1}_{C(T)B^k}(x_j) \\
\leq &CN^{-\frac{1}{9}+\sigma}N^{-\frac{1}{9}+2\sigma-(k(i)-1)\sigma}|\delta| \\
&+\left|\int_{t_k}^{t_{k+1}} N^{-\frac{1}{9}+2\sigma-k(i)\sigma}|\delta|\int_{\mathbb{R}^6}|Df({}^1\hat{q}_i(t)-{}^1\widetilde{z})|\mathbf{1}_{G^N(y)}(z)\mathbf{1}_{C(T)B^k}(z)k(z)\,\mathrm{d}z\,\mathrm{d}s\right| \\
\overset{2.5}{\leq} &CN^{-\frac{1}{9}+2\sigma-k(i)\sigma}|\delta| + CN^{-\frac{1}{9}+2\sigma-(k(i)-1)\sigma}N^{-3\sigma}\ln(N)|\delta| \\
\leq &\underline{N^{-\frac{1}{9}-k(i)\sigma+\sigma}|\delta|.}
\end{aligned}$$

Here we used the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|Df({}^1\hat{q}_i(t)-{}^1\widetilde{z})\right|\mathbf{1}_{G^N(y)}(z)k(z)\mathbf{1}_{C(T)B^k}(z)\,\mathrm{d}({}^1z,{}^2z) \\
\leq &\int_{|{}^2z-{}^2x_0|\leq CN^{-\sigma}}\int_{|{}^1z-{}^1x_0|\leq CN^{-\sigma}}\left|Df({}^1\hat{q}_i(t)-{}^1\widetilde{z})\right||k|_\infty\,\mathrm{d}({}^1z,{}^2z) \\
\leq &N^{-3\sigma}\int_{|{}^1z-{}^1\tilde{q}_i(t)|\leq c(N)} Cc(N)^3|k|_\infty\,\mathrm{d}\,{}^1z \\
&+N^{-3\sigma}\int_{c(N)\leq|{}^1z-{}^1\tilde{q}_i(t)|\leq 1} C\frac{1}{|{}^1z-{}^1\,\tilde{q}_i(t)|^3}|k|_\infty\,\mathrm{d}\,{}^1z \\
&+N^{-3\sigma}\int_{|{}^1z-{}^1\tilde{q}_i(t)|\geq 1|{}^1z-{}^1x_0|\leq N^{-\sigma}} C|k|_\infty\,\mathrm{d}\,{}^1z \\
\leq &CN^{-3\sigma}\ln(N).
\end{aligned}$$

For $k=\infty$, we have

$$\begin{aligned}
&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\hat{q}_i-\tilde{q}_j)\delta_j\right| \\
\leq &\frac{1}{N}\sum_{j\neq i,\,k(j)=\infty}\int_{t_k}^{t_{k+1}}\left|Df^N(\hat{q}_i-\tilde{q}_j)\right|\delta_g^\infty\mathbf{1}_{G^N(x_i)}(x_j) \\
\leq &CN^{-\frac{1}{9}+\sigma}\delta_g^\infty \\
&+\left|\int_{t_k}^{t_{k+1}}\delta_g^\infty(s)\int_{\mathbb{R}^6}\sup_{s\in[0,T]}|Df({}^1\hat{q}_i(t)-{}^1\widetilde{z})|\mathbf{1}_{G^N(x_i)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z\right| \\
\overset{2.5}{\leq} &CN^{-\frac{1}{9}+\sigma}\cdot\delta_g^\infty + \int_{t_k}^{t_{k+1}} C\ln(N)\cdot\delta_g^\infty\,\mathrm{d}s \\
\leq &\underline{N^{-\frac{1}{3}}|\delta| + \int_{t_k}^{t_{k+1}} C\ln(N)\cdot\delta_g^\infty\,\mathrm{d}s.}
\end{aligned}$$

Now it remains to estimate the effect of the good particles that are significantly closer than $x_i$, i.e. the ones with $x_j \in R^{k(j)}$ with $k(j) < k(i)-1$. We first consider the particles that still are reasonably far from $x_0$, i.e. the particles with $k(j) \geq \frac{1}{18\sigma}+2$.

Recall that

$$R^{\leqslant}_{\text{far}} = \bigcup_{k=\frac{1}{18\sigma}+2}^{k(i)-2} R^k.$$

As we have seen before, we know that $\delta_j \leq N^{-\frac{1}{9}+2\sigma-(\frac{1}{18\sigma}+2)\sigma}|\delta| = N^{-\frac{3}{18}}|\delta|$ and that the mean-field distance between $\hat{x}_i$ and $x_j$ has to stay at least of order $CN^{-\frac{2}{9}+\frac{1}{18}+2\sigma} \geq N^{-\frac{1}{6}+\sigma}$. This means that for $R = V = N^{-\frac{1}{6}+\sigma}$ we can add the indicator function for $M_{R,V}(\hat{x}_i)^C$ which then allows us to obtain the stronger estimate when switching to the integral using 6.20.

For the particles that are closer than $x_i$ but still are far from $x_0$ we can estimate for each $k$ with $R^k \subseteq R^{\leqslant}_{\text{far}}$

$$\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{\substack{j\neq i \text{ good} \\ x_j \in R^k}} Df^N(\hat{q}_i - \tilde{q}_j)\delta_g^{k(j)} \mathbf{1}_{M_{R,V}(\hat{x}_i)^C}(x_j) \right| \mathrm{d}t$$

$$\leq \frac{1}{N} \sum_{\substack{j\neq i \text{ good} \\ x_j \in R^k}} \int_{t_0}^{t_1} \left| Df^N(\hat{q}_i - \tilde{q}_j) \right| \delta_g^{k(j)} \mathbf{1}_{G^N(x_i)}(x_j) \cdot \mathbf{1}_{M_{R,V}(\hat{x}_i)^C}(x_j) \, \mathrm{d}t.$$

As we work on the complement of $\mathcal{B}_7$ and as we have the suitable indicator function, we can apply the stronger version of 6.24 here (see remark 2.15). This yields

$$\begin{aligned}
&\leq CN^{-\frac{3}{18}+\sigma}\delta_g^{k(j)}(t_1) \\
&\quad + \left| \int_{t_k}^{t_{k+1}} \delta_g^{k(j)}(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sup_{s\in[t_k,t_{k+1}]} |Df({}^1\hat{q}_i(t) - {}^1\tilde{z})| \mathbf{1}_{G^N(x_i)}(z) \, \mathrm{d}s k(z) \, \mathrm{d}z \right| \\
&\overset{(\star)}{\leq} CN^{-\frac{3}{18}+\sigma} \cdot N^{-\frac{3}{18}}|\delta| + \int_{t_k}^{t_{k+1}} CN^{-\frac{6}{9}+3k(j)\sigma} \ln(N) \cdot \delta_g^{k(j)}(s) \, \mathrm{d}s \\
&\leq CN^{-\frac{1}{3}+\sigma}|\delta| + CN^{-\frac{2}{3}+3k(j)\sigma}N^{-\frac{2}{9}-k(j)+2\sigma} \cdot |\delta| \\
&\leq CN^{-\frac{1}{3}+\sigma}|\delta| + CN^{-\frac{2}{3}+2k(j)\sigma}N^{-\frac{2}{9}+2\sigma} \cdot |\delta| \\
&\leq CN^{-\frac{1}{3}+\sigma}|\delta| + CN^{-\frac{2}{3}+2\cdot\frac{2}{9}}N^{-\frac{2}{9}+2\sigma} \cdot |\delta| \\
&\leq \underline{C(T)N^{-\frac{1}{3}+\sigma}|\delta|}.
\end{aligned}$$

In $(\star)$ we used the same estimate we already used when considering the case of a good particle $x_i$, i.e. section 4.1.

For the particles $x_j$ that are closer than $k(i)$ and also very close to $x_0$, we can estimate the integral over $Df$ with Lemma 2.7:

$$\begin{aligned}
&\delta_g^{k(j)} \frac{1}{N} \sum_{j<k(i)-1 \text{ close}} \int_{t_k}^{t_{k+1}} \left| Df^N(\hat{q}_i - \tilde{q}_j) \right| \mathrm{d}t \\
&\leq N^{-\frac{1}{9}+2\sigma-k(j)\sigma}|\delta| \frac{1}{N} \sum_{j<k(i)-1 \text{ close}} \frac{C}{c(N)^2 \max(\Delta q, \Delta p)} \\
&\leq N^{-\frac{1}{9}+2\sigma-k(j)\sigma} C|\delta| \frac{N^{\frac{2}{9}+\sigma-k(i)\sigma}}{c(N)^2 N} \\
&\leq C|\delta|N^{-\frac{1}{9}+2\sigma+\frac{8}{9}+2\sigma-1-k(i)\sigma} \leq \underline{C|\delta|N^{-\frac{1}{9}-k(i)\sigma}},
\end{aligned}$$

where we used that there are only very few close particles by Lemma 4.7 and that the distance is large enough by Lemma 4.8.

Next, we look at the effect that the bad particles have.

**Bad prts.** Note that we only have to care about bad particles that do not collide with $x_i$ as we already dealt with them in 2a. For better readability we will leave out the corresponding indicator function throughout this section.

We have to estimate

$$\int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i \text{ bad}} \int_0^1 \left| Df^N\big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\big) - Df^N(\hat{q}_i - \tilde{q}_j) \right| \mathrm{d}s \, |\delta_i(t) - \delta_j(t)| \, \mathrm{d}t$$

for which we now split the sum further using definition 2.10 with $r_0 = 4N^{-\frac{2}{9}-\sigma}$, $v_0 = 0.5N^{-\frac{2}{9}}$ and $\varepsilon = \sigma$ with $y = \hat{x}_i$.

We start with the classes where the minimal distance between $\hat{q}_i$ and $\tilde{q}_j$ is larger than $4N^{-\frac{2}{9}-\sigma}$.

For the bad particles we can estimate

$$\left|q_i - q_j + s(\delta_i(t) - \delta_j(t)) - (\hat{q}_i - \tilde{q}_j)\right| \leq |q_i - \hat{q}_i| + |q_j - \tilde{q}_j| + 2\delta_b \leq 2N^{-\frac{2}{9}-\sigma}.$$

This means that because $|\hat{q}_i - \tilde{q}_j| \geq 4N^{-\frac{2}{9}-\sigma}$ we can introduce $h$ by Lemma 2.4 and it remains to estimate

$$C\delta_b N^{-\frac{2}{9}-\sigma} \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i \text{ bad}} h^N(\hat{q}_i - \tilde{q}_j) \mathbf{1}_{M^N_{(4N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_i)}(x_j) \, \mathrm{d}t.$$

We want to use Lemma 2.13 and hence need to make sure that there are few bad particles. We have already seen this in Corollary 2.12. Note that this was a probabilistic result, so we have to remove a set of bad configurations again. Namely,

$$\begin{aligned} X \in &\mathcal{B}_{8,i,(r,R)(v,V)} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_8) \\ :\Longleftrightarrow &\Bigg( \sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(\hat{x}_i)}(x_j) > a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\ &\quad \vee |\mathcal{M}_b^N(X)| > a_N N^{\frac{2}{3}} \Bigg), \end{aligned}$$

where $a_N = \ln(N)^{\frac{1}{4}}$. We then define $\mathcal{B}_8$ as the union over these sets over all $i$ and all collision classes we are considering in definition 2.10. As the number of collision classes does not depend on $N$, by Corollary 2.12 we have that $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_8) \leq C_\gamma N^{-\gamma}$ for any $\gamma$ and suitable constant. From now on we work on the complement of $\mathcal{B}_8$, i.e. we assume that we can control the number of bad particles in each class.

Now we can apply Lemma 2.13 to estimate

$$\begin{aligned} &C\delta_b N^{-\frac{2}{9}-\sigma} \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i \text{ bad}} h^N(\hat{q}_i - \tilde{q}_j) \mathbf{1}_{M^N_{(4N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_i)}(x_j) \, \mathrm{d}t \\ \leq &CN^{-\frac{1}{36}+3\sigma} |\delta| N^{-\frac{2}{9}-\sigma} \cdot CN^{-\frac{1}{9}+5\sigma} \leq \underline{N^{-\frac{1}{3}}|\delta|}. \end{aligned}$$

We are left with the particles that come closer than $4N^{-\frac{2}{9}-\sigma}$ which are divided into the following collision classes (see definition 2.10) with $r_0 = 4N^{-\frac{2}{9}-\sigma}, v_0 = 0.5N^{-\frac{2}{9}}$:

1. $M^N_{(0,r_0),(0,v_0)}(\hat{x}_i)$
2. $M^N_{(0,r_0),(N^{l\delta}v_0,N^{(l+1)\delta}v_0)}(\hat{x}_i), \qquad 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\delta \ln(N)} \right\rfloor$
3. $M^N_{(0,r_0),(1,\infty)}(\hat{x}_i)$

Here we cannot introduce $h$. However, we already dealt with the first of these classes in 2a. As we know that the interaction between $x_j$ and $\hat{x}_i$ has to be relatively well behaved, we know that at their closest point their relative momentum has to be larger than $0.5N^{-\frac{2}{9}}$. This can be used to estimate $Df$ by Lemma 2.7 to obtain

$$\begin{aligned}
&C \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j \neq i \text{ bad}} \\
&\qquad \cdot \int_0^1 \left| Df^N\Big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\Big) - Df^N(\hat{q}_i - \tilde{q}_j) \right| \mathrm{d}s \, |\delta_i(t) - \delta_j(t)| \, \mathrm{d}t \\
\leq & C \int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j \neq i \text{ bad}} \\
&\qquad \cdot \int_0^1 \left| Df^N\Big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\Big) \right| + \left| Df^N(\hat{q}_i - \tilde{q}_j) \right| \mathrm{d}s \, |\delta_i(t) - \delta_j(t)| \, \mathrm{d}t \\
\leq & C \frac{1}{c(N)^2 v} \frac{1}{N} a_n \sum_{j \neq i \text{ bad}} (|\delta_j| + |\delta_i|).
\end{aligned}$$

Note that for the estimate of $\left|Df^N\Big(q_i - q_j + s(\delta_i(t) - \delta_j(t))\Big)\right|$ we need the second part of Lemma 2.7. We apply this with $y_1 = x_i + s\delta_i, y_2 = x_j + s\delta_j$ and $z_1 = \hat{x}_i, z_2 = x_j$. To verify the conditions, we use $|^1\hat{x}_i(t_{\min}) - {}^1\widetilde{x}_j(t_{\min})| \leq 4N^{-\frac{2}{9}-\sigma}$ and $|^2\hat{x}_i(t_{\min}) - {}^2\widetilde{x}_j(t_{\min})| \geq 0.5N^{-\frac{2}{9}}$.

This yields

$$N^{\frac{\sigma}{2}} |^1\hat{x}_i(t_{\min}) - {}^1\widetilde{x}_j(t_{\min})| \leq 0.5N^{-\frac{2}{9}} \leq |^2\hat{x}_i(t_{\min}) - {}^2\widetilde{x}_j(t_{\min})| = \Delta p$$

as well as

$$\sup_{t \in [t_k, t_{k+1}]} |\hat{x}_i(t) - x_i(t) - s\delta_i| \leq N^{-\frac{2}{9}-\sigma} + \delta_b \leq 2N^{-\frac{2}{9}-\sigma} \leq N^{-\frac{\sigma}{2}} \Delta p$$

as well as

$$\sup_{t \in [t_k, t_{k+1}]} |\widetilde{x}_j(t) - x_j(t) - s\delta_j| \leq N^{-\frac{2}{9}-\sigma} + \delta_b \leq 2N^{-\frac{2}{9}-\sigma} \leq N^{-\frac{\sigma}{2}} \Delta p.$$

Moving on with our estimate, we use $R = 4N^{-\frac{2}{9}-\sigma}$, $v \geq 0.5N^{-\frac{2}{9}}$ as well as $c(N)^2 \geq N^{-\frac{2}{3}-\frac{\sigma}{2}}$ to estimate the $\delta_i$ part first. This is done easily as the number of elements in the sum can be estimated by $a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V, R), 1)^4 \rceil$ using Corollary 2.12.

This gives us

$$\begin{aligned}
&C\frac{1}{c(N)^2 v}\frac{1}{N}a_n \sum_{j\neq i \text{ bad}} |\delta_i| \\
\leq& C\delta_i \frac{1}{c(N)^2 v}\frac{1}{N}a_N \left\lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \right\rceil \\
\leq& C\delta_b^{k(i)} \frac{1}{c(N)^2 v}\frac{1}{N}a_N N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 + C\delta_b^{k(i)} \frac{1}{c(N)^2 v}\frac{1}{N}a_N \\
\leq& C\delta_b^{k(i)} N^{-\frac{1}{9}+\sigma} \frac{\min(V,1)}{v} + C\delta_b^{k(i)} N^{-\frac{1}{9}+\sigma} \leq \underline{CN^{-\frac{1}{9}+2\sigma}\delta_b^{k(i)}}.
\end{aligned}$$

For the $\delta_b^{k(j)}$ we again take cases depending on the ring $x_j$ lies in. First, we consider the case of particles $x_j \in R^{k(j)}$ that are further out, i.e. $k(j) \geq k(i)-1$. This time we do not need to split off $k(j) = K+1 (= \infty)$. Again, we use 2.12 and that all the $\delta_b^{k(j)}$ in this case are bounded by $N^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta|$. Hence,

$$\begin{aligned}
&C\frac{1}{c(N)^2 v}\frac{1}{N}a_n \sum_{j\neq i \text{ bad}} |\delta_j| \\
\leq& CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta| \frac{1}{c(N)^2 v}\frac{1}{N}a_N \left\lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \right\rceil \\
\leq& CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta| \frac{1}{c(N)^2 v}\frac{1}{N}a_N N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \\
&+ CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta| \frac{1}{c(N)^2 v}\frac{1}{N}a_N \\
\leq& CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta| N^{-\frac{1}{9}+\sigma} \frac{\min(V,1)}{v} \\
&+ CN^{-\frac{1}{18}+2\sigma-(k(i)-1)\sigma}|\delta| N^{-\frac{1}{9}+\sigma} \\
\leq& \underline{N^{-\frac{1}{9}-k(i)\sigma}|\delta|}.
\end{aligned}$$

If on the other hand $k(j) < k(i)-1$, we know that $\Delta p \geq CN^{-\frac{2}{9}+k(i)\sigma-\sigma}$ by lemma 4.8. At the same time the corresponding particles stay close to $x_0$, i.e.

$$|\tilde{x}_j - \tilde{x}_0| \leq C\ln(N) N^{-\frac{2}{9}+k(j)\sigma}$$

by 4.5. We use that we work on the complement of $\mathcal{B}_4$. Here we need the second part, i.e. $\mathcal{B}_{4b}$ with $R_1, V_1 \leq C\ln(N)N^{-\frac{2}{9}+k(j)\sigma}$ to obtain a strong enough bound for the number of particles that are close to $x_0$ and in one of the corresponding classes with respect to $\hat{x}_i$. There is one small problem here: What we know is that for $\beta(N) = C\ln(N)N^{-\frac{2}{9}+k(j)\sigma}$

$$\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{\beta(N),\beta(N)}(x_0) \cap M^N_{R,V}(x_i)}(x_j) > a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \cdot \beta(N)^6 \rceil.$$

In our prove for this result it was essential that $x_i$ is a random variable; so we cannot simply replace it with $\hat{x}_i$. However, as $x_i$ and $\hat{x}_i$ are very close by design, we can simply modify our $R$ and $V$ a bit by multiplying by a small constant such that if $x_j$ is in $M^N_{\beta(N),\beta(N)}(x_0) \cap M^N_{R,V}(\hat{x}_i)$, it also lies in $M^N_{\beta(N),\beta(N)}(x_0) \cap M^N_{\tilde{R},\tilde{V}}(x_i)$. This then allows us to use this estimate for $\hat{x}_i$ (with a slightly larger constant) to obtain

$$
\begin{aligned}
&C\frac{1}{c(N)^2 v}\frac{1}{N} a_n \sum_{\substack{j\neq i \text{ bad}\\ k(j)<k(i)-1}} |\delta_j| \\
\leq& C\delta_b^{k(j)} \frac{1}{c(N)^2 v}\frac{1}{N} a_N \left[N^{\frac{2}{3}} R^2 \min(V,1) N^{6(-\frac{2}{9}+k(j)\sigma)} \ln(N)^6\right] \\
\leq& C\delta_b^{k(j)} \frac{\min(1,V)}{v} N^{\frac{2}{3}-\frac{4}{9}+\frac{2}{3}+\sigma-1} N^{6(-\frac{2}{9}+k(j)\sigma)} + C\delta_b^{k(j)} \frac{1}{c(N)^2 v}\frac{1}{N} a_N \cdot \ln(N)^6 \\
\leq& C\delta_b^{k(j)} N^{-\frac{1}{9}+2\sigma} N^{-\frac{12}{9}+6k(j)\sigma} + C\delta_b^{k(j)} N^{-\frac{1}{9}-k(i)\sigma+\sigma} \\
\leq& C N^{-\frac{1}{36}+2\sigma-k(j)\sigma}|\delta| \cdot N^{-\frac{1}{9}+2\sigma} N^{-\frac{12}{9}+6k(j)\sigma} + N^{-\frac{1}{9}-k(i)\sigma} \\
\leq& N^{-\frac{1}{9}-k(i)\sigma} N^{-\frac{12}{9}+5k(j)\sigma+k(i)\sigma+4\sigma}|\delta| + N^{-\frac{1}{9}-k(i)\sigma}|\delta| \\
\leq& \underline{2N^{-\frac{1}{9}-k(i)\sigma}|\delta|},
\end{aligned}
$$

where we used that

$$
\Big(5k(j)+k(i)\Big)\sigma \leq \left(5\Big(\frac{2}{9\sigma}-1\Big)+\frac{2}{9\sigma}\right)\sigma \leq \frac{12}{9}-5\sigma.
$$

Combining all of this, we obtain

$$
\begin{aligned}
&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i}\left(f(q_i-q_j)-f(q_i^\delta-q_j^\delta)\right) - \frac{1}{N}\sum_{j\neq i} Df^N(\hat{q}_i-\tilde{q}_j)(\delta_i-\delta_j)\,\mathrm{d}t\right| \\
&\leq \underline{\underline{C(T)\left(N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta| + N^{-\frac{1}{3}+\sigma}|\delta| + \int_{t_k}^{t_{k+1}} \ln(N)\cdot\delta_g^\infty(t)\,\mathrm{d}t\right)}}.
\end{aligned}
$$

Est. 2c) Clearly, we have

$$
\begin{aligned}
&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)(\delta_i-\delta_j)\,\mathrm{d}t - \int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_i\,\mathrm{d}t\right| \\
=&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\,\mathrm{d}t\right| \\
\leq&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq 0,i \text{ good}} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\,\mathrm{d}t\right| + \left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq 0,i \text{ bad}} Df^N(\tilde{q}_i-\tilde{q}_j)\delta_j\right| \\
&+\left|\int_{t_k}^{t_{k+1}} \frac{1}{N} Df^N(\tilde{q}_i-\tilde{q}_0)\delta_0\,\mathrm{d}t\right|.
\end{aligned}
$$

This works very similar to the corresponding estimate 1b in the case of good $x_i$. For the third summand we can apply 2.7 and estimate by

$$\begin{aligned}&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N} Df^N(\hat{q}_i - \tilde{q}_0)\delta_0 \,\mathrm{d}t\right| \leq C\delta_0(t_{k+1})\frac{1}{N}\frac{1}{c(N)^2\max(r,v)}\\ \leq& CN^{-\frac{1}{9}-+(k(i)-1)\sigma}|\delta| \leq \underline{CN^{-\frac{1}{9}-k(i)\sigma+1.5\sigma}|\delta|},\end{aligned}$$

where we used that for $x \in R^{k(i)}$ either $r \geq N^{-\frac{2}{9}+(k(i)-1)\sigma}$ or $v \geq N^{-\frac{2}{9}+(k(i)-1)\sigma}$.

The second summand can be estimated using 2.13 if $\Delta q > 6N^{-\frac{2}{9}-\sigma}$, and we have already estimated the remaining cases at the end of 1b. Combining this, we obtain

$$\begin{aligned}&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N} \sum_{j\neq i \text{ bad}} Df^N(\hat{q}_i - \tilde{q}_j)\delta_j \,\mathrm{d}t\right|\\ \leq& \underline{C|\delta|N^{-\frac{1}{9}+\sigma-k(i)\sigma} + CN^{-\frac{1}{3}}|\delta|}.\end{aligned}$$

The first summand is the summand we have already mentioned in our estimates for 2b and hence we have already seen this estimate.

Combining all of these estimates, we get the same estimate as for 2b, i.e.

$$\begin{aligned}&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\hat{q}_i - \tilde{q}_j)(\delta_i - \delta_j)\,\mathrm{d}t - \int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i} Df^N(\hat{q}_i - \tilde{q}_j)\delta_i \,\mathrm{d}t\right|\\ \leq& \underline{\underline{C(T)\Big(N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta| + N^{-\frac{1}{3}+\sigma}|\delta| + \int_{t_k}^{t_{k+1}} \ln(N)\cdot\delta_g^\infty(t)\,\mathrm{d}t\Big)}}.\end{aligned}$$

Est. 2d) After splitting of $x_0$, this can simply be estimated using the fact that we work on the complement of $\mathcal{B}_7$. This yields

$$\begin{aligned}&\Bigg|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i,0} \mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j) Df^N(\hat{q}_i - \tilde{q}_j)\delta_i \,\mathrm{d}t\\ &\quad - \int_{\mathbb{R}^6}\int_{t_k}^{t_{k+1}} \mathbf{1}_{\hat{G}^N(\hat{x}_i)}(y) Df^N(\hat{q}_i - \tilde{y})k(y)\delta_i \,\mathrm{d}t\,\mathrm{d}y\Bigg|\\ \leq& \underline{CN^{-\frac{1}{9}+\sigma}\delta_b^{k(i)}}.\end{aligned}$$

The effect of $x_0$ is also easily handled as we can simply apply Lemma 2.7 to estimate $Df$ yielding

$$\left|\int_{t_k}^{t_{k+1}} \frac{1}{N} Df^N(\hat{q}_i - \tilde{q}_0)\delta_i \,\mathrm{d}t\right| \leq \underline{N^{-\frac{1}{9}+\frac{3}{2}\sigma}\delta_b^{k(i)}}.$$

Est. 2e) The estimate of

$$\left|\int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^6} Df^N(\hat{q}_i - y)k(y)\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(y)\delta_i \,\mathrm{d}y\,\mathrm{d}t - \int_{t_k}^{t_{k+1}} \overline{Df}(\hat{q}_i)\delta_i \,\mathrm{d}t\right|$$

works analogously to what we have done for good particles:

$$\begin{aligned}
&\left|\int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^6} Df^N(\hat{q}_i - y)k(y)\mathbf{1}_{\hat{G}^N(\hat{x}_i)}(y)\delta_i \,\mathrm{d}y\,\mathrm{d}t - \int_{t_k}^{t_{k+1}} \overline{Df}(\hat{q}_i)\delta_i \,\mathrm{d}t\right| \\
\leq &\left|\int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^6} Df^N(\hat{q}_i - y)k(y)\mathbf{1}_{\hat{G}^N(\hat{x}_i)^C}(y)\delta_i \;\mathrm{d}y\,\mathrm{d}t\right| \\
\leq &C\delta_b^{k(i)} N^{1+\sigma}\Delta t \int_{t_k}^{t_{k+1}} k(y)\mathbf{1}_{\hat{G}^N(\hat{x}_i)^C}(y) \\
\leq &C\delta_b^{k} N^{1+\sigma}\Delta t\, \mathbb{P}(y \in \hat{G}^N(x_i)^C) \\
\leq &CN^{1+\sigma}\delta_b^{k(i)}\Delta t (N^{-\frac{2}{9}-\sigma})^2 (N^{-\frac{2}{9}})^4 \\
\leq &CN^{1-\frac{12}{9}+\sigma-2\sigma}\delta_b^{k(i)}\Delta t = \underline{\underline{CN^{-\frac{1}{3}-\sigma}\Delta t\delta_b^{k(i)}}},
\end{aligned}$$

where we used Lemma 2.11.

Est. 2f) As seen before, we have

$$\left|\int_{t_k}^{t_{k+1}} \overline{Df}(\hat{q}_i)\delta_i\right| \mathrm{d}t \leq C\int_{t_k}^{t_{k+1}} \ln(N)\cdot {}^1\delta_b^{k(i)}\,\mathrm{d}t.$$

This concludes the cumbersome part in our effort to prove that the stopping time cannot be triggered by a bad particle. Our Grönwall argument works almost identically as in the case for good particles. The only difference is that we had to artificially split the time frame $[t, t+\Delta t]$ into smaller intervals for our auxiliary particles to work. Hence, we now need to add our estimates up to obtain the estimate for the complete time $[t, t+\Delta t]$.

### 4.2.2 Grönwall estimate

We have proven that for all $k$ we have

$$\begin{aligned}&\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i}\left(f\Big(q_i(t)-q_j(t)\Big)-f\Big(q_i^\delta(t)-q_j^\delta(t)\Big)\right)\mathrm{d}t\right|\\ \leq&\int_{t_k}^{t_{k+1}} C\ln(N)\cdot {}^1\delta_b^{k(i)}(s)\,\mathrm{d}s + C(T)N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta|.\end{aligned}$$

So far the estimate looks very similar to the one for good particles. The estimate becomes a bit weaker now as we have to take into account that we divided the integral into $\approx N^{\frac{1}{18}}$ many segments. Hence, for any $t$ and $\Delta t$ we can use equidistant $t_k$ with $t_0 = t$ and $t_{k+1} = t_k + N^{-\frac{1}{18}}$ to obtain

$$\begin{aligned}&\left|\int_{t}^{t+\Delta t} \frac{1}{N}\sum_{j\neq i}\left(f\Big(q_i(t)-q_j(t)\Big)-f\Big(q_i^\delta(t)-q_j^\delta(t)\Big)\right)\mathrm{d}t\right|\\ \leq&\sum_{k=0}^{\lceil \Delta t N^{\frac{1}{18}}\rceil}\left|\int_{t_k}^{t_{k+1}} \frac{1}{N}\sum_{j\neq i}\left(f\Big(q_i(t)-q_j(t)\Big)-f\Big(q_i^\delta(t)-q_j^\delta(t)\Big)\right)\mathrm{d}t\right|\\ \leq&C\int_t^{t+\Delta t}\ln(N)({}^1\delta_b^{k(i)}) + C(T)N^{\frac{1}{18}}N^{-\frac{1}{9}+1.5\sigma-k(i)\sigma}|\delta|.\end{aligned}$$

Again, we use Grönwall. Fortunately, we can simply use our Lemma 3.2 with

$$v_1 = {}^1\delta_b^{k(i)}, \qquad v_2 = {}^2\delta_b^{k(i)} \qquad \text{and} \qquad C(T,N) = C(T)N^{-\frac{1}{18}+1.5\sigma-k(i)\sigma}|\delta|.$$

Hence, the lemma yields that ${}^2\delta_b^{k(i)}(\tau) \leq C(T)N^{-\frac{1}{18}+1.6\sigma-k(i)\sigma}$. This in turn yields

$${}^1\delta_b^{k(i)} \leq T\cdot {}^2\delta_b \leq TC(T)N^{-\frac{1}{18}+1.6\sigma-k(i)\sigma}.$$

Thus, for large enough $N$ (depending on $C(T)$ and $\sigma$) we have

$$\delta_b^{k(i)}(\tau) \ll N^{-\frac{1}{18}+2\sigma-k(i)\sigma}\delta.$$

We conclude that the stopping time cannot be triggered by a bad particle. Now only the most interesting case remains. We need to show that the moved particle does not trigger the stopping time.

## 4.3 Third case: $\mathbf{x_i}$ is the moved particle $\mathbf{x_0}$

Here we follow a slightly different approach. Instead of estimating $\delta_0$ directly, we are going to show that $\delta_0$ stays close to $\tilde{\delta}$. The arguments and ideas remain mostly the same, and we start by estimating the corresponding difference in force.

### 4.3.1 Estimating the difference in force

Similar to $\delta_g$ and $\delta_b$, the relevant force for $\delta_0$ is given as

$$\frac{1}{N}\sum_{j\neq 0} f^N(q_0^\delta - q_j^\delta) - f^N(q_0 - q_j).$$

On the other hand, the force for $\tilde{\delta}$ is given as

$$\tilde{f}^N(\tilde{q}_0^\delta) - \tilde{f}^N(\tilde{q}_0) = \int_{\mathbb{R}^6} f(\tilde{q}_0^\delta - {}^1y) - f^N(\tilde{q}_0 - {}^1y)k(y)\,\mathrm{d}y.$$

As we want to estimate $|\delta_0 - \tilde{\delta}|$, we consider

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0} f(q_0^\delta - q_j^\delta) - f(q_0 - q_j) - \left(\tilde{f}(\tilde{q}_0^\delta) - \tilde{f}^N(\tilde{q}_0)\right)\mathrm{d}t\right| \\
\leq &\left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0} \left(f(q_0^\delta - q_j^\delta) - f(q_0 - q_j) - (f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\tilde{q}_0 - \tilde{q}_j))\right)\mathrm{d}t\right| \qquad (3a)\\
&+ \left|\int_{t_0}^{t_1} \frac{1}{N}\sum_{j\neq 0} \left(f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\tilde{q}_0 - \tilde{q}_j)\right)\right. \\
&\quad \left. - \int_{\mathbb{R}^6} \left(f^N(\tilde{q}_0^\delta - y) - f^N(\tilde{q}_0 - y)\right)\mathbf{1}_{G^N(x_0)}(y)k(y)\,\mathrm{d}y\,\mathrm{d}t\right| \qquad (3b)\\
&+ \left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6} \left(f^N(\tilde{q}_0^\delta - y) - f^N(\tilde{q}_0 - y)\right)(1 - \mathbf{1}_{G^N(x_0)}(y))k(y)\,\mathrm{d}y\,\mathrm{d}t\right| \qquad (3c)\\
\leq &CN^{-\frac{1}{18}+3\sigma}|\delta| + \int_{t_0}^{t_1} \ln(N)|{}^1\delta_0 - {}^1\tilde{\delta}|\,\mathrm{d}t.
\end{aligned}$$

Again, the estimate in the last line will be our goal for the next section. While many arguments should feel familiar, we will distinguish between two new cases: Namely, we will need different estimates depending on whether $|\delta| \ll c(N)$ or $|\delta| \approx c(N)$.

Est. 3a) As usual we split the sum into the effect of the good and the bad particles. The effect of the bad particles can be estimated very similar to the previous cases, so we consider them first.

**Bad prts.** We have to work with weaker estimates when considering the bad particles. However, we know that there are few of them, which can be leveraged to our adventage.

Using the mean-value theorem on both sides followed by triangle-inequality, we estimate

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} f(q_0^\delta - q_j^\delta) - f(q_0 - q_j) - \left(f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\tilde{q}_0 - \tilde{q}_j)\right) \mathrm{d}t\right| \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \int_0^1 \Big| Df^N\Big(q_0 - q_j + s(\delta_0 - \delta_j)\Big)(\delta_0 - \delta_j) \\
&\qquad - Df^N\Big(\tilde{q}_0 - \tilde{q}_j + s\tilde{\delta}_0\Big)\tilde{\delta}_0\Big| \,\mathrm{d}s\ \mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \int_0^1 \Big| Df^N\Big(q_0 - q_j + s(\delta_0 - \delta_j)\Big)(\delta_0 - \delta_j) \\
&\qquad - Df^N(\tilde{q}_0 - \tilde{q}_j)(\delta_0 - \delta_j)\Big| \,\mathrm{d}s\ \mathrm{d}t \qquad (4.2) \\
&+ \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \Big| Df^N(\tilde{q}_0 - \tilde{q}_j)(\delta_0 - \delta_j) - Df^N(\tilde{q}_0 - \tilde{q}_j)\tilde{\delta}_0\Big| \,\mathrm{d}s\ \mathrm{d}t \qquad (4.3) \\
&+ \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \int_0^1 \Big| Df^N(\tilde{q}_0 - \tilde{q}_j)\tilde{\delta}_0 - Df^N(\tilde{q}_0 - \tilde{q}_j + s\tilde{\delta})\tilde{\delta}_0\Big| \,\mathrm{d}s\ \mathrm{d}t \qquad (4.4)
\end{aligned}$$

We note that the second summand 4.3 can be estimated by

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \Big| Df^N(\tilde{q}_0 - \tilde{q}_j)(\delta_0 - \delta_j) - Df^N(\tilde{q}_0 - \tilde{q}_j)\tilde{\delta}_0\Big| \,\mathrm{d}s\ \mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \Big| Df^N(\tilde{q}_0 - \tilde{q}_j)\Big| \cdot C|\delta| \\
\leq &\underline{CN^{-\frac{1}{9}+5\sigma}|\delta|},
\end{aligned}$$

where we used Lemma 2.13 with $\alpha = 3$ in the last step. Note that this is applicable as we work on the compliment of $\mathcal{B}_4$.

The last summand 4.4 can be estimated similarly using $|\tilde{\delta}| \leq C(T)|\delta| \leq C(T)c(N)$ and Lemma 2.13:

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \int_0^1 \Big| Df^N(\tilde{q}_0 - \tilde{q}_j)\tilde{\delta} - Df^N(\tilde{q}_0 - \tilde{q}_j + s\tilde{\delta})\tilde{\delta}\Big| \,\mathrm{d}s\ \mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} h^N(\tilde{q}_0 - \tilde{q}_j)|\tilde{\delta}|^2 \\
\leq &C(T)N^{\frac{2}{9}+6\sigma}c(N)|\tilde{\delta}| \\
\leq &C(T)N^{-\frac{1}{9}+7\sigma}|\tilde{\delta}| \\
\leq &\underline{C(T)N^{-\frac{1}{9}+7\sigma}|\delta|}.
\end{aligned}$$

Only the first summand 4.2 remains, i.e. we need to estimate

$$\begin{aligned}&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \\ &\qquad \cdot \int_0^1 \left| Df^N\Big(q_0 - q_j + s(\delta_0 - \delta_j)\Big)(\delta_0 - \delta_j) - Df^N(\tilde{q}_0 - \tilde{q}_j)(\delta_0 - \delta_j)\right| \mathrm{d}s \ \mathrm{d}t \\ \leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} \int_0^1 \left| Df^N\Big(q_0 - q_j + s(\delta_0 - \delta_j)\Big) - Df^N(\tilde{q}_0 - \tilde{q}_j)\right|(\delta_0 + \delta_b) \,\mathrm{d}s \ \mathrm{d}t.\end{aligned}$$

We now split the sum even further using the partition introduced in definition 2.10 with $r = 6N^{-\frac{2}{9}-\sigma}$, $v = N^{-\frac{2}{9}}$ and $\delta = \sigma$. First, we will consider the case where the minimal distance between $\tilde{q}_i$ and $\tilde{q}_j$ is more than $6N^{-\frac{2}{9}-\sigma}$. We note that for the bad particles we can estimate

$$|q_0 - q_j + s(\delta_0(t) - \delta_j(t)) - (\tilde{q}_0 - \tilde{q}_j)| \leq |q_0 - \tilde{q}_0| + |q_j - \tilde{q}_j| + 2\max(\delta_0, \delta_b) \leq 3N^{-\frac{2}{9}-\sigma}.$$

This means that in this case we can actually introduce $h$ by Lemma 2.4 and it remains to estimate:

$$C|\delta|N^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} h^N(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{M^N_{(N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_i)}(x_j) \,\mathrm{d}t.$$

As we work on the complement of $\mathcal{B}_4$, we can apply Lemma 2.13 to estimate

$$\begin{aligned}&C|\delta|N^{-\frac{2}{9}-\sigma} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ bad}} h^N(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{M^N_{(N^{-\frac{2}{9}-\sigma},\infty),(0,\infty)}(x_i)}(x_j) \,\mathrm{d}t \\ \leq &C|\delta|N^{-\frac{2}{9}-\sigma} \cdot CN^{-\frac{1}{9}+5\sigma} \leq \underline{CN^{-\frac{1}{3}+4\sigma}|\delta|}\end{aligned}$$

We are left with the particles that come closer than $6N^{-\frac{2}{9}-\sigma}$. Here we introduce the collision classes from definition 2.10 again:

1. $M^N_{(0,r_0),(0,v_0)}(x_i)$
2. $M^N_{(0,r_0),(N^{l\delta}v_0,N^{(l+1)\delta}v_0)}(x_i), 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\delta \ln(N)} \right\rfloor$
3. $M^N_{(0,r_0),(1,\infty)}(x_i)$

In these cases we cannot introduce $h$. However, because $x_0$ is a good particle, the relative momentum at their closest point has to be larger than $N^{-\frac{2}{9}}$. This means that we do not need to care about the first of these classes. It can also be used to estimate

$Df$ by Lemma 2.7 to obtain

$$\begin{aligned}
&C\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0 \text{ bad}}\\
&\qquad\cdot\int_0^1\left|Df^N\Big(q_0-q_j+s(\delta_0(t)-\delta_j(t))\Big)-Df^N(\tilde{q}_0-\tilde{q}_j)\right|\mathrm{d}s\left|\delta_0(t)-\delta_j(t)\right|\mathrm{d}t\\
\leq& C\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0 \text{ bad}}\\
&\qquad\cdot\int_0^1\left|Df^N\Big(q_0-q_j+s(\delta_0(t)-\delta_j(t))\Big)\right|+\left|Df^N(\tilde{q}_0-\tilde{q}_j)\right|\mathrm{d}s\left|\delta_0(t)-\delta_j(t)\right|\mathrm{d}t\\
\leq& C(\delta_b+\delta_0)\frac{1}{c(N)^2v}\frac{1}{N}a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil.
\end{aligned}$$

Note that for the estimate of $\left|Df^N\Big(q_i-q_j+s(\delta_i(t)-\delta_j(t))\Big)\right|$ we need the second part of Lemma 2.7. We apply this with $y_1=x_0+s\delta_0, y_2=x_j+s\delta_j$ and $z_1=x_0, z_2=x_j$ and a bit confusingly $\sigma=\frac{\sigma}{2}$. Note that we can apply b) here as

$$\begin{aligned}
&N^{\frac{\sigma}{2}}|{}^1\widetilde{x}_0-{}^1\widetilde{x}_j|\leq N^{-\frac{2}{9}}\leq|{}^2\widetilde{x}_0-{}^2\widetilde{x}_j|=\Delta p\\
&\text{as well as}\\
&\sup|\widetilde{x}_0(t)-x_0(t)+s\delta_i|\leq N^{-\frac{2}{9}-\sigma}+N^{-\frac{1}{3}}\leq 2N^{-\frac{2}{9}-\sigma}\leq N^{-\frac{\delta}{2}}\Delta p\\
&\text{as well as}\\
&\sup|\widetilde{x}_j(t)-x_j(t)+s\delta_j|\leq N^{-\frac{2}{9}-\sigma}+N^{-\frac{1}{3}}\leq 2N^{-\frac{2}{9}-\sigma}\leq N^{-\frac{\delta}{2}}\Delta p.
\end{aligned}$$

Using $R=6N^{-\frac{2}{9}-\sigma}$, $v\geq N^{-\frac{2}{9}}$ as well as $c(N)^2\geq N^{-\frac{2}{3}-\sigma}$, we estimate

$$\begin{aligned}
&C(\delta_b+\delta_0)\frac{1}{c(N)^2v}\frac{1}{N}a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil\\
\leq& C(\delta_b+\delta_0)\frac{1}{c(N)^2v}\frac{1}{N}a_NN^{\frac{2}{3}}R^2\min(\max(V,R),1)^4+C(\delta_b+\delta_0)\frac{1}{c(N)^2v}\frac{1}{N}a_N\\
\leq& C(\delta_b+\delta_0)N^{-\frac{1}{9}+\sigma}\frac{\min(V,1)}{v}+C(\delta_b+\delta_0)N^{-\frac{1}{9}-\sigma}\leq\underline{CN^{-\frac{1}{9}+2\sigma}|\delta|},
\end{aligned}$$

where in the last estimate we used that we can estimate $\frac{\min(V,1)}{v}\leq N^{\sigma}$. If we are in one of the second classes, we always have $\frac{V}{v}=N^{\sigma}$. If we are in the last class instead, we know that $v=N^{-\sigma}$. Combining all of these estimates, we obtain

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0 \text{ bad}}\left(f(q_0^\delta-q_j^\delta)-f(q_0-q_j)-\Big(f^N(\tilde{q}_0^\delta-\tilde{q}_j)-f^N(\tilde{q}_0-\tilde{q}_j)\Big)\right)\mathrm{d}t\right|\\
\leq&\underline{C(T)N^{-\frac{1}{9}+7\sigma}|\delta|}.
\end{aligned}$$

**Good prts.** Now we estimate the effect of the good particles, i.e.

$$\left|\int_{t_0}^{t_1}\frac{1}{N}\sum_{j\neq 0 \text{ good}}\left(f(q_0^\delta-q_j^\delta)-f(q_0-q_j)-\Big(f^N(\tilde{q}_0^\delta-\tilde{q}_j)-f^N(\tilde{q}_0-\tilde{q}_j)\Big)\right)\mathrm{d}t\right|.$$

For this section we need to remove a set of bad starting configurations:

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{9a,i} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_9)\\ &\quad \iff \Bigg(\exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j\neq 0} \int_{t_1}^{t_2} h^N(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}s \\ &- \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N(\tilde{q}_0 - {}^1\tilde{z}) \mathbf{1}_{G^N(x_0)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z \Bigg| > 4N^{-\frac{1}{9}+\sigma} c(N)^{-1} \Bigg) \end{aligned}$$

and

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{9b,i} \subseteq \mathbb{R}^{6(N-1)}\\ &\quad \iff \Bigg(\exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j\neq 0} \int_{t_1}^{t_2} Df(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}s \\ &- \int_{\mathbb{R}^6} \int_{t_1}^{t_2} Df(\tilde{q}_0 - {}^1\tilde{z}) \mathbf{1}_{G^N(x_0)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z \Bigg| > 4N^{-\frac{1}{9}+\sigma} \Bigg) \end{aligned}$$

as well as

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{9c,i} \subseteq \mathbb{R}^{6(N-1)}\\ &\quad \iff \Bigg(\exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j\neq 0} \int_{t_1}^{t_2} f(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}s \\ &- \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f(\tilde{q}_0 - {}^1\tilde{z}) \mathbf{1}_{G^N(x_0)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z \Bigg| > N^{-\frac{4}{9}+\sigma} \Bigg). \end{aligned}$$

Defining $\mathcal{B}_9$ as the union of all these sets, we can estimate $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_{9a,i}) \leq C_\gamma N^{-\gamma}$ for any $\gamma > 0$ and suitable $C_\gamma$ as the same is true for each of the individual sets by 2.14.

We distinguish two cases: Our ansatz depends on whether $\delta$ is very close to our cut-off $c(N)$ or $\delta$ is significantly smaller than the cut-off. This is the case we will start with.

$\delta \ll c(N)$ Assume $|\delta| \leq N^{-\frac{1}{18}} c(N)$. In this case we start by introducing $Df$ via the mean-value theorem:

$$\begin{aligned} &\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} f^N(q_0 - q_j) - f^N(q_0^\delta - q_j^\delta) - (f^N(\tilde{q}_0 - \tilde{q}_j) - f^N(\tilde{q}_0^\delta - \tilde{q}_j))\,\mathrm{d}t \right| \\ &= \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 Df^N\Big(q_0 - q_j + s(\delta_0 - \delta_j)\Big)({}^1\delta_0 - {}^1\delta_j) - Df^N(\tilde{q}_0 - \tilde{q}_j + s\tilde{\delta})\tilde{\delta}\,\mathrm{d}s\,\mathrm{d}t \right| \\ &\leq \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 Df^N\Big(q_0 - q_j + s(\delta_0 - \delta_j)\Big)({}^1\delta_0 - {}^1\delta_j) - Df^N(\tilde{q}_0 - \tilde{q}_j)({}^1\delta_0 - {}^1\delta_j)\,\mathrm{d}s\,\mathrm{d}t \right| \\ &\quad + \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} Df^N(\tilde{q}_0 - \tilde{q}_j)|\delta_j|\,\mathrm{d}t \right| \\ &\quad + \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} Df^N(\tilde{q}_0 - \tilde{q}_j)({}^1\delta_0 - {}^1\tilde{\delta})\,\mathrm{d}t \right| \\ &\quad + \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 \Big(Df^N(\tilde{q}_0 - \tilde{q}_j) - Df^N(\tilde{q}_0 - \tilde{q}_j + s\tilde{\delta})\Big)\tilde{\delta}\,\mathrm{d}t \right| \end{aligned}$$

We note that by our assumption

$$|q_0 - q_j + s(\delta_0 - \delta_j) - (\tilde{q}_0 - \tilde{q}_j)| \leq 2N^{-\frac{4}{9}+3\sigma} + \delta_0 + \delta_g \leq C(T)N^{-\frac{1}{18}}c(N).$$

Hence, we can introduce $h$ to estimate the first summand by

$$\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} h^N(\tilde{q}_0 - \tilde{q}_j)\,\mathrm{d}t \leq C(T)N^{-\frac{1}{18}}c(N)|\delta_0 - \delta_j| \leq \underline{C(T)N^{-\frac{1}{18}}|\delta|},$$

where we applied Lemma 6.24 with $\beta = 1$, which is possible as we work on the complement of $\mathcal{B}_9$. The second summand can be estimated similarly using Lemma 6.24 again with $\beta = 1$ as

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \left|Df^N(\tilde{q}_0 - \tilde{q}_j)\right| \cdot |\delta_j|\,\mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \mathbf{1}_{G^N(x_0)}(x_j)\left|Df^N(\tilde{q}_0 - \tilde{q}_j)\right|\delta_g\,\mathrm{d}t \\
\leq &C(T)\ln(N)\delta_g \\
\leq &\underline{C(T)N^{-\frac{2}{9}+5\sigma}|\delta|}.
\end{aligned}$$

Following the pattern, the third term can be estimated by

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} |Df^N(\tilde{q}_0 - \tilde{q}_j)| \cdot |{}^1\delta_0 - {}^1\tilde{\delta}|\,\mathrm{d}t \\
\leq &\underline{C(T)N^{-\frac{1}{9}+\sigma}|{}^1\delta_0 - {}^1\tilde{\delta}| + \int_{t_0}^{t_1} C\ln(N)|{}^1\delta_0 - {}^1\tilde{\delta}|\,\mathrm{d}t},
\end{aligned}$$

where we this time used $\beta = |\delta_0 - \tilde{\delta}|$. We apply Lemma 6.24 one last time for the last term to estimate

$$\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} h^N(\tilde{q}_0 - \tilde{q}_j)|\tilde{\delta}|^2\,\mathrm{d}t \leq C(T)c(N)^{-1}|\tilde{\delta}|^2 \leq \underline{C(T)N^{-\frac{1}{18}}|\tilde{\delta}|},$$

where we used our assumption $|\delta| \leq N^{-\frac{1}{18}}c(N)$ as well as $|\tilde{\delta}| \leq C(T)|\delta|$. Combining all these estimates, we have proven that if $|\delta| \leq N^{-\frac{1}{18}}c(N)$:

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} f(q_0 - q_j) - f(q_0^\delta - q_j^\delta) - \left(f^N(\tilde{q}_0 - \tilde{q}_j) - f^N(\tilde{q}_0^\delta - \tilde{q}_j)\right)\mathrm{d}t\right| \\
&\leq \underline{\underline{C(T)N^{-\frac{1}{18}}|\delta| + \int_{t_0}^{t_1} C\ln(N)|{}^1\delta_0 - {}^1\tilde{\delta}|\,\mathrm{d}t}}.
\end{aligned}$$

$\delta \approx c(N)$ If $|\delta| > N^{-\frac{1}{18}}c(N)$, we change our approach slightly. This time, we first estimate the

distance between the particles and their mean-field counterparts.

$$
\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} f^N(q_0-q_j) - f^N(q_0^\delta - q_j^\delta) - \left(f^N(\tilde q_0-\tilde q_j) - f^N(\tilde q_0^\delta - \tilde q_j)\right) \mathrm{d}t\right| \\
=&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} f^N(q_0-q_j) - f^N(\tilde q_0-\tilde q_j) + f^N(\tilde q_0^\delta - \tilde q_j) - f^N(q_0^\delta - q_j^\delta) \,\mathrm{d}t\right| \\
\leq&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 \left|Df^N\left(\tilde q_0-\tilde q_j + s(q_0-\tilde q_0 - q_j + \tilde q_j)\right)\right| \cdot 2N^{-\frac{4}{9}+3\sigma} \,\mathrm{d}s\,\mathrm{d}t \\
&+\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 Df^N\left(\tilde q_0^\delta-\tilde q_j + s(q_0^\delta-\tilde q_0^\delta - q_j^\delta + \tilde q_j)\right)\left({}^1q_0^\delta - {}^1\tilde q_0^\delta - {}^1q_j^\delta + {}^1\tilde q_j\right) \mathrm{d}s\,\mathrm{d}t\right| \\
\leq&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \left|Df^N(\tilde q_0-\tilde q_j)\right| \cdot 2N^{-\frac{4}{9}+3\sigma} \,\mathrm{d}t \\
&+\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 \left|Df^N\left(\tilde q_0-\tilde q_j + s(q_0-\tilde q_0 - q_j + \tilde q_j)\right) - Df^N(\tilde q_0-\tilde q_j)\right| 2N^{-\frac{4}{9}+3\sigma} \,\mathrm{d}s\,\mathrm{d}t \\
&+\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 Df^N\left(\tilde q_0^\delta-\tilde q_j + s(q_0^\delta-\tilde q_0^\delta - q_j^\delta + \tilde q_j)\right)\left({}^1q_0^\delta - {}^1\tilde q_0^\delta - {}^1q_j^\delta + {}^1\tilde q_j\right) \mathrm{d}s\,\mathrm{d}t\right|
\end{aligned}
$$

where we used that $|q_i - \tilde q_i| \leq N^{-\frac{4}{9}+3\sigma}$.

The first summand is easily handled by Lemma 6.24 with $\beta = 1$ yielding

$$\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} |Df^N(\tilde q_0-\tilde q_j)| 2N^{-\frac{4}{9}+3\sigma} \,\mathrm{d}t \leq C(T)\ln(N) N^{-\frac{4}{9}+3\sigma} \leq \underline{N^{-\frac{1}{18}+3.5\sigma}|\delta|},$$

where we used $|\delta| > N^{-\frac{1}{18}} c(N) = N^{-\frac{7}{18}} (\ln(N))^{-\frac{1}{4}}$ in the last estimate.

In the second summand we can introduce $h$ followed by Lemma 6.24 to obtain

$$
\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} \int_0^1 \left|Df^N(\tilde q_0-\tilde q_j + s(q_0-\tilde q_0 - q_j + \tilde q_j)) - Df^N(\tilde q_0-\tilde q_j)\right| 2N^{-\frac{7}{18}+\sigma} \,\mathrm{d}s\,\mathrm{d}t \\
\leq&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} \int_0^1 \left|h^N(\tilde q_0-\tilde q_j)\right| (2N^{-\frac{7}{18}+\sigma})^2 \,\mathrm{d}s\,\mathrm{d}t \\
\leq&C(T)(N^{-\frac{7}{18}+\sigma})^2 c(N)^{-1} \leq \underline{N^{-\frac{1}{18}+3\sigma}|\delta|},
\end{aligned}
$$

where we again used $|\delta| > N^{-\frac{1}{18}} c(N)$.

We want to use similar estimates for the last remaining summand. There is one small inconvenience here, namely that to apply Lemma 6.24 we would need our law of large numbers for $\tilde x_0^\delta$, and hence our probability estimate would depend on $\delta$. To avoid this, we cover the area around $x_0$, or more precisely the area around $\tilde x_0$, with auxiliary particles.

We consider the ball with radius $c(N)$ around $x_0$ in phase-space and fill it with particles that have distance $N^{-\frac{1}{9}} c(N)^4$ from each other. We need about $CN^{\frac{1}{9}} c(N)^{-3}$ of these particles. We now find a particle that at $t = 0$ has a distance of less than $N^{-\frac{1}{9}} c(N)^4$ to $x_0^\delta$. We reference its mean-field trajectory by $\hat x_0^\delta$. We then know that for all times until $T$ the distance of $\hat x_0^\delta$ and $\tilde x_0^\delta$ is less than $C(T)N^{-\frac{1}{9}} c(N)^4$ by Lemma 2.6.

Using these auxiliary particles, we can now deal with the last summand. First, we note that

$$|q_j^\delta - \tilde{q}_j| = |q_j^\delta - q_j| + |q_j - \tilde{q}_j| \leq N^{-\frac{1}{9}+2\sigma}|\delta| + N^{-\frac{4}{9}+3\sigma} \leq 2N^{-\frac{4}{9}+3\sigma}$$

which implies

$$\begin{aligned}|{}^1q_0^\delta - {}^1\tilde{q}_0^\delta - {}^1q_j^\delta + {}^1\tilde{q}_j| &\leq |{}^1q_0^\delta - {}^1q_0 + {}^1q_0 - {}^1\tilde{q}_0 + {}^1\tilde{q}_0 - {}^1\tilde{q}_0^\delta| + |{}^1q_j^\delta - {}^1\tilde{q}_j| \\ &\leq 3N^{-\frac{4}{9}+\sigma} + |{}^1\delta_0 - {}^1\tilde{\delta}|.\end{aligned}$$

This can be used to estimate

$$\begin{aligned}&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 Df^N\Big(\tilde{q}_0^\delta - \tilde{q}_j + s(q_0^\delta - \tilde{q}_0^\delta - q_j^\delta + \tilde{q}_j)\Big)({}^1q_0^\delta - {}^1\tilde{q}_0^\delta - {}^1q_j^\delta + {}^1\tilde{q}_j)\,\mathrm{d}s\,\mathrm{d}t\right| \\ \leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \Big|Df^N(\hat{q}_0^\delta - \tilde{q}_j)\Big|(3N^{-\frac{4}{9}+3\sigma} + |{}^1\delta_0 - {}^1\tilde{\delta}|)\,\mathrm{d}t \qquad (4.5) \\ &+ \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0 \text{ good}} \int_0^1 \Big|Df^N(\tilde{q}_0^\delta - \tilde{q}_j + s(q_0^\delta - \tilde{q}_0^\delta - q_j^\delta + \tilde{q}_j))\,\mathrm{d}s \qquad (4.6) \\ &\qquad - Df^N(\hat{q}_0^\delta - \tilde{q}_j)\Big| \cdot (3N^{-\frac{4}{9}+3\sigma} + |\delta_0 - \tilde{\delta}|)\,\mathrm{d}t.\end{aligned}$$

Next, we want to apply 6.24. We run into a similar problem as with the auxiliary particles we used for our bad particles. While $x_0$ is good and hence we can simply add the indicator function $\mathbf{1}_{G^N(x_0)}(x_j)$, the trajectory of $\hat{x}_0^\delta$ might come too close to some particles. For that reason we again use $\hat{G}^N(\hat{x}_0^\delta) = G^N_{4N^{-\frac{2}{9}-\sigma}, 0.5N^{-\frac{2}{9}}}(\hat{x}_0^\delta)$.

As $|x_0 - \hat{x}_0^\delta| \leq C(T)c(N)$, we can prove that $G^N(x_0) \subseteq \hat{G}^N(\hat{x}_0^\delta)$. Before we can finally apply Lemma 6.24, we need to remove some bad starting configurations

$$\begin{aligned}&X^{\geq 1} \in \mathcal{B}_{10a,i} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_{10}) \\ &\quad \iff \Bigg(\exists t_1, t_2 \in [0,T] : \left|\frac{1}{N}\sum_{j\neq 0}\int_{t_1}^{t_2} Df\Big((\hat{q}_0^\delta)_i - \tilde{q}_j\Big)\mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(x_j)\,\mathrm{d}s\right. \\ &\left. - \int_{\mathbb{R}^6}\int_{t_1}^{t_2} Df\Big((\hat{q}_0^\delta)_i - {}^1\tilde{z}\Big)\mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z\right| > 4N^{-\frac{1}{9}+\sigma}\Bigg)\end{aligned}$$

and

$$\begin{aligned}&X^{\geq 1} \in \mathcal{B}_{10b,i} \subseteq \mathbb{R}^{6(N-1)} \\ &\quad \iff \Bigg(\exists t_1, t_2 \in [0,T] : \left|\frac{1}{N}\sum_{j\neq 0}\int_{t_1}^{t_2} h^N\Big((\hat{q}_0^\delta)_i - \tilde{q}_j\Big)\mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(x_j)\,\mathrm{d}s\right. \\ &\left. - \int_{\mathbb{R}^6}\int_{t_1}^{t_2} h^N\Big((\hat{q}_0^\delta)_i - {}^1\tilde{z}\Big)\mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z\right| > 4N^{-\frac{1}{9}+\sigma}c(N)^{-1}\Bigg),\end{aligned}$$

where the $i$ index all of our $CN^{\frac{1}{9}}c(N)^{-3}$ auxiliary particles. Defining $\mathcal{B}_{10}$ as the union, we find for any $\gamma > 0$ a $C_\gamma > 0$ such that $\mathbb{P}(X^{\geq 1} \in \mathcal{B}_{10}) \leq C_\gamma N^{-\gamma}$ using Corollary 2.14.

Working on the complement of this set, we can finally apply Lemma 6.24 with $\beta = 3N^{-\frac{4}{9}+3\sigma} + |\delta_0 - \tilde{\delta}|$ to estimate the first summand 4.5 yielding

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} \left| Df^N(\hat{q}_0^\delta - \tilde{q}_j) \right| \cdot \left( 3N^{-\frac{4}{9}+3\sigma} + |{}^1\delta_0 - {}^1\tilde{\delta}| \right) \mathrm{d}t \\
=&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} \mathbf{1}_{\hat{G}^N(\hat{x}_i)}(x_j) \left| Df^N(\hat{q}_0^\delta - \tilde{q}_j) \right| (3N^{-\frac{4}{9}+3\sigma} + |{}^1\delta_0 - {}^1\tilde{\delta}|) \, \mathrm{d}t \\
&\leq C(T) N^{-\frac{1}{9}+\sigma} (3N^{-\frac{4}{9}+3\sigma} + |\delta_0 - \tilde{\delta}|) + \int_{t_0}^{t_1} C \ln(N) (3N^{-\frac{4}{9}+3\sigma} + |{}^1\delta_0 - {}^1\tilde{\delta}|) \\
&\leq C(T) \ln(N) N^{-\frac{4}{9}+3\sigma} + \int_{t_0}^{t_1} C \ln(N) |{}^1\delta_0 - {}^1\tilde{\delta}| \\
&\leq \underline{N^{-\frac{1}{18}+3.5\sigma} |\delta| + \int_{t_0}^{t_1} C \ln(N) |{}^1\delta_0 - {}^1\tilde{\delta}|},
\end{aligned}$$

where we used our assumption $|\delta| > N^{-\frac{1}{18}} c(N)$ in the last step.

It only remains to estimate 4.6. This can be done by introducing $h$ (we have already seen above that this is okay) and then estimating $h$ by Lemma 6.24. This yields:

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} \\
&\cdot \int_0^1 \left| Df^N(\tilde{q}_0^\delta - \tilde{q}_j + s(q_0^\delta - \tilde{q}_0^\delta - q_j^\delta + \tilde{q}_j)) - Df^N(\hat{q}_0^\delta - \tilde{q}_j) \right| (3N^{-\frac{4}{9}+3\sigma} + |\delta_0 - \tilde{\delta}|) \, \mathrm{d}s \, \mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} h^N(\hat{q}_0^\delta - \tilde{q}_j) (3N^{-\frac{4}{9}+3\sigma} + |\delta_0 - \tilde{\delta}|)^2 \, \mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)} h^N(\hat{q}_0^\delta - \tilde{q}_j) (4N^{-\frac{1}{18}+4\sigma} |\delta|)^2 \, \mathrm{d}t \\
\leq &C(T) N^{-\frac{1}{9}+8\sigma} |\delta|^2 c(N)^{-1} \\
\leq &\underline{C(T) N^{-\frac{1}{9}+8\sigma} |\delta|},
\end{aligned}$$

where we used $|\delta_0 - \tilde{\delta}| \leq N^{-\frac{1}{18}+4\sigma}$ and in the end $|\delta| \leq c(N)$.

$$\begin{aligned}
&\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \text{ good}} f(q_0 - q_j) - f(q_0^\delta - q_j^\delta) - \left( f^N(\tilde{q}_0 - \tilde{q}_j) - f^N(\tilde{q}_0^\delta - \tilde{q}_j) \right) \mathrm{d}t \right| \\
&\leq \underline{\underline{C(T) N^{-\frac{1}{18}+3.5\sigma} |\delta| + \int_{t_0}^{t_1} C \ln(N) |{}^1\delta_0 - {}^1\tilde{\delta}| \, \mathrm{d}t.}}
\end{aligned}$$

Est. 3b) Next, we want to estimate

$$\begin{aligned}
&\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0} (f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\tilde{q}_0 - \tilde{q}_j)) \right. \\
&\left. - \int_{\mathbb{R}^6} \left( f(\tilde{q}_0^\delta - y) - f^N(\tilde{q}_0 - y) \right) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}y \, \mathrm{d}t \right|.
\end{aligned}$$

For this we again need to consider the cases $|\delta_0| \approx c(N)$ and $|\delta_0| \ll c(N)$ separately. As we are already working with our auxiliary particle, we start with the case $\delta_0 \approx c(N)$ this time.

$\delta \approx c(N)$ If $|\delta| \geq N^{-\frac{1}{18}} c(N)$, we can use our auxiliary particles to estimate

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} \left( f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\tilde{q}_0 - \tilde{q}_j) \right) \mathrm{d}t \right. \\
&\left. - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} \left( f^N(\tilde{q}_0^\delta - {}^1\widetilde{y}) - f^N(\tilde{q}_0 - {}^1\widetilde{y}) \right) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}y \, \mathrm{d}t \right| \\
\leq &\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} f^N(\tilde{q}_0 - \tilde{q}_j) \, \mathrm{d}t - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\tilde{q}_0 - {}^1\widetilde{y}) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}t \, \mathrm{d}y \right| \\
&+ \left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0} \left( f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\hat{q}_0^\delta - \tilde{q}_j) \right) \mathrm{d}t \right| \\
&+ \left| \int_{t_0}^{t_1} \int_{\mathbb{R}^6} \left( f^N(\tilde{q}_0^\delta - {}^1\widetilde{y}) - f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \right) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}y \, \mathrm{d}t \right| \\
&+ \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - \tilde{q}_j) \quad - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}t \, \mathrm{d}y \right|.
\end{aligned}$$

The first summand can be estimated by $\underline{4N^{-\frac{4}{9}+\sigma}}$ using Corollary 2.14, which is applicable as we work on the complement of $\mathcal{B}_9$.

The second and third summand can be estimated quite brutally by leveraging that $|\hat{x}_0^\delta - \tilde{x}_0^\delta| \leq C(T) c(N)^4 N^{-\frac{1}{9}}$. This yields

$$\begin{aligned}
&\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0} f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\hat{q}_0^\delta - \tilde{q}_j) \, \mathrm{d}t \right| \\
&+ \left| \int_{t_0}^{t_1} \int_{\mathbb{R}^6} \left( f^N(\tilde{q}_0^\delta - {}^1\widetilde{y}) - f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \right) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}y \, \mathrm{d}t \right| \\
\leq &C(T) c(N)^4 N^{-\frac{1}{9}} \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq 0} |Df|_\infty \, \mathrm{d}t + C(T) c(N)^4 N^{-\frac{1}{9}} \int_{t_0}^{t_1} \int_{\mathbb{R}^6} |Df|_\infty k(y) \, \mathrm{d}y \, \mathrm{d}t \\
\leq &C(T) c(N)^4 N^{-\frac{1}{9}} c(N)^{-3} \leq C(T) c(N) N^{-\frac{1}{9}} \leq \underline{C(T) N^{-\frac{1}{18}} |\delta|},
\end{aligned}$$

where we used $|\delta| \geq N^{-\frac{1}{18}} c(N)$.

It remains to estimate

$$\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - \tilde{q}_j) \, \mathrm{d}t - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}t \, \mathrm{d}y \right|.$$

At first, this looks like an easy application of our law of large numbers, i.e. Corollary 2.14. However, on the right side we have the wrong indicator-function. Hence, we estimate

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - \tilde{q}_j) \, \mathrm{d}t \quad - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \mathbf{1}_{G^N(x_0)}(y) k(y) \, \mathrm{d}t \, \mathrm{d}y \right| \\
\leq &\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - \tilde{q}_j) \, \mathrm{d}t \quad - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(y) k(y) \, \mathrm{d}t \, \mathrm{d}y \right| \\
&+ \left| \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \left( \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(y) - \mathbf{1}_{G^N(x_0)}(y) \right) k(y) \, \mathrm{d}t \, \mathrm{d}y \right|.
\end{aligned}$$

We start with the more interesting second summand. As $G^N(x_0) \subseteq \hat{G}^N(\hat{x}_0^\delta)$, we can estimate by

$$\int_{\mathbb{R}^6} \int_{t_0}^{t_1} \left| f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \right| \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta) \cap G^N(x_0)^C}(y) k(y) \, \mathrm{d}t \, \mathrm{d}y.$$

Using that we work on $\hat{G}^N(\hat{x}_0^\delta)$ as well as Lemma 2.7, we can estimate by

$$\begin{aligned}
&\leq \int_{\mathbb{R}^6} C(T) \frac{1}{c(N)^2 N^{-\frac{2}{9}}} \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta) \cap G^N(x_0)^C}(y) k(y) \, \mathrm{d}y \\
&\leq C(T) \frac{1}{c(N)^2 N^{-\frac{2}{9}}} \, \mathbb{P}\left( X \in G^N(x_0)^C \right) \\
&\leq C(T) c(N)^{-2} N^{\frac{2}{9}} (N^{-\frac{2}{9}-\sigma})^2 (N^{-\frac{2}{9}})^4 \\
&\leq C(T) N^{\frac{8}{9}+\sigma} N^{-\frac{12}{9}-2\sigma} \\
&\leq C(T) N^{-\frac{4}{9}-\sigma} \leq C(T) N^{-\frac{1}{9}} c(N) \leq \underline{C(T) N^{-\frac{1}{18}} |\delta|},
\end{aligned}$$

where in the last step we used our assumption $|\delta| \geq N^{-\frac{1}{18}} c(N)$.
To estimate

$$\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - \tilde{q}_j) \quad - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(y) k(y) \, \mathrm{d}y \right|,$$

we need to introduce a last set of bad starting configurations.

$$\begin{aligned}
&X^{\geq 1} \in \mathcal{B}_{11,i} \subseteq \mathbb{R}^{6(N-1)} \qquad (\mathcal{B}_{11}) \\
&\quad \iff \Bigg( \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} f\Big( (\hat{q}_0^\delta)_i - \tilde{q}_j \Big) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(x_j) \, \mathrm{d}s \right. \\
&\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f\Big( (\hat{q}_0^\delta)_i - {}^1\widetilde{z} \Big) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(z) \, \mathrm{d}s k(z) \, \mathrm{d}z \right| > 4N^{-\frac{4}{9}+\sigma} \Bigg),
\end{aligned}$$

where the $i$ indexes all of our $CN^{\frac{1}{9}} c(N)^{-3}$ auxiliary particles.
By Corollary 2.14 we can estimate the probability $\mathcal{B}_{11} = \bigcup \mathcal{B}_{11,i}$ by

$$\mathbb{P}(X^{\geq 1} \in \mathcal{B}_{11}) \leq C_\gamma N^{-\gamma}.$$

On the complement of $\mathcal{B}_{11}$ we, of course, have

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - \tilde{q}_j) \, \mathrm{d}t \; - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} f^N(\hat{q}_0^\delta - {}^1\widetilde{y}) \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(y) k(y) \, \mathrm{d}t \, \mathrm{d}y \right| \\
\leq & 4N^{-\frac{4}{9}+\sigma} \leq \underline{N^{-\frac{1}{18}+2\sigma} |\delta|}.
\end{aligned}$$

$\delta_0 \ll c(N)$ If $|\delta| \le N^{-\frac{1}{18}} c(N)$, we can estimate

$$\begin{aligned}
&\Big| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} \big( f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\tilde{q}_0 - \tilde{q}_j) \big) \\
&\quad - \int_{\mathbb{R}^6} \big( f(\tilde{q}_0^\delta - y) - f^N(\tilde{q}_0 - y) \big) \mathbf{1}_{G^N(x_0)}(y) k(y) \,\mathrm{d}y \,\mathrm{d}t \Big| \\
=&\Big| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} \int_0^1 Df^N(\tilde{q}_0 - \tilde{q}_j + s\tilde{\delta}) \tilde{\delta} \,\mathrm{d}s \\
&\quad - \int_{\mathbb{R}^6} \int_0^1 Df^N(\tilde{q}_0 - y + s\tilde{\delta}) \mathbf{1}_{G^N(x_0)}(y) k(y) \tilde{\delta} \,\mathrm{d}s \,\mathrm{d}y \,\mathrm{d}t \Big| \\
\le& \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} \Big| \int_0^1 Df^N(\tilde{q}_0 - \tilde{q}_j + s\tilde{\delta}) - Df^N(\tilde{q}_0 - \tilde{q}_j) \Big| \,|\tilde{\delta}| \,\mathrm{d}t \\
&\quad + \int_{t_0}^{t_1} \int_{\mathbb{R}^6} \int_0^1 \Big| Df^N(\tilde{q}_0 - y + s\tilde{\delta}) - Df^N(\tilde{q}_0 - y) \Big| \,\mathrm{d}s \mathbf{1}_{G^N(x_0)}(y) k(y) |\tilde{\delta}| \,\mathrm{d}y \,\mathrm{d}t \\
&\quad + \Big| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} Df^N(\tilde{q}_0 - \tilde{q}_j) - \int_{t_0}^{t_1} \int_{\mathbb{R}^6} Df^N(\tilde{q}_0 - y) \mathbf{1}_{G^N(x_0)}(y) k(y) \,\mathrm{d}t \Big| \,\tilde{\delta}(t_1) \\
\le& \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} \Big| h^N(\tilde{q}_0 - \tilde{q}_j) \Big| \,|\tilde{\delta}(t)|^2 \,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^6} \int_{t_0}^{t_1} \Big| h^N(\tilde{q}_0 - y) \Big| \,|\tilde{\delta}(t)|^2 \,\mathrm{d}t \,\mathrm{d}y \\
&\quad + \Big| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} Df^N(\tilde{q}_0 - \tilde{q}_j) \,\mathrm{d}t - \int_{\mathbb{R}^6} \int_{t_0}^{t_1} Df^N(\tilde{q}_0 - y) \mathbf{1}_{G^N(x_0)}(y) k(y) \,\mathrm{d}t \Big| \,|\tilde{\delta}(t_1)|.
\end{aligned}$$

Here the first summand can be estimated by $Cc(N)^{-1}\tilde{\delta}^2 \le \underline{CN^{-\frac{1}{18}}\delta}$ using Lemma 6.24 (with $\beta = 1$); the second summand can be estimated by $Cc(N)^{-1}\tilde{\delta}^2 \le \underline{CN^{-\frac{1}{18}}\delta}$ using Lemma 2.5. Finally, the last summand can be estimated by $\underline{4N^{-\frac{1}{9}+\sigma}\tilde{\delta}}$, as we work on the complement of $\mathcal{B}_9$.

Combining these estimates, we have proven

$$\begin{aligned}
&\Big| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq 0} \big( f^N(\tilde{q}_0^\delta - \tilde{q}_j) - f^N(\tilde{q}_0 - \tilde{q}_j) \big) \\
&\quad - \int_{\mathbb{R}^6} \big( f(\tilde{q}_0^\delta - y) - f^N(\tilde{q}_0 - y) \big) \mathbf{1}_{G^N(x_0)}(y) k(y) \,\mathrm{d}y \,\mathrm{d}t \Big| \\
\le& \underline{\underline{C(T) N^{-\frac{1}{18}+2\sigma} |\delta|}}.
\end{aligned}$$

Est. 3c) We already had a similar estimate:

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6}\left(f^N(\tilde{q}_0^\delta - y) - f^N(\tilde{q}_0 - y))(1 - \mathbf{1}_{G^N(x_0)}(y)\right)k(y)\,\mathrm{d}y\,\mathrm{d}t\right| \\
\leq& C|\tilde{\delta}|\left|\int_{t_0}^{t_1}\int_{\mathbb{R}^6}\int_0^1 Df^N(\tilde{q}_0 - y + s\tilde{\delta})k(y)\mathbf{1}_{G^N(x_0)^C}(y)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}t\right| \\
\leq& C|\tilde{\delta}|Tc(N)^{-3}\left|\int_{\mathbb{R}^6} k(y)\mathbf{1}_{G^N(x_0)^C}(y)\,\mathrm{d}y\right| \\
\leq& CT|\tilde{\delta}|N^{1+\sigma}\,\mathbb{P}(y \in G^N(x_0)^C) \leq CTN^{1+\sigma}|\tilde{\delta}|(N^{-\frac{2}{9}-\sigma})^2(N^{-\frac{2}{9}})^4 \\
\leq& CTN^{1-\frac{12}{9}-2\sigma}|\tilde{\delta}| = \underline{CTN^{-\frac{1}{3}-\sigma}|\tilde{\delta}|.}
\end{aligned}$$

### 4.3.2 Grönwall estimate

We have proven that

$$\begin{aligned}
&\int_{t_0}^{t_1}\frac{1}{N}f^N(q_0 - q_j) - f^N(q_0^\delta - q_j^\delta) - \left(\tilde{f}^N(\tilde{q}_j) - \tilde{f}^N(\tilde{q}_0^\delta)\right)\mathrm{d}t \\
\leq& CN^{-\frac{1}{18}+3.5\sigma}|\delta| + C\int_{t_0}^{t_1}\ln(N)|{}^1\delta_0 - {}^1\tilde{\delta}|\,\mathrm{d}t.
\end{aligned}$$

Finally, we can use our Grönwall argument Lemma 3.2 one last time. This time with $v_1 = |{}^1\delta_0 - {}^1\tilde{\delta}|$ and $v_2 = |{}^2\delta_0 - {}^2\tilde{\delta}|$ as well as $C(T,N) = CN^{-\frac{1}{18}+3.5\sigma}|\delta|$. The lemma then yields that

$$|{}^2\delta_0(\tau) - {}^2\tilde{\delta}(\tau)| \leq CN^{-\frac{1}{18}+(3.5+\varepsilon)\sigma}|\delta|.$$

Hence, for large enough $N$ we can conclude that

$$|\delta_0(\tau) - \tilde{\delta}(\tau)| \leq N^{-\frac{1}{18}+4\sigma}|\delta|.$$

This means, the stopping time was not triggered by the added and moved particle $x_0$.

Combining this with the other results from this chapter, we can conclude that $\tau = T$. Of course, this only holds true on the complement of all the $\mathcal{B}$. As

$$\mathbb{P}\Big(\bigcup_{k=1}^{11}\mathcal{B}_i\Big) \leq C_\gamma N^{-\gamma},$$

we conclude that $\mathbb{P}(\tau \neq T) < C_\gamma N^{-\gamma}$ for a suitable constant. This directly proves our main theorem.

## 4.4 Discussion of the main result

One could interpret our main result as a form of stability of the system. Following this idea, the result can be summarized in one sentence as:

> If you change the starting position of one particle in a many-particle system by a bit, the system as a whole typically stays stable.

Inspired by this interpretation, we will call a configuration $X^{\geq 1}$ stable with respect to $x_0$ if it satisfies all the estimates from our main result, i.e. 4.2.

This point of view also motivates some mathematical results: In the next chapter we will prove that if $X^{\geq 1}$ is stable with respect to $x_0$, even after moving $x_0$ a bit, the mean-field trajectories will still be a good estimate for the real trajectories in this disturbed setting. In the first order sense, i.e. as in Graß's result, this is an easy corollary. However, this is even true in the second order sense, but that proof will be slightly more involved.

Comparing our result to Graß, he was able to prove that typically the mean-field trajectories, i.e. the $\varphi(x)$, are a good estimate for the real trajectories $\phi(x)$. We have proven in Theorem 4.2 that in a typical configuration this approximation still holds when considering the derivative (with respect to the starting position of $x$ in phase-space). We will make this precise in the following chapter, but it is already intuitively clear that this is just a rephrasing of our result. We studied how $\phi(x+\delta)-\phi(x)$ behaves and were able to show that $\varphi(x+\delta)-\varphi(x)$ is a very good estimate in the sense that the error is smaller than $N^{-\frac{1}{18}+4\sigma}|\delta|$. So, after dividing by $|\delta|$ and taking the limit, this becomes a statement about derivatives. One, of course, first has to check that both functions are differentiable, which we will do in the next chapter.

This approximation of first order yields many useful results. For a nice enough configuration $X^{\geq 1}$ and a fixed $t \in [0,T]$ we will compare the map $\varphi_t(x_0)$ with the map $\psi_t(x_0)$, which adds $x_0$ to the configuration and then applies the Newtonian time evolution.

The mean-field flow $\varphi_t$ is a diffeomorphism, and with our new estimates we will be able to lift this property onto $\psi_t$. This can be applied when estimating probabilities. In fact, we will show later on that in certain cases the probability of the Newtonian trajectory passing through a given area after some time can be estimated using mean-field trajectories.

Before we move on to some of the consequences and corollaries we have already mentioned, we want to briefly discuss the strengths and shortcomings of our result.

We start with the cut-off $c(N) = N^{-\frac{1}{3}}\ln(N)^{-\frac{1}{4}}$. Our intention in the beginning was to build on the results by Graß in [9], and hence we at first considered a cut-off at $N^{-\frac{7}{18}+\sigma}$ for small $\sigma > 0$. This cut-off, however, was not tenable. The problem is that we deal with higher derivatives of the force, which are more singular. This can be seen very clearly in our estimate for the bad particles, more precisely at the beginning when estimating the term 2a. Most of the time the cut-off is not so relevant in the estimates, because we have other stronger conditions, like particles being good particles. In this estimate we are considering exactly the effect of bad interactions, and hence the only estimate we can do relies on the cut-off. Because we have to extract our $\delta$, we have to consider $Df$. Comparing this to the same estimate in Graß's paper, he only has to deal with the force $f$ itself, making the estimate quite trivial even with the larger cut-off.

One of Graß's stated goals was to bring the cut-off below $N^{-\frac{1}{3}}$, which is the mean inter-particle distance in 3-dimensional space. Our cut-off of $N^{-\frac{1}{3}}\ln(N)^{-\frac{1}{4}}$, at least barely, achieves this goal.

While on the topic of cut-off sizes, we also repeat the following observation by Graß. Following the ansatz both we and Graß used, it is impossible to shrink the cut-off behind an order of $N^{-\frac{1}{2}}$. This stems from the fact that the law of large numbers at best yields a control of order $N^{-\frac{1}{2}}$ around the expectation value. For a more detailed discussion see [9].

Finally, we want to give an outlook on how our result could be improved. First, any ansatz that improves on Graß's result is at least very promising to also yield a better cut-off or better estimates for us. One such example is Feistl's approach in [5] to not only consider good and bad particles but rather three types of particles. With this approach she was able to shrink the cut-off and a similar idea could work in our setting.

On a related note, we kept the definition of good and bad particles from Graß but changed our cut-off. It is plausible that definitions tailored directly to our configuration could strengthen our result.

Furthermore, we only considered the effects of moving one good particle. However, when carefully reading our proof, one will realize that moving $m \in \mathbb{N}$ good particles for a fixed integer $m$ should yield similar results. In fact, we believe that moving $\approx \ln(N)$ good particles should still be manageable with our approach. Considering multiple particles will make the notion of rings around the moved particles more complicated, and one probably has to take into account how close the moved particles are to each other. If the moved particles are too close to each other, the rings might overlap making estimates more involved. As a first step one could drop the notion of rings. This is the same as assuming all particles are in the first ring around $x_0$ and should still yield interesting results.

Finally, we are confident that one can also move bad particles. The main difference is that the disturbed bad particle will infect its bad collision partners. After the interaction the infected particles will behave like disturbed particles themselves. But we know that typically the amount of bad particles in the cluster around one bad particle can be estimated by $\approx \ln(N)$; so we are hopeful that this can be resolved. Of course, we expect that moving a bad particle has more effect on the system than moving a good particle.

# 5 Applications of the main result

## 5.1 Interpretation as stability

One consequence of our result is a form of stability to our system. As a direct corollary of our main result we obtain that our result from Chapter 3 not only holds typically for any $x_0 \in \mathbb{R}^6$, but in fact typically holds for all potential $x_0$ inside a small sphere.

With a bit more effort we will then also show that our main result satisfies a similar stability.

### 5.1.1 Stability of our result in Chapter 3

In this section we will quickly prove the following corollary as a combination of theorems 4.2 and 3.1.

**Corollary 5.1.** *et $\sigma > 0$ and let $f^N : \mathbb{R}^3 \to \mathbb{R}^3$ be the force from our set up and $k$ be a density satisfying the conditions we introduced earlier. Let $T > 0$ and $\gamma > 0$. We find a constant $C_\gamma$ such that for any $y \in \mathbb{R}^6$ we have for all $x_0 \in B_{N^{-\frac{7}{18}-2\sigma}}(y)$ we have*

$$\mathbb{P}\left( \sup_{x_i \in X \text{ good}} \sup_{0\le s\le T} |x_i(s) - \tilde{x}_i(s)| \le N^{-\frac{4}{9}+3\sigma} \,\middle|\, y \text{ is good particle in } X^{\ge 1} \cup \{x_0\} \right) < C_\gamma N^{-\gamma},$$

$$\mathbb{P}\left( \sup_{x_i \in X \text{ bad}} \sup_{0\le s\le T} |x_i(s) - \tilde{x}_i(s)| \le N^{-\frac{1}{3}+\sigma} \,\middle|\, y \text{ is good particle in } X^{\ge 1} \cup \{x_0\} \right) < C_\gamma N^{-\gamma}.$$

*Here all probabilities are taken with respect to $X^{\ge 1} \in \mathbb{R}^{6(N-1)}$, so importantly $x_0$, unlike all the other $x_i$, is not a random variable.*

*Proof.* Let $X^{\ge 1} \in \mathbb{R}^{6(N-1)}$ be a configuration which is stable with respect to $y$, i.e. that satisfies the estimate from our main result 4.2 when moving $y$. As always, we assume that $y$ is a good particle in $X = X^{\ge 1} \cup \{y\}$.

This implies directly that our result from Chapter 3 holds, i.e. we know that for the good particles (including $y$) we have

$$\sup_{0\le s\le T} |x_i(s) - \tilde{x}_i(s)| \le N^{-\frac{4}{9}+3\sigma},$$

and for bad $x_i$ we still have

$$\sup_{0\le s\le T} |x_i(s) - \tilde{x}_i(s)| \le N^{-\frac{1}{3}+\sigma}.$$

Using the newly proven stability of the system, these estimates hold true if we would have added any $x_0 = y + \delta$ with small enough $\delta \in \mathbb{R}^6$ instead. Note that our result allows us to choose any $\delta$ with $|\delta| \le c(N)$, but for this result we will only need to consider $|\delta| \le N^{-\frac{7}{18}-2\sigma}$.

The idea is to now interpret the system $X^{\ge 1} \cup \{x_0\}$ as $\left(X^{\ge 1} \cup \{y\}\right)^\delta$.

To avoid weakening the estimates, we start with the ever so slightly stronger bounds that came up in the proof of 3.1, namely, for good particles

$$\sup_{0\le s\le T}|x_i(s)-\tilde{x}_i(s)|\le C(T)N^{-\frac{4}{9}+3\sigma-\varepsilon},$$

and for bad $x_i$

$$\sup_{0\le s\le T}|x_i(s)-\tilde{x}_i(s)|\le C(T)N^{-\frac{1}{3}+\sigma-\varepsilon},$$

for some small $\varepsilon>0$.

With this in mind, we obtain the following estimates for the three categories of particles:

$\underline{x_0}$ As $y$ is good we have $\sup_{s\in[0,T]}|y-\tilde{y}|\le CN^{-\frac{4}{9}+3\sigma-\varepsilon}$. Using that

$$|y_0^\delta-y_0-(\tilde{y}_0^\delta-y_0^\delta)|=|\delta_0-\tilde{\delta}|\le N^{-\frac{1}{18}+4\sigma}|\delta|\le N^{-\frac{1}{18}+4\sigma}N^{-\frac{7}{18}-2\sigma}=N^{-\frac{4}{9}+2\sigma},$$

we obtain

$$\begin{aligned}|x_0-\tilde{x}_0|=|y^\delta-\tilde{y}^\delta|&\le|y_0^\delta-y_0-(\tilde{y}_0^\delta-y_0^\delta)|+|y_0-\tilde{y}_0|\\&\le N^{-\frac{4}{9}+2\sigma}+CN^{-\frac{4}{9}+3\sigma-\varepsilon}\le(C+1)N^{-\frac{4}{9}+3\sigma-\varepsilon}.\end{aligned}$$

For large enough $N$ this, of course, is bounded by $N^{-\frac{4}{9}+3\sigma}$, which is a good enough estimate. Note that we do not care whether $x_0$ is a good or a bad estimate.

$\underline{x_i \text{ good}}$ For the other good particles we have $|x_i-\tilde{x}_i|\le CN^{-\frac{4}{9}+3\sigma-\varepsilon}$ in $X^{\ge1}\bigcup\{y\}$. This time we also know that $|x_i-x_i^\delta|\le N^{-\frac{1}{9}+2\sigma}\delta\le N^{-\frac{1}{9}+2\sigma}N^{-\frac{7}{18}-2\sigma}=N^{-\frac{4}{9}}$. This allows us to estimate:

$$|x_i^\delta-\widetilde{x}_i|\le|x_i^\delta-x_i|+|x_i-\widetilde{x}_i|\le N^{-\frac{4}{9}}+CN^{-\frac{4}{9}+3\sigma-\varepsilon}\le(C+1)CN^{-\frac{4}{9}+3\sigma-\varepsilon}.$$

This again is a suitable estimate for large enough $N$.

$\underline{x_i \text{ bad}}$ For bad particles the estimate is weaker and we only know $|x_i-\widetilde{x}_i|\le C(T)N^{-\frac{1}{3}+\sigma-\varepsilon}$. At the same time we have $|x_i-x_i^\delta|\le N^{-\frac{1}{18}+2\sigma}|\delta|\le N^{-\frac{4}{9}}$. Combining these estimates, we obtain

$$|x_i^\delta-\widetilde{x}_i|\le|x_i^\delta-x_i|+|x_i-\widetilde{x}_i|\le N^{-\frac{4}{9}}+CN^{-\frac{1}{3}+\sigma-\varepsilon}\le(C+1)CN^{-\frac{1}{3}+\sigma-\varepsilon},$$

which – for large enough $N$ – is bounded by $N^{-\frac{1}{3}+\sigma}$.

This means that for large enough $N$ for all good particles in $X^{\ge1}\cup\{x_0\}=\left(X^{\ge1}\cup\{y\}\right)^\delta$ we have $|x_i-\tilde{x}_i|\le N^{-\frac{4}{9}+3\sigma}$ and for bad particles the distance to the mean-field trajectory can be estimated by $N^{-\frac{1}{3}+\sigma}$. □

The same argument can be applied to our alternative version 4.3 where we have proven that for suitable configurations any good particle can be moved. In this setting we obtain that even after moving any good particle in such a configuration the estimates from 3.1 still hold. Note that we needed the estimates provided by our new result. Hence, not every configuration, that satisfies 3.1 with respect to $x_0$, will automatically satisfy this local version.

### 5.1.2 Stability of our main result

In this section we give a sketch on how to prove that our main result is stable in the same sense. By this we mean that for $x_0 \in \mathbb{R}^6$ a typical configuration is not only stable with respect to $x_0$ but stable to all $y$ in a small neighborhood around $x_0$.

Ideally, this should have been integrated in our version of the result, but due to time constraints we handle it here. Note that most of the times our estimates did not depend on $x_0$ and that all the conditions on the starting configurations were handled in the $\mathcal{B}$ sets we removed.

Here, we are going to prove that all these $\mathcal{B}$ can be modified to work with all $y$ in a $N^{-\frac{7}{18}-2\sigma}$ ball around $x_0$. Of course, one does need to carefully check the whole proof to make sure that no other problems arise, but we are very confident that this is the case.

One important aspect is that most of our definitions use the mean-field trajectories. In particular, adding a good particle $y$ to a configuration, will not change which particles are good and bad.

to $\mathcal{B}_1$ This is just our version of Graß's result, which, as we just have seen, typically holds on a $N^{-\frac{7}{18}-2\sigma}$ ball around $x_0$.

to $\mathcal{B}_2$ In $\mathcal{B}_{2a}$ we considered

$$\begin{aligned} X^{\geq 1} \in \mathcal{B}_{2a,i} &\subseteq \mathbb{R}^{6(N-1)} \\ :\iff \left( \sum_{j \text{ bad}} \mathbf{1}_{M^N_{N^{-\frac{1}{9}}, N^{-\frac{1}{9}}}(x_i)}(x_j) > \ln(N) \right). \end{aligned}$$

Here we had to make sure that the clusters of connected bad particles stay relatively small. For $x_i \neq x_0$ this does not depend on $x_0$. In the case of $x_i = x_0$, of course this depends on $x_0$. However, a typical configuration will satisfy

$$\sum_{j \text{ bad}} \mathbf{1}_{M^N_{2N^{-\frac{1}{9}}, 2N^{-\frac{1}{9}}}(x_0)}(x_j) \leq \ln(N)$$

by Lemma 2.12. In such a configuration we have for all $y$ in the $N^{-\frac{7}{18}-2\sigma}$ ball around $x_0$ that

$$\sum_{j \text{ bad}} \mathbf{1}_{M^N_{N^{-\frac{1}{9}}, N^{-\frac{1}{9}}}(y)}(x_j) \leq \ln(N).$$

For $\mathcal{B}_{2b}$ recall that

$$X^{\geq 1} \in \mathcal{B}_{2b} \subseteq \mathbb{R}^{6(N-1)} :\iff \left| \bigcup_{k=0}^{\frac{1}{18}\sigma+2} R^k \right| > CN^{12\sigma}.$$

First, note that by Lemma 4.5 we know that these particles $x_i$ will stay in their corresponding $CN^{-\frac{1}{6}+2\sigma}$ ball around $x_0$. Estimating the expected number of particles inside this sphere works similarly to our arguments for $\mathcal{B}_{2b}$.

Note that if we would center our rings around a $y$ with $|x_0 - y| \leq N^{-\frac{7}{18}-2\sigma}$, we still have the same estimate (with a modified constant).

Let $x_j \in \bigcup_{k=0}^{\frac{1}{18}\sigma+2} R_y^k$ where the $y$ in the index reflects that these rings are centered around $y$.

This then implies that the trajectory of $x_j$ stays in the $CN^{-\frac{1}{6}+2\sigma}$ sphere around $y$. By the triangle inequality and as the mean-field trajectories are Lipschitz by Lemma 2.6, this means that it also stays in the $\widetilde{C}N^{-\frac{1}{6}+2\sigma}$ ball around $x_0$.

With the same arguments we used in our proof, we can show that for a typical configuration the number of particles in the $\widetilde{C}N^{-\frac{1}{6}+2\sigma}$ ball around $x_0$ can be estimated by $CN^{12\sigma}$. This then implies the corresponding result for any $y$ close enough to $x_0$.

to $\mathcal{B}_3$ Recall from $\mathcal{B}_3$ that

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{3a,i} \subseteq \mathbb{R}^{6(N-1)} \\ &:\iff \exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j \neq 0,i} \int_{t_1}^{t_2} \Big| Df^N(\tilde{q}_i - \tilde{q}_j) \Big| \mathbf{1}_{G^N(x_i)}(x_j)\, \mathrm{d}s \\ &- \int_{\mathbb{R}^6} \int_{t_1}^{t_2} \Big| Df^N(\tilde{q}_i - {}^1\tilde{y}) \Big| \mathbf{1}_{G^N(x_i)}(y)\, \mathrm{d}s k(y)\, \mathrm{d}y \Bigg| > N^{-\frac{1}{9}+\sigma}, \end{aligned}$$

as well as for $R = V = N^{-\frac{1}{6}+\sigma}$,

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{3b,i} \subseteq \mathbb{R}^{6(N-1)} \\ &:\iff \exists t_1, t_2 \in [0,T] : \Bigg| \frac{1}{N} \sum_{j \neq 0,i} \int_{t_1}^{t_2} \Big| Df^N(\tilde{q}_i - \tilde{q}_j) \Big| \mathbf{1}_{M_{R,V}^N(x_i)^C}(x_j)\, \mathrm{d}s \\ &- \int_{\mathbb{R}^6} \int_{t_1}^{t_2} \Big| Df^N(\tilde{q}_i - \tilde{q}_j) \Big| \mathbf{1}_{M_{R,V}^N(x_i)^C}(y)\, \mathrm{d}s k(y)\, \mathrm{d}y \Bigg| > N^{-\frac{1}{6}+\sigma}. \end{aligned}$$

Note that $x_0$ is actively avoided here, and we are only considering the mean-field trajectories of the other particles which do not depend on the added particle $x_0$.

to $\mathcal{B}_4$ Recall from $\mathcal{B}_4$ that

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{4a,i,(r,R)(v,V)} \subseteq \mathbb{R}^{6(N-1)} \qquad \text{with } i \in \{0, \dots, N-1\} \\ &:\iff \Bigg( \sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(x_i)}(x_j) > a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\ &\qquad\qquad \vee |\, \mathcal{M}_b^N(X)| > a_N N^{\frac{2}{3}} \Bigg) \end{aligned}$$

and

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{4b,i,j(r,R)(v,V)} \subseteq \mathbb{R}^{6(N-1)} \qquad \text{with } i \in \{1, \dots, N-1\} \\ &:\iff \Bigg( \sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j) \\ &\qquad > a_N \left\lceil N^{\frac{2}{3}} R_1^2 R_2^2 \min(\max(V_1,R_1),1)^4 \cdot \min(\max(V_2,R_2),1)^4 \right\rceil \Bigg), \end{aligned}$$

where $a_N = \ln(N)^{\frac{1}{4}}$.

We start with $\mathcal{B}_{4a}$. For $x_i \neq x_0$ this does not depend on $x_0$, as we assume $x_0$ to be good. Hence, it does not change the amount of bad particles, as these definitions only depend on the mean-field trajectories.

If $x_i = x_0$, we use the argument from $\mathcal{B}_2$ again. Here we need to be careful as we consider many different $R$ and $V$, namely

(i) $M^N_{(0,r_0),(0,v_0)}(y)$

(ii) $M^N_{(0,r_0),(N^{l\varepsilon}v_0,N^{(l+1)\varepsilon}v_0)}(y), \quad 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\varepsilon \ln(N)} \right\rfloor$

(iii) $M^N_{(0,r_0),(1,\infty)}(y)$

(iv) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(0,v_0)}(y), \quad 0 \leq k \leq \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor$

(v) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(N^{l\varepsilon}v_0,N^{(l+1)\varepsilon}v_0)}(y), \quad 0 \leq k \leq \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor, \quad 0 \leq l \leq \left\lfloor \frac{\ln(\frac{1}{v_0})}{\varepsilon \ln(N)} \right\rfloor$

(vi) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(1,\infty)}(y), \quad 0 \leq k \leq \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor$

(vii) $M^N_{(N^{-\varepsilon},\infty),(0,\infty)}(y)$

with $r_0 = 6N^{-\frac{2}{9}-\sigma}, v_0 = N^{-\frac{2}{9}}$ and $\varepsilon = \delta$. Hence, the smallest (and most problematic) class is given by $M^N_{(0\,,\,6N^{-\frac{2}{9}-\sigma})\,,\,(0\,,\,N^{-\frac{2}{9}})}$ and our trick works again. We only consider configurations where we can control how many bad particles there are in $M^N_{(0\,,\,12N^{-\frac{2}{9}-\sigma})\,,\,(0\,,\,2N^{-\frac{2}{9}})}(x_0)$. By the triangle inequality this then allows us to control how many bad particles there are in $M^N_{(0\,,\,6N^{-\frac{2}{9}-\sigma})\,,\,(0\,,\,N^{-\frac{2}{9}})}(y)$ for $y$ in the $N^{-\frac{7}{18}-2\sigma}$ ball around $x_0$. This works analogously for all other classes as well.

One needs to be a little careful as the constants in the estimates change slightly, but importantly the order of $N$ in the estimate stays the same.

The argument for $\mathcal{B}_{4b}$ is basically the same. To obtain a bound for

$$\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(y) \cap M^N_{R_2,V_2}(x_i)}(x_j),$$

we first get a bound on

$$\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{2R_1,2V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j).$$

Then, as $|x_0 - y| \leq N^{-\frac{7}{18}-2\sigma}$, it follows that

$$\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(y) \cap M^N_{R_2,V_2}(x_i)}(x_j) \leq \sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{2R_1,2V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j).$$

This argument only works if $R_1, V_1 \gg N^{-\frac{7}{18}-2\sigma}$. We only apply this result once with $R = V = C\ln(N)N^{-\frac{2}{9}+k(j)\sigma} \gg N^{-\frac{7}{18}-2\sigma}$.

Again, the estimate we get is slightly worse by a factor of $2^6$, but the order of $N$ stays the same.

to $\mathcal{B}_5$ Recall from $\mathcal{B}_5$ that

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{5,i,k_1,\dots,k_6} \subseteq \mathbb{R}^{6(N-1)} \\ &:\iff \sum_{j\neq i} \mathbf{1}_{M^N_{6N^{-\frac{2}{9}-\sigma},N^{-\frac{2}{9}}}(x_{i,k_1,\dots,k_6})}(x_j) \geq \ln(N)^{\frac{1}{4}}. \end{aligned}$$

In this case we only considered bad $x_i$ and estimated how many bad interactions one bad particle can have. So, whether we add $x_0$ or another close particle $y$ does not change anything as long as the added particle is good.

to $\mathcal{B}_6$ Recall from $\mathcal{B}_6$ that

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{6,i} \subseteq \mathbb{R}^{6(N-1)} \qquad\qquad \text{with } i \in \{1,\dots N-1\} \\ :\iff &\left| \frac{1}{N} \sum_{j\neq 0,i} \int_{t_1}^{t_2} |h^N(\hat{q}_i - \tilde{q}_j)| \mathbf{1}_{\mathcal{M}_G(X)}(x_j) \right. \\ &\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} |h^N(\hat{q}_i - {}^1\tilde{y})| \,\mathrm{d}s \mathbf{1}_{\mathcal{M}_G(X)}(y) k(y) \,\mathrm{d}y \right| > 4N^{-\frac{1}{9}+\sigma} c(N)^{-1}. \end{aligned}$$

Note that $x_0$ is actively left out of the sum and that we only consider the auxiliary cloud of particles around bad particles here, i.e. not $x_0$.

to $\mathcal{B}_7$ Recall from $\mathcal{B}_7$ that

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{7,i} \subseteq \mathbb{R}^{6(N-1)} \\ &:\iff \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j\neq 0,i} \int_{t_1}^{t_2} |Df^N(\hat{q}_i - \tilde{q}_j)| \mathbf{1}_{G^N(x_i)}(x_j) \,\mathrm{d}s \right. \\ &\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} |Df^N(\hat{q}_i - \tilde{q}_j)| \mathbf{1}_{G^N(x_i)}(y) \,\mathrm{d}s \; k(y) \,\mathrm{d}y \right| > 4N^{-\frac{1}{9}+\sigma}. \end{aligned}$$

This works exactly the same as $\mathcal{B}_6$.

to $\mathcal{B}_8$ Recall from $\mathcal{B}_8$ that

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{8,i,(r,R)(v,V)} \subseteq \mathbb{R}^{6(N-1)} \\ :\iff &\left( \sum_{j\neq i \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(\hat{x}_i)}(x_j) > a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \right. \\ &\qquad\qquad \left. \text{or } |\mathcal{M}^N_b(X)| > a_N N^{\frac{2}{3}} \right). \end{aligned}$$

We only consider the auxiliary clouds around the bad particles and, as the sum only counts bad particles, the choice of $x_0$ has no influence.

to $\mathcal{B}_{10}$ We first consider $\mathcal{B}_{10}$, as we need its estimates for $\mathcal{B}_9$. Recall from $\mathcal{B}_{10}$ that

$$\begin{aligned} &X^{\geq 1} \in \mathcal{B}_{10a,i} \subseteq \mathbb{R}^{6(N-1)} \\ &\iff \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j\neq 0} \int_{t_1}^{t_2} Df\Big((\hat{q}^\delta_0)_i - \tilde{q}_j\Big) \mathbf{1}_{\hat{G}^N(\hat{x}^\delta_0)}(x_j) \,\mathrm{d}s \right. \\ &\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} Df\Big((\hat{q}^\delta_0\Big)_i - {}^1\tilde{z}) \mathbf{1}_{\hat{G}^N(\hat{x}^\delta_0)}(z) \,\mathrm{d}s \; k(z) \,\mathrm{d}z \right| > 4N^{-\frac{1}{9}+\sigma} \end{aligned}$$

and

$$X^{\geq 1} \in \mathcal{B}_{10b,i} \subseteq \mathbb{R}^{6(N-1)}$$
$$\iff \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N\Big((\hat{q}_0^\delta)_i - \tilde{q}_j\Big) \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(x_j)\,\mathrm{d}s \right.$$
$$\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N\Big((\hat{q}_0^\delta)_i - {}^1\tilde{z}\Big) \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(z)\,\mathrm{d}s\; k(z)\,\mathrm{d}z \right| > 4N^{-\frac{1}{9}+\sigma} c(N)^{-1}.$$

Note that here we used $\hat{q}_0$ to cover an area around our particle. To account for other particles $y$ near $x_0$, we simply cover a larger area. This way we always find a suitable auxiliary particle.

More precisely, we still want the auxiliary particles to have a distance to each other of order $N^{-\frac{1}{9}} c(N)^4$ but this time to fill a ball with radius of $2c(N)$ instead of just $c(N)$ around $x_0$.

As we will apply this for $\mathcal{B}_9$, we also want our configurations to satisfy:

$$X^{\geq 1} \in \mathcal{B}_{10c,i} \subseteq \mathbb{R}^{6(N-1)}$$
$$:\iff \Bigg( \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} f^N((\hat{q}_0^\delta)_i - \tilde{q}_j) \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(x_j)\,\mathrm{d}s \right.$$
$$\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f^N((\hat{q}_0^\delta)_i - {}^1\tilde{z}) \mathbf{1}_{\hat{G}^N(\hat{x}_0^\delta)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z \right| > 4N^{-\frac{1}{9}+\sigma} c(N)^{-1} \Bigg).$$

This is the only modification we need here as the auxiliary particles do not directly depend on the added particle, and we only consider their mean-field trajectories.

to $\mathcal{B}_9$ Recall from $\mathcal{B}_9$ that

$$X^{\geq 1} \in \mathcal{B}_{9a,i} \subseteq \mathbb{R}^{6(N-1)}$$
$$:\iff \Bigg( \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}s \right.$$
$$\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N(\tilde{q}_0 - {}^1\tilde{z}) \mathbf{1}_{G^N(x_0)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z \right| > 4N^{-\frac{1}{9}+\sigma} c(N)^{-1} \Bigg)$$

and

$$X^{\geq 1} \in \mathcal{B}_{9b,i} \subseteq \mathbb{R}^{6(N-1)}$$
$$:\iff \Bigg( \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} Df(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}s \right.$$
$$\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} Df(\tilde{q}_0 - {}^1\tilde{z}) \mathbf{1}_{G^N(x_0)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z \right| > 4N^{-\frac{1}{9}+\sigma} \Bigg)$$

as well as

$$X^{\geq 1} \in \mathcal{B}_{9c,i} \subseteq \mathbb{R}^{6(N-1)}$$
$$:\iff \Bigg( \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} f(\tilde{q}_0 - \tilde{q}_j) \mathbf{1}_{G^N(x_0)}(x_j)\,\mathrm{d}s \right.$$
$$\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f(\tilde{q}_0 - {}^1\tilde{z}) \mathbf{1}_{G^N(x_0)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z \right| > N^{-\frac{4}{9}+\sigma} \Bigg).$$

These estimates clearly depend on the choice of $x_0$. We use our auxiliary particles one more time. For any particle $y$ close enough to $x_0$ we find a auxiliary particle $\hat{x}_0$ with $|\tilde{y} - \hat{x}_0| < N^{-\frac{1}{9}} c(N)^4$.

With this we can estimate

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j) \, \mathrm{d}s - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N({}^1\tilde{y} - {}^1\widetilde{z}) \mathbf{1}_{G^N(y)}(z) \, \mathrm{d}s k(z) \, \mathrm{d}z \right| \\
\leq &\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N(\hat{q}_0 - \tilde{q}_j) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(x_j) \, \mathrm{d}s - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N(\hat{q}_0 - {}^1\widetilde{z}) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(z) \, \mathrm{d}s k(z) \, \mathrm{d}z \right| \\
&+ \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j) \, \mathrm{d}s - \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N(\hat{q}_0 - \tilde{q}_j) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(x_j) \, \mathrm{d}s \right| \\
&+ \left| \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N({}^1\tilde{y} - {}^1\widetilde{z}) \mathbf{1}_{G^N(y)}(z) \, \mathrm{d}s \, k(z) \, \mathrm{d}z - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N(\hat{q}_0 - {}^1\widetilde{z}) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(z) \, \mathrm{d}s \, k(z) \, \mathrm{d}z \right|.
\end{aligned}$$

Here the first summand can be estimated by $4N^{-\frac{1}{9}+\sigma} c(N)^{-1}$ after removing a suitable set of bad starting configurations (i.e. $\mathcal{B}_{10}$). The second and third summand can be estimated using that $|\tilde{q}_0 - \hat{q}_0|$ is very small. However, we need to be careful as the indicator functions do not match.

In the case of the second summand this is no problem as we have $G^N(y) \subseteq \hat{G}^N(\hat{x}_0)$ and all $x_j$ are good with respect to $y$ which means that they also lie in $G^N(\hat{x}_0)$.

Hence, we can estimate the second summand by

$$\left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N({}^1\tilde{y} - \tilde{q}_j) - h^N(\hat{q}_0 - \tilde{q}_j) \, \mathrm{d}s \right|.$$

This is the first time we have to estimate how $h$ behaves under small changes in the variable. Looking back at the definition of $h$ (see Lemma 2.4), this can be estimated by the function

$$h'^N(q) = \begin{cases} C_1 c(N)^{-5} & \text{if } |q| \leq 3\,\mathbf{c}\, c(N), \\ C_1 |q|^{-5} & \text{else }, \end{cases}$$

for suitable constant $C_1$. More precisely we have:

**Lemma 5.2.** *If $\delta \in \mathbb{R}^6$ satisfies $|\delta| \leq \mathbf{c}\, c(N)$ or $|\delta| < \frac{1}{2}|q|$, it holds that*

$$\left| h^N(q) - h^N(q + \delta) \right| \leq h'^N(q) |\delta|.$$

*Proof.* Let $\delta \in \mathbb{R}^6$ be given such that $|\delta| \leq \mathbf{c}\, c(N)$ or $|\delta| \leq \frac{1}{2}|q|$. The estimate is trivial for $|q| \leq 3\,\mathbf{c}\, c(N)$. If $|q| \geq 3\,\mathbf{c}\, c(N)$, then we can conclude that $|q + s\delta| \geq 2\,\mathbf{c}\, c(N)$ for all $s \in [0, 1]$ in both cases.

Hence, we can estimate

$$\begin{aligned}\left|h^N(q)-h^N(q+\delta)\right| &= \left|(2\,\mathbf{c})^4 C(|q|^{-4}-|q+\delta|^{-4})\right| \\ &\le C\max_{0\le s\le 1}\left|\frac{1}{|q+s\delta|^5}\right|\cdot|\delta| \\ &\le C|\delta|\frac{1}{\left|q-\frac{q}{|q|}|\delta|\right|^5} \\ &\le C|\delta|\frac{1}{|q|^5}\cdot\frac{1}{\left(1-\frac{|\delta|}{|q|}\right)^5} \le C\frac{1}{|q|^5}\cdot 2^5|\delta| \le h'^N(q)|\delta|.\end{aligned}$$

In the last estimate we used that $|\delta|\le\frac{1}{2}|q|$. □

As $|{}^1\tilde{y}-\hat{q}_0|\le N^{-\frac{1}{9}}c(N)^4 \ll \mathbf{c}\,c(N)$, this result can be applied to obtain

$$\begin{aligned}&\left|\frac{1}{N}\sum_{j\neq 0}\int_{t_1}^{t_2} h^N({}^1\tilde{y}-\tilde{q}_j)-h^N(\hat{q}_0-\tilde{q}_j)\,\mathrm{d}s\right| \\ \le&\frac{1}{N}\sum_{j\neq 0}\int_{t_1}^{t_2} h'^N({}^1\tilde{y}-\tilde{q}_j)\cdot N^{-\frac{1}{9}}c(N)^4\,\mathrm{d}s \le CN^{-\frac{1}{9}}c(N)^{-1},\end{aligned}$$

where we estimated $h'$ by $Cc(N)^5$.

For the third summand, i.e.

$$\left|\int_{\mathbb{R}^6}\int_{t_1}^{t_2} h^N({}^1\tilde{y}-{}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z-\int_{\mathbb{R}^6}\int_{t_1}^{t_2} h^N(\hat{q}_0-{}^1\widetilde{z})\mathbf{1}_{\hat{G}^N(\hat{x}_0)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z\right|,$$

we have to take care of the indicator function. As mentioned before, we know that $G^N(y)\subseteq\hat{G}^N(\hat{x}_0)$, and so on $G^N(y)$ we can estimate by

$$\int_{\mathbb{R}^6}\int_{t_1}^{t_2}\left|h^N(\tilde{y}-{}^1\widetilde{z})-h^N(\hat{q}_0-{}^1\widetilde{z})\right|\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z,$$

where we can introduce $h'$ again. Hence, this term is bounded by

$$\int_{\mathbb{R}^6}\int_{t_1}^{t_2} h'^N(\tilde{y}-{}^1\widetilde{z})\cdot N^{-\frac{1}{9}}c(N)^4\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z\le CN^{-\frac{1}{9}}c(N)^{-1}.$$

However, we still have to estimate

$$\left|\int_{\mathbb{R}^6}\int_{t_1}^{t_2} h^N(\hat{q}_0-{}^1\widetilde{z})\mathbf{1}_{\hat{G}^N(\hat{x}_0)\setminus G^N(y)}(z)k(z)\,\mathrm{d}s\,\mathrm{d}z\right|.$$

The set we are integrating over is contained in $G^N(y)^C$, and we have already seen that $\mathbb{P}(z\in G^N(y)^C)\le N^{-\frac{4}{3}-2\sigma}$ by 2.11. Combining this with 2.7, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^6}\int_{t_1}^{t_2} h^N(\hat{q}_0-{}^1\widetilde{z})\mathbf{1}_{\hat{G}^N(\hat{x}_0)\setminus G^N(y)}(z)\,\mathrm{d}s k(z)\,\mathrm{d}z\right| \\ \le&C\left|\int_{\mathbb{R}^6} c(N)^{-3}N^{\frac{2}{9}+\sigma}\mathbf{1}_{\hat{G}^N(\hat{x}_0)\setminus G^N(y)}(z)k(z)\,\mathrm{d}z\right| \\ \le&Cc(N)^{-3}N^{\frac{2}{9}+\sigma}N^{-\frac{4}{3}-2\sigma}\ll N^{-\frac{1}{9}}c(N)^{-1}.\end{aligned}$$

Combining all these estimates, we have proven that for a typical configuration we have for all $y$ close enough to $x_0$:

$$\forall t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} h^N(\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j) \, \mathrm{d}s \right.$$
$$\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h^N(\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z) \, \mathrm{d}s k(z) \, \mathrm{d}z \right| \leq 4N^{-\frac{1}{9}+\sigma} c(N)^{-1}.$$

The other two estimates work analogously.

to $\mathcal{B}_{11}$ Regarding $\mathcal{B}_{11}$, we recall that

$$X^{\geq 1} \in \mathcal{B}_{11,i} \subseteq \mathbb{R}^{6(N-1)}$$
$$\iff \left( \exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j \neq 0} \int_{t_1}^{t_2} f((\hat{q}_0^\delta)_i - \tilde{q}_j) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(x_j) \, \mathrm{d}s \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f((\hat{q}_0^\delta)_i - {}^1\tilde{z}) \mathbf{1}_{\hat{G}^N(\hat{x}_0)}(z) \, \mathrm{d}s k(z) \, \mathrm{d}z \right| > 4N^{-\frac{4}{9}+\sigma} \right),$$

which works exactly the same as $\mathcal{B}_{10}$, i.e. we have to account for the slightly larger area covered with auxiliary particles.

## 5.2 Interpretation as a higher order approximation for the trajectory

Our result can also be interpreted as a higher order approximation. We have proven that the mean-field trajectories are not only a good approximation themselves, but the way they behave under small changes is a good estimate for the corresponding behavior of the real trajectories as well.

To make this more precise, we consider the following map:

**Definition 5.3.** Let $X^{\geq 1} \in \mathbb{R}^{6(N-1)}$ be any starting configuration. For any $z \in \mathbb{R}^6$ and $t \geq 0$ we define the map $\psi_t^{X^{\geq 1}} : \mathbb{R}^6 \to \mathbb{R}^6$ by setting $\psi_t^{X^{\geq 1}}(z)$ as the position in phase-space of $z$ after applying the Newtonian time evolution in $X = X^{\geq 1} \cup \{z\}$, i.e.

$$\psi_t^{X^{\geq 1}}(z) = \phi_t^X(z).$$

This is the Newtonian-flow of one added particle.

We introduce this new notation to emphasize that the function depends on $z$ in a sophisticated way. This becomes clear when comparing it to the corresponding mean-field version.

Adding the particle has almost no effect on the mean-field force, and hence $\varphi_t(z)$ behaves very predictable and has many nice properties, which can be summarized in $\varphi^N : \mathbb{R}^6 \to \mathbb{R}^6$ being a diffeomorphism. When considering $\psi(z)$, things are more complicated, and it is apriori not obvious whether $\psi(z)$ satisfies some of the same properties as $\varphi_t(z)$.

While the Newtonian flow of the whole system is reversible and hence bijective, this map with focus on only one particle does not need to be bijective apriori.

Consider the situation, where two different particles when added $X^{\geq 1}$ create distinct end states, but these two particles end up at the exactly same position in phase space.

This is where our main result proves useful. Restating it by using our new function, we have the following:

If $X^{\geq 1}$ is stable on $M \subseteq \mathbb{R}^6$, then for any $z \in M$ that becomes a good particle in $X^{\geq 1} \cup \{z\}$ we have:

$$|\psi(z) - \varphi(z)| \leq N^{-\frac{4}{9}+3\sigma},$$

and for $\delta \in \mathbb{R}^6$ with $|\delta| < c(N)$ we have

$$|\psi(z+\delta) - \psi(z) - (\varphi(z+\delta) - \varphi(z))| \leq N^{-\frac{1}{18}+4\sigma}\delta.$$

In the next few sections we will use these estimates to prove nice properties for $\psi$. Note that these arguments only work if $z$ is good in $X^{\geq 1}$ and all these results are probabilistic in nature, as we assume $X^{\geq 1}$ to be stable.

The results can be nicely summarized as

**Corollary 5.4.** *We work in our usual setup with $\sigma > 0$ and a reasonable force $f^N$ and density $k$. Let $T > 0$ and $x_0 \in \mathbb{R}^6$ such that $X^{\geq 1}$ is stably stable on the ball with radius $N^{-\frac{7}{18}-2\sigma}$ around $x_0$, i.e. on $B = B_{N^{-\frac{7}{18}-2\sigma}}(x_0)$. We assume, that all $z \in B$ would be good particles in $X^{\geq 1} \cup \{z\}$.*

*Then for large enough $N$ we have for any $t \in [0,T]$ we have*

*1.) $\psi_t$ is injective on $B$*

*2.) $\varphi_t(B_{N^{-\frac{7}{18}-3\sigma}}(x_0)) \subseteq \psi_t(B)$*

*3.) $\psi_t$ is Lipschitz-continuous on $B$*

*4.) $\psi_t$ is a diffeomorphism onto its image almost everywhere on $B$.*

This corollary will be proven throughout the next sections.

## $\psi$ is typically locally injective

Fix $x_0 \in \mathbb{R}^6$ such that $X^{\geq 1}$ is stable on the $N^{-\frac{7}{18}-2\sigma}$ ball around $x_0$ and such that all $z$ in this ball become good particles in $X^{\geq 1} \cup \{z\}$.

Then the map $z \mapsto \psi_t(z)$ from the ball around $x_0$ is injective. Let $z_1, z_2$ with $|z_i - x_0| < N^{-\frac{7}{18}-2\sigma}$ and assume that $\psi_t(z_1) = \psi_t(z_2)$. We know $z_2 = z_1 + \delta$ with $\delta < 2N^{-\frac{7}{18}-2\sigma}$. Hence, we have

$$\begin{aligned}
|\psi_t(z_2) - \psi_t(z_1)| &= |\psi_t(z_1+\delta) - \psi_t(z_1)| \\
&\geq |\varphi_t(z_1+\delta) - \varphi_t(z_1)| - N^{-\frac{1}{18}+4\sigma}|\delta| \\
&\geq \left(C(T) - N^{-\frac{1}{18}+4\sigma}\right)|\delta|.
\end{aligned}$$

As the constant does not depend on $N$, for large enough $N$ this implies $\delta = 0$. But that means $z_1 = z_2$ and $\psi_t$ is locally injective.

This idea can be extended. Even for an unbounded set $M$ we can apply similar arguments as long as $X^{\geq 1}$ is stable with respect to all $z \in M$ and the $z$ would be good particles in $X^{\geq 1} \cup \{z\}$.

Let $z_1, z_2 \in M$ and assume that $\psi_t(z_1) = \psi_t(z_2)$. We write $z_2 = z_1 + \delta$. If $\delta < c(N)$, we can simply apply the argument from above and again obtain

$$\begin{aligned}|\psi_t(z_2) - \psi_t(z_1)| &= |\psi_t(z_1+\delta) - \psi_t(z_1)| \\ &\geq |\varphi_t(z_1+\delta) - \varphi_t(z_1)| - N^{-\frac{1}{18}+4\sigma}|\delta| \\ &\geq \Big(C(T) - N^{-\frac{1}{18}+4\sigma}\Big)|\delta|.\end{aligned}$$

If $\delta > c(N)$, we can use the first order approximation instead, i.e.

$$|\psi(z_2) - \psi(z_1)| \geq |\varphi(z_2) - \varphi(z_1)| - 2N^{-\frac{4}{9}+3\sigma} \geq C|\delta| - 2N^{-\frac{4}{9}+3\sigma} \geq Cc(N) - 2N^{-\frac{4}{9}+3\sigma}.$$

For large enough $N$ we know that $Cc(N) \gg 2N^{-\frac{4}{9}+\sigma}$ and hence obtain injectivity.
This proves part a) of Corollary 5.4.

## $\psi$ is typically locally surjective

Let $M \in \mathbb{R}^6$ and $X^{\geq 1}$ be a configuration that is stably stable (in the sense of section 5.1.2) with respect to all $z \in M$ and such that the $z \in M$ would be good particles in $X^{\geq 1} \cup \{z\}$. Then $\psi_t^{X^{\geq 1}} : \mathbb{R}^6 \to \mathbb{R}^6$ is surjective onto $\varphi_t(M)$. We fix $y \in M$ and consider $\varphi_t(y)$. We know that $|\varphi_t(y) - \psi_t(y)| \leq N^{-\frac{4}{9}+3\sigma}$.

We define $y_0 = y$ and construct $y_1$ such that $\psi_t(y_1)$ comes closer to $\varphi_t(y)$ than $\psi_t(y_0)$.
For this we set

$$\delta_1 = \varphi^{-1}\Big(\varphi_t(y_0) + \varphi_t(y) - \psi_t(y_0)\Big) - y_0$$

Rewriting $y_0 = \varphi_t^{-1}(\varphi_t(y_0))$, we can use that $\varphi^{-1}$ is Lipschitz by Lemma 2.6 in order to obtain

$$|\delta_1| \leq C|\varphi_t(y) - \psi_t(y_0)| \leq CN^{-\frac{4}{9}+3\sigma}.$$

This choice might seem a bit random, but it is done such that $\varphi_t(y_0 + \delta_1) = \varphi_t(y_0) + \varphi_t(y) - \psi_t(y_0)$, which will be perfect for our next estimate.

We claim that $\psi_t(y_1) = \psi_t(y_0 + \delta_1)$ comes closer to $\varphi(y)$ than $\psi_t(y_0)$.

Using the choice of $\delta_1$ followed by our main result in combination with the fact that $\delta$ is reasonably small, we obtain

$$\begin{aligned}&\Big|\psi(y_0+\delta_1) - \varphi(y)\Big| \\ =&\Big|\psi(y_0+\delta_1) - \psi(y_0) - \Big(\varphi(y_0+\delta_1) - \varphi(y_0)\Big) + \psi(y_0) + \Big(\varphi(y_0+\delta_1) - \varphi(y_0)\Big) - \varphi(y)\Big| \\ =&\Big|\psi(y_0+\delta_1) - \psi(y_0) - \Big(\varphi(y_0+\delta_1) - \varphi(y_0)\Big)\Big| \leq N^{-\frac{1}{18}+4\sigma}|\delta_1| \leq CN^{-\frac{1}{18}+4\sigma}|\varphi(y_0) - \psi(y_0)|.\end{aligned}$$

As we started with $|\varphi_t(y) - \psi_t(y)| \leq N^{-\frac{4}{9}+3\sigma}$, this implies that

$$|\psi(y_0+\delta_1) - \varphi(y)| \leq N^{-\frac{1}{18}+4\sigma}N^{-\frac{4}{9}+3\sigma}.$$

Next, we can define $\delta_j$ inductively by

$$\delta_{j+1} = \varphi^{-1}\Big(\varphi(y_j) + \varphi(y) - \psi(y_j)\Big) - y_j$$

and then

$$y_{j+1} = y_j + \delta_{j+1}.$$

Following our arguments from above, we know that

$$\begin{aligned}\delta_{j+1} &= \varphi^{-1}\Big(\varphi(y_j) + \varphi(y) - \psi(y_j)\Big) - y_j \\ &= \varphi^{-1}\Big(\varphi(y_j) + \varphi(y) - \psi(y_j)\Big) - \varphi^{-1}(\varphi(y_j)) \leq C|\varphi(y) - \psi(y_j)|.\end{aligned}$$

Furthermore, we have $\varphi(y_j + \delta_{j+1}) = \varphi(y_j) + \varphi(y) - \psi(y_j)$ and

$$\begin{aligned}&\Big|\psi(y_j + \delta_{j+1}) - \varphi(y)\Big| \\ =&\Big|\psi(y_j + \delta_{j+1}) - \psi(y_j) - \Big(\varphi(y_j + \delta_{j+1}) - \varphi(y_j)\Big) + \psi(y_j) + \Big(\varphi(y_j + \delta_{j+1}) - \varphi(y_j)\Big) - \varphi(y)\Big| \\ =&\Big|\psi(y_j + \delta_{j+1}) - \psi(y_j) - \Big(\varphi(y_j + \delta_{j+1}) - \varphi(y_j)\Big)\Big| \\ \leq& N^{-\frac{1}{18}+4\sigma}|\delta_j| \leq CN^{-\frac{1}{18}+4\sigma}|\varphi(y_j) - \psi(y_j)|.\end{aligned}$$

With a small induction this implies that

$$|\psi(y_j) - \varphi(y)| \leq C(N^{-\frac{1}{18}+4\sigma})^j N^{-\frac{4}{9}+3\sigma},$$

and

$$|\delta_j| \leq C(N^{-\frac{1}{18}+4\sigma})^{j-1} N^{-\frac{4}{9}+3\sigma}.$$

For this to be meaningful, we need to check if the sequence $(y_j)_{j\in\mathbb{N}}$ converges.

For $j > l$ we have

$$|y_j - y_l| \leq \sum_{k=l+1}^{j} \delta_k \leq N^{-\frac{4}{9}+3\sigma} \sum_{k=l+1}^{j} (N^{-\frac{1}{18}+4\sigma})^{k-1},$$

which clearly makes $(y_j)_{j\in\mathbb{N}}$ Cauchy and hence convergent in $\mathbb{R}^6$. However, by our estimates we know that for the limit $y_\infty$ we have $\psi_t(y_\infty) = \varphi_t(y)$ making $\psi_t$ surjective onto $\varphi_t(M)$.

There is one thing we need to be careful about. We use our arguments not only for $y_0 \in M$ but also for the $y_j$ which do not need to be in $M$. However, we assumed $X^{\geq 1}$ to be stably stable with respect to $y$, and hence it suffices to check that our $y_j$ stay close enough to $y$. This is the case as we have:

$$|y_j - y| \leq \sum_{k=1}^{j} \delta_k \leq N^{-\frac{4}{9}+3\sigma} \sum_{k=1}^{j} (N^{-\frac{1}{18}+4\sigma})^{k-1} \leq CN^{-\frac{4}{9}+3\sigma} \ll N^{-\frac{7}{18}-2\sigma}.$$

In particular, if we define $\widetilde{M} = \Big\{x \in R^6 \mid \operatorname{dist}(x, M) \leq CN^{-\frac{4}{9}+3\sigma}\Big\}$, we have that $\varphi_t(M) \subseteq \psi_t(\widetilde{M})$. This proves part b) of Corollary 5.4.

## $\psi$ is typically Lipschitz

Let $M \in \mathbb{R}^6$ and $X^{\geq 1}$ be a configuration that is stable with respect to all $z \in M$ and such that the $z \in M$ would be good particles in $X^{\geq 1} \cup \{z\}$. Then $\psi_t^{X^{\geq 1}} : M \to \mathbb{R}^6$ is Lipschitz. Let $z_1, z_2 \in M$ and define $\delta = z_2 - z_1$. If $|\delta| < c(N)$, we have

$$|\psi_t(z_1 + \delta) - \psi_t(z_1)| = |\varphi_t(z_1 + \delta) - \varphi_t(z_1)| + N^{-\frac{1}{18}+4\sigma}|\delta| \leq (C(T) + N^{-\frac{1}{18}+4\sigma})|\delta|.$$

If $|\delta| \geq c(N)$, we have

$$\begin{aligned}|\psi_t(z_1+\delta)-\psi_t(z_1)| &\leq |\varphi_t(z_1+\delta)-\varphi_t(z_1)| + 2N^{-\frac{4}{9}+3\sigma} \\ &\leq C|\delta| + N^{-\frac{1}{9}+4\sigma}|\delta| \leq (C(T)+N^{-\frac{1}{9}+4\sigma})|\delta|.\end{aligned}$$

In both cases this can be estimated by $C(T)|\delta|$, which proves the Lipschitz continuity.

This proves part c) of Corollary 5.4.

**Remark 5.5.** Here and in our argument for injectivity one can clearly see the strengths and shortcomings of the result $|\varphi(z)-\psi(z)| < N^{-\frac{4}{9}+3\sigma}$. It is a very strong approximation as long as we are interested in distances and estimates that are of smaller order than $N^{-\frac{4}{9}+3\sigma}$. However, if particles come closer than that, this estimate is not very useful. This is exactly where our main result has the edge, as it can deal with arbitrarily small distances.

In fact, both results complement each other very well as our main result cannot deal with particles that are too far apart.

## $\psi$ is differentiable

Let $M \in \mathbb{R}^6$ and $X^{\geq 1}$ be a configuration that is stable with respect to all $z \in M$ and such that the $z \in M$ would be good particles in $X^{\geq 1} \cup \{z\}$.

As a direct consequence of the Lipschitz-continuity we obtain that $\phi$ is almost everywhere differentiable on $M$ by Rademacher's Theorem.

We already have an idea what the differential should look like, namely $D\varphi_t(x_0)$, which is given by

$$D\varphi_t(x_0) := e^{\int_0^t A(t)},$$

where

$$A(t) = \begin{pmatrix} 0 & E_3 \\ \widetilde{Df}_N({}^1\tilde{x}_0(t)) & 0 \end{pmatrix}.$$

To prove this, we want to show that

$$|\varphi_t(x_0+\delta)-\varphi_t(x_0)-D\varphi_t(x_0)\delta| = |\tilde{\delta}-D\varphi_t(x_0)\delta| \ll |\delta|.$$

Note that in all our arguments $\delta$ was in a way interlocked with $N$. In this case, we first want to prove that for any $N$ this result holds.

As usual, we use a Grönwall-Argument and hence consider the difference in force, i.e.

$$\left|\int_0^t \tilde{f}^N(\tilde{x}_0+\tilde{\delta}_t)-\tilde{f}^N(\tilde{x}_0)-\widetilde{Df}(\tilde{x}_0)\cdot {}^1D\varphi_t(x_0(t))\delta \,\mathrm{d}t\right|.$$

Recall that $\widetilde{f}(x_0) = \int_{\mathbb{R}^{3N}} f({}^1x_0-y)\tilde{k}_t(y)\,\mathrm{d}y$ is a convolution. As convolutions are compatible with the differential, i.e. $D\widetilde{f} = \widetilde{Df}$, we can use mean value theorem followed by the triangle inequality to obtain

$$\begin{aligned}&\left|\int_{t_0}^{t_1} \tilde{f}^N(\tilde{x}_0+\tilde{\delta}_t)-\tilde{f}^N(\tilde{x}_0)-\widetilde{Df}(\tilde{x}_0)\cdot {}^1D\varphi_t(x_0)\delta \,\mathrm{d}t\right| \\ =&\left|\int_{t_0}^{t_1}\int_0^1 \widetilde{Df}(\tilde{x}_0+s\tilde{\delta}_t){}^1\tilde{\delta}_t - \widetilde{Df}(\tilde{x}_0)\cdot {}^1D\varphi_t(x_0)\delta \,\mathrm{d}s\,\mathrm{d}t\right| \\ \leq&\left|\int_{t_0}^{t_1} \widetilde{Df}(\tilde{x}_0)({}^1\tilde{\delta}_t - {}^1D\varphi_t(x_0)\delta)\right| + \left|\int_0^t\int_0^1 \left(\widetilde{Df}(\tilde{x}_0+s^1\tilde{\delta}_t)-\widetilde{Df}(\tilde{x}_0)\right){}^1\delta_t\right|.\end{aligned}$$

The first term can be estimated using Lemma 6.24, which is applicable as we work on a configuration that is well behaved with respect to $x_0$, yielding

$$\left|\int_{t_0}^{t_1} \widetilde{Df}(\widetilde{x}_0)(\tilde{\delta}_t - D\varphi_t(x_0)\delta)\right| \leq \int_{t_0}^{t_1} \ln(N)|\tilde{\delta}_t - D\varphi_t(x_0)\delta|.$$

The second term can be estimated by introducing $h$ by Lemma 2.4

$$\begin{aligned}
&\left|\int_{t_0}^{t_1}\int_0^1 \left(\widetilde{Df}(\widetilde{x}_0 + s\tilde{\delta}_t) - \widetilde{Df}(\widetilde{x}_0)\right)\tilde{\delta}_t\right| \mathrm{d}s\,\mathrm{d}t \\
\leq &\int_{t_0}^{t_1}\int_0^1\int_{\mathbb{R}^6} \left|Df^N(\widetilde{x}_0 + s\tilde{\delta}_t - y) - Df^N(\tilde{x}_0 - y)\right| k_t(y)\,\mathrm{d}y \cdot |\tilde{\delta}_t|\,\mathrm{d}s\,\mathrm{d}t \\
\leq &\int_{t_0}^{t_1}\int_0^1\int_{\mathbb{R}^6} \left|h^N(\widetilde{x}_0 - y)\right| \cdot |\tilde{\delta}_t| \cdot k_t(y)\,\mathrm{d}y \cdot |\tilde{\delta}_t|\,\mathrm{d}s\,\mathrm{d}t \\
= &\int_{t_0}^{t_1} \tilde{h}(\tilde{x}_0)|\tilde{\delta}_t|^2\,\mathrm{d}t \\
\leq &\int_{t_0}^{t_1} \ln(N)c(N)^{-1}|\delta|^2.
\end{aligned}$$

This means that we can apply our Grönwall Lemma 3.2 one last time to obtain that

$$|\varphi_t(x_0+\delta) - \varphi_t(x_0) - D\varphi_t(x_0)\delta| \leq C(T)N^{\epsilon}c(N)^{-1}|\delta|^2,$$

where $\varepsilon > 0$ can be arbitrarily small. We conclude that $\varphi_t(x)$ for any fixed $N$ is differentiable in its position in phase-space.

This is helpful as for the derivative $D\psi$ we know that it is close to $D\varphi$ (at least where it is defined). For any $\delta$ with $|\delta| < c(N)$, we have

$$\begin{aligned}
&|D\psi_t(z)\delta - D\varphi_t(z)\delta| \\
\leq &|(\psi_t(z+\delta) - \psi_t(z)) - D\psi(z)\cdot\delta| + |(\psi_t(z+\delta) - \psi_t(z)) - D\varphi(z)\cdot\delta| \\
\leq &|(\psi_t(z+\delta) - \psi_t(z)) - D\psi_t(z)\cdot\delta| + N^{-\frac{1}{18}+4\sigma}|\delta| + |(\varphi_t(z+\delta) - \varphi_t(z)) - D\varphi_t(z)\cdot\delta| \\
\leq &R_\delta + N^{-\frac{1}{18}+4\sigma}|\delta| + R_\delta,
\end{aligned}$$

where $\frac{R_\delta}{\delta} \to 0$ for $\delta \to 0$. As this holds true for all $\delta$, after dividing by $|\delta|$, we obtain that the difference between $D\psi$ and $D\varphi$ is bounded by $N^{-\frac{1}{18}+4\sigma}$.

Our next goal is to show that $\psi$ locally is a diffeomorphism (almost everywhere) onto its image. For this, it is sufficient to show that $D\psi \neq 0$.

Note that $D\varphi$ is defined by the Matrix $e^{\int_0^t A(t)}$. Its determinant is given by $e$ to the power of the trace of $\int_0^t A(s)$ which – considering the form of $A$ – is simply $e^0 = 1$. Hence, our approximation for the derivative vanishes nowhere.

We can say even more. We know that for any $\delta$ with $|\delta| = 1$ we have

$$|D\varphi_t(x_0)\delta| = \frac{|D\varphi_t(x_0)\lambda\delta|}{|\lambda\delta|} \geq \frac{|\varphi_t(x_0) - \varphi_t(x_0+\lambda\delta)|}{|\lambda\delta|} - R_\lambda \geq C(T) - R_\lambda,$$

where $R_\lambda \to 0$ for $\lambda \to 0$. Hence, we obtain

$$|D\varphi(x_0)\delta| \geq C(T)$$

for all $\delta$ with $|\delta| = 1$. This means that for these $\delta$ we also have

$$|D\psi\delta| \geq |D\varphi\delta| - N^{-\frac{1}{18}+4\sigma}|\delta| \geq (C(T) - N^{-\frac{1}{18}+4\sigma}) > 0.$$

So, for large enough $N$ the differential does not vanish, and hence for suitable configurations $\psi : \mathbb{R}^6 \to \mathbb{R}^6$ is locally a diffeomorphism onto its image.

This proves part d) of Corollary 5.4.

## Reverse direction

As we have seen many times in this paper, it is very helpful that $\varphi$ is reversible and that the reverse direction still satisfies strong estimates.

We have now seen that with the right configuration $\psi$ is also – at least locally – bijective. Hence, we want to prove a corresponding estimate for $\psi^{-1}$.

Let $M \in \mathbb{R}^6$ and $X^{\geq 1}$ be a configuration that is stably stable with respect to all $z \in M$ and such that the $z \in M$ would be good particles in $X^{\geq 1} \cup \{z\}$.

This implies that $\psi$ is injective on $M$ and surjective onto $\varphi(M)$. Let $y \in \mathbb{R}^6$ and $|\varepsilon| \leq N^{-\frac{1}{18}}c(N) \leq N^{-\frac{7}{18}}$ such that $y$ and $y+\varepsilon$ both lie in $\varphi(M)$ as well as $\psi^{-1}(y)$ and $\psi^{-1}(y+\varepsilon)$ both lie in $M$. Then, using that $|\varphi_t(x) - \psi_t(x)| < CN^{-\frac{4}{9}+3\sigma}$ for all nice enough $x$ and large enough $N$, we have

$$\begin{aligned}
|\psi_t^{-1}(y+\varepsilon) - \psi_t^{-1}(y)| &\leq C(T)|\varphi_t(\psi_t^{-1}(y+\varepsilon)) - \varphi_t(\psi_t^{-1}(y))| \\
&\leq C(T)|\psi_t(\psi_t^{-1}(y+\varepsilon)) - \psi_t(\psi_t^{-1}(y))| + C(T)N^{-\frac{4}{9}+3\sigma} \\
&\leq c(N).
\end{aligned}$$

This means our main result is applicable to $\delta = \psi_t^{-1}(y+\varepsilon) - \psi_t^{-1}(y)$ yielding

$$\begin{aligned}
|\psi_t^{-1}(y+\varepsilon) - \psi_t^{-1}(y)| &= |\varphi_t^{-1}(\varphi_t(\psi_t^{-1}(y+\varepsilon))) - \varphi_t^{-1}(\varphi_t(\psi_t^{-1}(y)))| \\
&\leq C(T)|\varphi_t(\psi_t^{-1}(y+\varepsilon)) - \varphi_t(\psi_t^{-1}(y))| \\
&\leq C(T)|\psi_t(\psi_t^{-1}(y+\varepsilon)) - \psi_t(\psi_t^{-1}(y))| + C(T)N^{-\frac{1}{18}+4\sigma}|\delta| \\
&\leq C|\varepsilon| + CN^{-\frac{1}{18}+4\sigma}|\varepsilon| \leq C(T)|\varepsilon|,
\end{aligned}$$

which implies that $\psi_t^{-1}$ is also Lipschitz continuous (for points that are close).

If the points are further away we can use that $\psi^{-1}$ can be estimated by $\varphi^{-1}$, i.e.

$$\begin{aligned}
|\psi_t^{-1}(y) - \varphi_t^{-1}(y)| &\leq C(T)|\varphi_t(\psi_t^{-1}(y)) - \varphi_t(\varphi_t^{-1}(y))| \\
&= C(T)|\varphi_t(\psi_t^{-1}(y)) - \psi_t(\psi_t^{-1}(y))| \\
&\leq C(T)N^{-\frac{4}{9}+3\sigma}.
\end{aligned}$$

With this we obtain

$$\begin{aligned}
|\psi_t^{-1}(y+\varepsilon) - \psi_t^{-1}(y)| &\leq |\phi_t^{-1}(y+\varepsilon) - \phi_t^{-1}(y)| + C(T)N^{-\frac{4}{9}+3\sigma} \\
&\leq C(T)|\varepsilon| + C(T)N^{-\frac{4}{9}+3\sigma},
\end{aligned}$$

which becomes a good enough estimate for $|\varepsilon| > N^{-\frac{1}{18}}c(N)$.

Finally, we also want to prove a corresponding reverse direction for our main result 4.2.

For this we consider $|\varepsilon| \leq N^{-\frac{1}{18}}c(N)$ again and define $\delta = \psi_t^{-1}(y+\varepsilon) - \psi_t^{-1}(y)$ and $\tilde{\delta} = \varphi_t^{-1}(y+\varepsilon) - \varphi_t^{-1}(y)$. As $\psi^{-1}$ is Lipschitz continuous we know $|\delta| \leq C_0(T)|\varepsilon|$. For this constant $C_0(T)$ we have

$$\begin{aligned}
&N^{-\frac{1}{18}+4\sigma}C_0(T)|\varepsilon| \\
=&\left|\psi_t(\psi_t^{-1}(y+\varepsilon)) - \psi_t(\psi_t^{-1}(y)) - \varepsilon\right| + N^{-\frac{1}{18}+4\sigma}C_0(T)|\varepsilon| \\
=&\left|\psi_t(\psi_t^{-1}(y)+\delta) - \psi_t(\psi_t^{-1}(y)) - \Big(\varphi_t(\varphi_t^{-1}(y+\varepsilon)) - \varphi_t(\varphi_t^{-1}(y))\Big)\right| + N^{-\frac{1}{18}+4\sigma}C_0(T)|\varepsilon| \\
\geq&\left|\varphi_t(\psi_t^{-1}(y)+\delta) - \varphi_t(\psi_t^{-1}(y)) - \Big(\varphi_t(\varphi_t^{-1}(y+\varepsilon)) - \varphi_t(\varphi_t^{-1}(y))\Big)\right|.
\end{aligned}$$

In the estimate at the end we used $|\delta| \leq C_0(T)|\varepsilon|$ and hence can apply our main result to estimate:

$$\begin{aligned}
&\left|\psi_t(\psi_t^{-1}(y)+\delta) - \psi_t(\psi_t^{-1}(y)) - \Big(|\varphi_t(\psi_t^{-1}(y)+\delta) - \varphi_t(\psi_t^{-1}(y))\Big)\right| \\
\leq&N^{-\frac{1}{18}+4\sigma}|\delta| \leq N^{-\frac{1}{18}+4\sigma}C_0(T)|\varepsilon|.
\end{aligned}$$

Moving on with the estimate, we can now introduce $D\varphi$ while using our estimate from the section above.

$$\begin{aligned}
&\left|\varphi_t(\psi_t^{-1}(y)+\delta) - \varphi_t(\psi_t^{-1}(y)) - \Big(\varphi_t(\varphi_t^{-1}(y)+\tilde{\delta}) - \varphi_t(\varphi_t^{-1}(y))\Big)\right| \\
\geq&\left|D\varphi(\psi_t^{-1}(y))\delta - D\varphi(\varphi^{-1}(y))\tilde{\delta}\right| - C(T)c(N)^{-1}N^{\sigma}(\delta^2+\tilde{\delta}^2)
\end{aligned}$$

Using the triangle inequality for the first summand, and noting that $\delta \geq C(T)\varepsilon$ and $\tilde{\delta} \geq C(T)\varepsilon$ because $\psi$ and $\varphi$ are both Lipschitz, we obtain

$$\begin{aligned}
&\left|D\varphi(\psi_t^{-1}(y))\delta - D\varphi(\varphi^{-1}(y))\tilde{\delta}\right| - C(T)c(N)^{-1}N^{\sigma}(\delta^2+\tilde{\delta}^2) \\
\geq&\left|D\varphi(\varphi_t^{-1}(y))\delta - D\varphi(\varphi_t^{-1}(y))\tilde{\delta}\right| - \left|D\varphi(\psi_t^{-1}(y)) - D\varphi(\varphi_t^{-1}(y))\right||\delta| - C(T)c(N)^{-1}N^{\sigma}|\varepsilon|^2 \\
\geq&\left|D\varphi(\varphi_t^{-1}(y))(\delta-\tilde{\delta})\right| - \left|e^{\int_0^t A_s(\varphi_t^{-1}(y))} - e^{\int_0^t A_s(\psi_t^{-1}(y))}\right| \cdot |\delta| - C(T)N^{-\frac{1}{18}+\sigma}|\varepsilon| \\
\overset{(\star)}{\geq}&C(T)|\delta-\tilde{\delta}| - C(T)N^{-\frac{1}{9}+4\sigma}|\varepsilon| - C(T)N^{-\frac{1}{18}+\sigma}|\varepsilon|.
\end{aligned}$$

Before we go into details for our estimates in $(\star)$, we first finish our argument. We have just proven, that

$$N^{-\frac{1}{18}+4\sigma}C_0(T)|\varepsilon| \geq C(T)|\delta-\tilde{\delta}| - C(T)N^{-\frac{1}{9}+4\sigma}|\varepsilon| - C(T)N^{-\frac{1}{18}+\sigma}|\varepsilon|.$$

After rearranging and simplifying, we obtain

$$|\delta-\tilde{\delta}| \leq C(T)N^{-\frac{1}{18}+4\sigma}|\varepsilon|.$$

Rewriting this without $\delta$ and $\tilde{\delta}$, one clearly sees that this is the desired reverse direction of our main result:

$$\left|\psi_t^{-1}(y+\varepsilon) - \psi_t^{-1}(y) - \Big(\varphi_t^{-1}(y+\varepsilon) - \varphi_t^{-1}(y)\Big)\right| < C(T)N^{-\frac{1}{18}+4\sigma}|\varepsilon|.$$

We still need to prove our estimates in $(\star)$. There we estimated $|D\varphi|$ and $|e^{\int_0^t A(\varphi_t^{-1}(y))} - e^{\int_0^t A(\psi_t^{-1}(y))}|$.

For the first estimate we want to show that

$$|D\varphi(\varphi_t^{-1}(y))(\delta - \tilde{\delta})| \geq C(T)|\delta - \tilde{\delta}|.$$

If $(\delta - \tilde{\delta}) = 0$, the estimate is trivial; otherwise recall that we have just seen that for any normalized vector $v$ we have

$$|D\varphi_t(\varphi_t^{-1}(y)) \cdot v| \geq C(T).$$

Normalizing $(\delta - \tilde{\delta})$ and applying this estimate gives us exactly what we want.

For the second estimate we note that as the matrix exponential is Lipschitz, we have

$$\left|e^{\int_0^t A(\varphi_t^{-1}(y))} - e^{\int_0^t A(\psi_t^{-1}(y))}\right| \leq \left|\int_0^t A(\varphi_t^{-1}(y)) - A(\psi_t^{-1}(y))\,\mathrm{d}t\right| \cdot e^{\left|\int_0^t A(\varphi_t^{-1}(y))\right|} \cdot e^{\left|\int_0^t A(\psi_t^{-1}(y))\right|}.$$

The second and third factor are each bounded by $e^{C(T)}$ as

$$A_s(x) = \begin{pmatrix} 0 & E_3 \\ \widetilde{Df}_N({}^1\widetilde{x}(s)) & 0 \end{pmatrix}.$$

This means it suffices to note that

$$\left\|\int_0^t \widetilde{Df}_N({}^1\widetilde{x}(s))\right\| \leq C(T),$$

where we simply applied 2.5.

The first factor can be estimated by

$$\begin{aligned}
&\left|\int_0^t A_s(\varphi_t^{-1}(y)) - A_s(\psi_t^{-1}(y))\,\mathrm{d}s\right| \\
=&\left|\int_0^t \widetilde{Df}_N\left({}^1\varphi_s(\varphi_t^{-1}(y))\right) - \widetilde{Df}_N\left({}^1\varphi_s(\psi_t^{-1}(y))\right)\mathrm{d}s\right| \\
\leq&\int_0^t \tilde{h}\left({}^1\varphi_t(\varphi_s^{-1}(y))\right) \cdot C(T)N^{-\frac{4}{9}+3\sigma}\,\mathrm{d}s \\
\leq&C(T)c(N)^{-1}N^{-\frac{4}{9}+3\sigma} \leq C(T)N^{-\frac{1}{9}+4\sigma},
\end{aligned}$$

where we applied 2.5 again.

## 5.3 Estimating probabilities

We consider a sequence of sets $M_N \subseteq \mathbb{R}^6$ where $M_N = B_{N^\alpha}(0)$ with $\alpha > 0$.

We then want to estimate

$$\mathbb{P}\left(X^{\geq 1} \text{ is stable with respect to every } z \in M_N\right).$$

The idea is that for any $y \in M_N$ the probability that $X^{\geq 1}$ is stable with respect to all $z \in B_{N^{-\frac{7}{18}-2\sigma}}(y)$ can be estimated by $C_\gamma N^{-\gamma}$ for every $\gamma > 0$. Importantly, $C_\gamma$ does not depend on $y$.

This means that we can cover $M_N$ by approximately $CN^{\alpha}N^{\frac{7}{18}+2\sigma}$ such spheres and hence can find a set of $CN^{\alpha}N^{\frac{7}{18}+2\sigma}$ points $y_n$ such that it is sufficient to be stably stable around all the $y_n$. Hence, for any $\gamma > 0$ we find a $C_\gamma$ such that

$$\begin{aligned}&\mathbb{P}(\exists z \in M_N \text{ such that } X^{\geq 1} \text{ is not stable with respect to } z)\\ \leq &\mathbb{P}(\exists y_n \in M_N \text{ such that } X^{\geq 1} \text{ is not stably stable with respect to } y_n)\\ \leq &CN^{\alpha}N^{\frac{7}{18}+2\sigma} \cdot C_\gamma N^{-\gamma}.\end{aligned}$$

If we apply this with $\hat{\gamma} = \gamma + \alpha + \frac{7}{18} + 2\sigma$, we obtain that

$$\mathbb{P}(\exists z \in M_N \text{ such that } X^{\geq 1} \text{ is not stable with respect to } z) \leq C_\gamma N^{-\gamma}$$

for every $\gamma > 0$.

As we have seen, this means that we have many strong estimates on $M_N$. Furthermore, note that the volume of $M_N$ goes to infinity for $N \to \infty$. Hence, for suitable densities $k$ we can choose $M_N$ in such a way, that typically almost all the particles will lie in $M_N$. As we are mostly working with probabilistic results, we can typically work on a set where all our new results hold.

As a final outlook we want to give one example on how our main result can be used to obtain strong estimates for probabilities. We are going to give a proof sketch – with one additional assumption, which we will discuss in a second – that for a typical configuration $X^{\geq 1}$ and under certain conditions we have

$$\mathbb{P}\left(\psi_t(x) \in B_{N^{-\alpha}}(z)\right) \approx \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(z)\right).$$

In words, this means that under the right conditions the probability that the real trajectory of a newly added particle passes close to the point $z$ at the time $t$ can be estimated using the mean-field trajectories.

To make this more precise, let $\alpha > 0$. We then consider sets $M_N \subseteq \mathbb{R}^6$ such that $\mathbb{P}(x \notin M_N) \ll N^{-6\alpha}$ and only consider configurations $X^{\geq 1} = (x_1, \dots, x_{N-1})$ that are stable on all of $M_N$. As we have just seen, this is a reasonable assumption, and in fact a typical configuration will satisfy this (for very large $N$).

However, we will need one additional assumption as we would run into a problem otherwise. We want to add a new randomly placed particle $x_0$ and estimate the probability on where it ends up. While we have many tools for good particles, we have no control for bad particles as all our results assumed that the added particle was good. For this proof sketch we will only consider configurations $X^{\geq 1}$, where, if we add a bad particle $x_0 \in M_N$ to $X^{\geq 1}$, we still know that $\sup_{0\leq s\leq t} |\varphi_s(x_0) - \psi_s(x_0)| < N^{-\frac{1}{3}+\sigma}$.[1]

Let $z = \varphi_t(y_0)$ and let $X^{\geq 1}$ be a configuration that satisfies the condition from above. Furthermore, we assume that

$$B_{C(T)(N^{-\alpha}+N^{-\frac{1}{3}+\sigma})}(y_0) \subseteq M_N$$

[1]Note that this assumption is at least somewhat reasonable. $N^{-\frac{1}{3}+\sigma}$ was exactly the margin we have proven for bad particles in 3.1. Of course, in that theorem we always assumed that the newly added particle becomes a good particle in the system. There are some new problems one has to account for if one drops this assumption. For example, one has to be careful, because adding a particle that becomes bad might flip an existing good particle to become bad as well. Still we are hopeful that this assumption – or at least some variation of it – should be provable.

and that any $z$ from that set becomes a good particle in $X^{\geq 1} \cup \{z\}$.

The constant $C(T)$ is chosen such that $\varphi$ is Lipschitz on $[0,T]$ with Lipschitz-constant $C(T)$, which is possible by 2.6.

Under these assumptions we will prove that

$$\left|\mathbb{P}\left(\psi_t(x) \in B_{N^{-\alpha}}(z)\right) - \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(z)\right)\right| \ll N^{-6\alpha}.$$

Here $N^{-6\alpha}$ is the expected order of the probability $\mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(z)\right)$ because $\varphi$ is a probability preserving diffeomorphism.

*Proof sketch.* The first step is to show that for reasonable $X^{\geq 1}$ both $\varphi^{-1}(B_{N^{-\alpha}}(\varphi_t(y)))$ and $\psi^{-1}(B_{N^{-\alpha}}(\varphi_t(y)))$ are contained in

$$B_{C(T)(N^{-\alpha}+N^{-\frac{1}{3}+\sigma})}(y).$$

This is very helpful as it gives us a playing field we can focus on. However, this claim is not necessarily true. The problem comes from particles outside from $M_N$, as for such particles $y \notin M_N$ we have no control on how $\psi_t(y)$ behaves. So we will only consider particles inside $M_N$. Note that from a probabilistic point of view this is not really a problem, as $\mathbb{P}(x \notin M_N) \ll N^{-6\alpha}$ and hence the error coming from particles outside of $M_N$ is of small enough order. From now on, whenever we consider sets and their probabilities, we always consider the intersection with $M_N$ unless otherwise specified.

Even without this assumption the argument for $\varphi$ is quite easy, as it is Lipschitz continuous by 2.6, i.e.

$$\varphi^{-1}(B_{N^{-\alpha}}(\varphi_t(y))) \subseteq B_{C(T)N^{-\alpha}}(y) \subseteq B_{C(T)(N^{-\alpha}+N^{-\frac{1}{3}+\sigma})}(y).$$

Things become a bit more involved for the pullback under $\psi$.

We do this by contraposition. For this $x \notin B_{C(T)(N^{-\alpha}+N^{-\frac{1}{3}+\sigma})}(y)$, i.e. $|x-y| \geq C(T)(N^{-\alpha}+N^{-\frac{1}{3}+\sigma})$. As we mentioned, we assume $x \in M_N$, and hence we know that $|\psi(x)-\varphi(x)| < N^{-\frac{1}{3}+\sigma}$. However, at the same time we have

$$C(T)|\varphi_t(x)-\varphi_t(y)| \geq |x-y| \geq C(T)(N^{-\alpha}+N^{-\frac{1}{3}+\sigma})$$

because $\varphi^{-1}$ is Lipschitz.

This is exactly the constant we have chosen, and hence we obtain

$$|\varphi(x)-\varphi(y)| \geq (N^{-\alpha}+N^{-\frac{1}{3}+\sigma}).$$

Combining this with $|\psi(x)-\varphi(x)| \leq N^{-\frac{1}{3}+\sigma}$, we obtain

$$|\psi(x)-\varphi(y)| \geq |\varphi(x)-\varphi(y)| - N^{-\frac{1}{3}+\sigma} \geq (N^{-\alpha}+N^{-\frac{1}{3}+\sigma}) - N^{-\frac{1}{3}+\sigma} = N^{-\alpha}.$$

In words: $\psi(x)$ does not lie in the $N^{-\alpha}$ ball around $\varphi_t(y)$, and hence $x$ does not lie in the corresponding pullback under $\psi$.

In conclusion, we have proven that both $\varphi^{-1}\Big(B_{N^{-\alpha}}(\varphi_t(y))\Big)$ and $\psi^{-1}\Big(B_{N^{-\alpha}}(\varphi_t(y))\Big)$ are contained in

$$B_{C(T)(N^{-\alpha}+N^{-\frac{1}{3}+\sigma})}(y).$$

Moving on, we consider two cases regarding the order of $N^{-\alpha}$: First, we assume that $N^{-\alpha} \geq N^{-\frac{1}{18}}c(N) > N^{-\frac{7}{18}-\sigma}$, and later we will consider the case $N^{-\alpha} < N^{-\frac{1}{18}}c(N)$.

If $N^{-\alpha} \geq N^{-\frac{1}{18}} c(N)$, we can use that particles stay close to their mean-field trajectories (on $M_N$), namely $|\psi_t(x) - \varphi_t(x)| \leq N^{-\frac{4}{9}+3\sigma} = \epsilon$. This means that

$$\varphi_t(x) \in B_{N^{-\alpha}-\epsilon}(\varphi_t(y)) \implies \psi_t(x) \in B_{N^{-\alpha}}(\varphi_t(y)) \implies \varphi_t(x) \in B_{N^{-\alpha}+\epsilon}(\varphi_t(y)).$$

These implications follow directly from $|\psi_t(x) - \varphi_t(x)| \leq N^{-\frac{4}{9}+3\sigma} = \epsilon$ (on $M_N$). Let $x$ be a particle such that $\varphi_t(x) \in B_{N^{-\alpha}-\epsilon}(\varphi_t(y))$. As we have seen, we can conclude $x \in M_N$, and hence we obtain

$$|\psi_t(x) - \varphi_t(y)| = |\psi_t(x) - \varphi_t(x) + \varphi_t(x) - \varphi_t(y)| \leq N^{-\frac{4}{9}+3\sigma} + N^{-\alpha} - N^{-\frac{4}{9}+3\sigma} = N^{-\alpha}.$$

The argument for the second implication works analogously.

Considering the possibilities, we note that

$$\begin{aligned}
\mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}+\epsilon}(\varphi_t(y))\right) &= \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(\varphi_t(y))\right) + \int_{B_{N^{-\alpha}+\epsilon}(\varphi_t(y)) \setminus B_{N^{-\alpha}}(\varphi_t(y))} k_t(y) \\
&\leq \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(\varphi_t(y))\right) + C\Big((N^{-\alpha}+\epsilon)^6 - N^{-6\alpha}\Big)|k_t|_\infty \\
&\leq \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(\varphi_t(y))\right) + C|k_t|_\infty \epsilon N^{-5\alpha} \\
&\leq \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(\varphi_t(y))\right) + C(T) N^{-\frac{1}{18}+4\sigma} N^{-6\alpha}.
\end{aligned}$$

Here we used $N^{-\alpha} \geq N^{-\frac{1}{18}} c(N) > N^{-\frac{7}{18}-\sigma} = N^{\frac{1}{18}-4\sigma}\epsilon$. Analogously, we get

$$\mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}-\epsilon}(\varphi_t(y))\right) \geq \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(\varphi_t(y))\right) - C(T) N^{-\frac{1}{18}+4\sigma} N^{-6\alpha}.$$

Combining these estimates, we have proven:

$$\left|\mathbb{P}\left(\psi_t(x_0) \in B_{N^{-\alpha}}(\varphi_t(y))\right) - \mathbb{P}\left(\varphi_t(x) \in B_{N^{-\alpha}}(\varphi_t(y))\right)\right| \leq C(T) N^{-\frac{1}{18}+4\sigma} N^{-6\alpha},$$

which concludes our estimates if $N^{-\alpha} \geq N^{-\frac{1}{18}} c(N)$.

Next, we consider the remaining case $r := N^{-\alpha} \leq N^{-\frac{1}{18}} c(N)$.

The idea is to consider the pullback of $B_r(\varphi_t(y))$ under both $\varphi$ and $\psi$ and show that both pullbacks are close to each other.

As the mean-field trajectories are a good estimation for the real trajectory, we can conclude that the center-points of these pullbacks (i.e. the pullbacks of $\varphi(t)$) stay reasonably close to each other.

Our new result adds that both pullbacks have a similar shape. Combining these two results, we get the following picture 5.1:

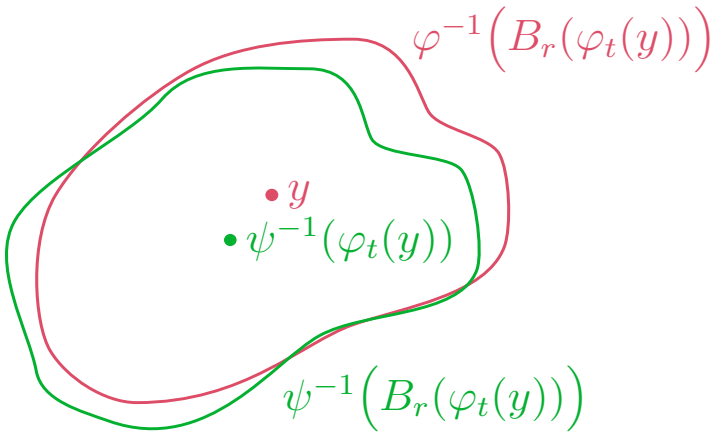


Figure 5.1: Pullback of $B_r(\varphi_t(y))$ under $\varphi$ and $\psi$

Let us now formalize this idea. We first note that by choice of $X^{\geq 1}$ and $M_N$ we find exactly one point (in $M_N$) $\psi_t^{-1}(\varphi_t(y))$. This allows us to rewrite the pullback under $\psi$.

$$\begin{aligned}\mathbb{P}\left(\psi_t(x) \in B_r(\varphi_t(y))\right) &= \int_{\psi_t^{-1}(B_r(\varphi_t(y)))} k_0(x)\,\mathrm{d}x \\ &= \int_{\psi_t^{-1}(B_r(\varphi_t(y)))-\psi^{-1}(\varphi_t(y))} k_0\left(\psi^{-1}(\varphi_t(y))+\delta\right)\mathrm{d}\delta.\end{aligned}$$

Here $\psi^{-1}\Big(B_r(\varphi_t(y))\Big)$ consists exactly of the $x \in \mathbb{R}^6$ that, if added to $X^{\geq 1}$, pass through the $\delta$-ball around $\varphi_t(y)$ at the time $t$.

The transformation of variables in the last step shifts our focus to the relative position around the center of the ball. Our goal is to show that this relative position is similar to the mean-field version of this integral, which we will consider next:

$$\mathbb{P}\left(\varphi_t(x) \in B_r(\varphi_t(y))\right) = \int_{\varphi^{-1}(B_r(\varphi_t(y)))} k(x)\,\mathrm{d}x = \int_{\varphi_t^{-1}(B_r(\varphi_t(y)))-y} k(y+\delta)\,\mathrm{d}\delta.$$

We want to estimate the difference between both integrals.

First, we consider the intersection

$$U = \left(\varphi_t^{-1}(B_r(\varphi_t(y))) - y\right) \cap \left(\psi^{-1}(B_r(\varphi_t(y))) - \psi^{-1}(\varphi_t(y))\right).$$

The volume of $U$ is bounded by $\varphi^{-1}\Big(B_r(\varphi_t(y))\Big)$ which in turn is bounded by $C(T)r^6$ by 2.6 as the original ball had radius $\delta$.

Inside the integral we have to estimate:

$$\left|k(y+\delta) - k(\psi^{-1}(\varphi_t(y))+\delta)\right|.$$

This difference in density can be estimated as the density $k$ is Lipschitz by our conditions in the very beginning 1.2. This means we only need to estimate

$$|y+\delta-(\psi_t^{-1}(\varphi_t(y))+\delta)| = |\psi_t^{-1}(\varphi_t(y)) - \varphi_t^{-1}(\varphi_t(y))| \leq C(T)N^{-\frac{4}{9}+3\sigma}.$$

The last estimate uses the fact that mean-field trajectories and real trajectories stay close. However, here we used the reverse direction we have proven at the end of the last section.

Combining these estimates, we obtain

$$\int_U \left| k(\psi_t^{-1}(\varphi_t(y)) + \delta) - k(y + \delta) \right| \leq C(T) r^6 N^{-\frac{4}{9}+3\sigma}.$$

On the intersection we leveraged that the difference in density is small. Away from the intersection we have no similar option. Instead we will prove that the intersection makes up most of both sets. For this we will need our main result.

To this end, we write $B_\psi = \psi^{-1}\Big(B_r(\varphi_t(y))\Big) - \psi^{-1}\Big(\varphi_t(y)\Big)$ and $B_\varphi = \varphi^{-1}\Big(B_r(\varphi_t(y))\Big) - y$. First, we estimate the volume of $B_\psi \setminus B_\varphi$.

For any point $\delta \in B_\psi$ we have $\delta = \psi_t^{-1}\Big(\varphi_t(y) + \epsilon\Big) - \psi_t^{-1}\Big(\varphi_t(y)\Big)$ for suitable $\varepsilon$ with $|\varepsilon| \leq r$. We know that $\tilde{\delta} = \varphi^{-1}\Big(\varphi_t(y) + \varepsilon\Big) - \varphi_t^{-1}\Big(\varphi_t(y)\Big)$ is a very good estimate in the sense that

$$|\delta - \tilde{\delta}| < C(T) N^{-\frac{1}{18}+4\sigma} r.$$

Here we simply applied the reversed version of our main result, which is applicable as we assumed $|r| \leq N^{-\frac{1}{18}} c(N)$.

This already seems very promising as for any $\delta \in B_\psi$ we find a very close $\tilde{\delta} \in B_\varphi$. But we need some regularity on $B_\varphi$. Otherwise, this closeness in distance does not imply a closeness in volume, as can be seen in 5.2.

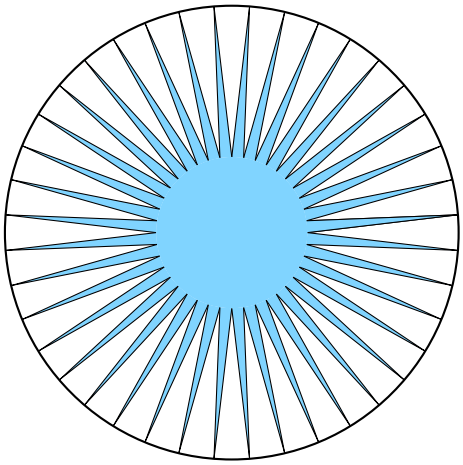

Figure 5.2: A sea urchin like set inside a ball

The sea urchin like set satisfies the closeness in distance to the points inside the ball but can have arbitrarily small volume.

As $B_\varphi$ is simply the pullback of a sphere via the diffeomorphism $\varphi_t$, it has enough regularity (and cannot be a sea urchin).

One possibility to make this this rigorous is the following: We consider $\varphi(B_\psi \setminus B_\varphi)$. Because $\varphi$ is bijective, this is simply $\varphi(B_\psi) \setminus B_r(0)$. Note that we still know that points in this set have a distance of at most $C(T)N^{-\frac{1}{18}+4\sigma} r$ to $\varphi(B_\varphi) = B_r(0)$.

Since $B_r(0)$ is a simple ball, this means that $\varphi(B_\psi) \setminus B_r(0)$ is contained in the $r + C(T)N^{-\frac{1}{18}+4\sigma}\delta$ ball around $0 \in \mathbb{R}^6$. This means the volume can be estimated by

$$(r + C(T)N^{-\frac{1}{18}+4\sigma} r)^6 - r^6 \leq C(T) N^{-\frac{1}{18}+4\sigma} r^6.$$

So now it only remains to compare the volume of $B_\psi \setminus B_\varphi$ with the one of $\varphi_t(B_\psi \setminus B_\varphi)$.

However, as $\varphi_t$ is differentiable and the differential has to be bounded by a constant $C(T)$ by 2.6, we know that

$$V(B_\psi \setminus B_\varphi) \leq C(T)V(\varphi_t(B_\psi \setminus B_\varphi)) \leq C(T)N^{-\frac{1}{18}+4\sigma}r^6.$$

As the density is bounded by a constant as well, this yields a suitable estimate.

It now remains to estimate $V(B_\varphi \setminus B_\psi)$, which works very similar. However, this time we need the differentiability of $\psi$ instead of $\varphi$.

We consider $\psi(B_\varphi \setminus B_\psi)$. Since $\psi$ is bijective on this set, this pushforward is given by $\psi(B_\varphi) \setminus B_r(0)$. As points in $B_\varphi$ had distance of at most $C(T)N^{-\frac{1}{18}+4\sigma}r$ to $B_\psi$ and as $\psi$ is Lipschitz, we obtain that the points in $\psi(B_\varphi)$ have distance of at most $C(T)N^{-\frac{1}{18}+4\sigma}r$ to $B_r(0)$.

Just like before, this implies that the volume of the difference $\psi(B_\varphi) \setminus B_r(0)$ can be estimated by $C(T)N^{-\frac{1}{18}+4\sigma}r^6$.

Finally, as $\psi_t$ is differentiable almost everywhere and the differential is bounded by $C(T)$, we obtain

$$V(B_\varphi \setminus B_\psi) \leq C(T)V(\psi_t(B_\varphi \setminus B_\psi)) \leq C(T)N^{-\frac{1}{18}+4\sigma}r^6.$$

In the first estimate we simply removed the points where $\psi_t$ is not differentiable before applying the estimate that required the bounded differential. Because the set of these points has a volume of 0, this does not effect the estimate.

In conclusion, we were able to prove that under suitable conditions

$$\left|\mathbb{P}\left(\psi_t(x) \in B_r(z)\right) - \mathbb{P}\left(\varphi_t(x) \in B_r(z)\right)\right| \ll r^6.$$

□

## 5.4 Final remarks

Through out the section, we always had to consider sets of good particles. $\psi$ being injective, surjective and differentiable and our probability estimate just now – all of them had to assume that we work with good particles. In the last section we already had to assume that we have some control on bad particles. Hence, extending our main result to also include newly added bad particles, would strengthen it significantly.

Considering our last result, i.e. that under the right conditions, we have

$$\left|\mathbb{P}\left(\psi_t(x) \in B_r(z)\right) - \mathbb{P}\left(\varphi_t(x) \in B_r(z)\right)\right| \ll r^6.$$

It becomes clear that the most interesting case would require dealing with bad particles.

As it stands, we have to restrict ourselves to configurations where the considered $x \in \mathbb{R}^6$ are good particles when added. This means that this estimate is not suited to answer the question whether a newly added particle has a collision or not. As collisions are of upmost importance to understand the system as a whole, extending this result could entail new approaches for the derivation of the Vlasov equation.

As we discussed previously, we do believe that our arguments can be extended to include the case of $x_0$ being a bad particle with some effort. We are hopeful that solving this next problem will entail new interesting results.

# 6 Appendix

In the appendix we go into more details for the results we simply stated in Chapter 2.

## 6.1 Proofs and further explanations for Chapter 2

We start by calculating the derivative of $f^N$. We have

$$Df^N(q) = \frac{1}{|q|^3}\begin{pmatrix} 1-\frac{3x^2}{|q|^2} & \frac{-3xy}{|q|^2} & \frac{-3xz}{|q|^2} \\ \frac{-3xy}{|q|^2} & 1-\frac{3y^2}{|q|^2} & \frac{-3xy}{|q|^2} \\ \frac{-3xz}{|q|^2} & \frac{-3yz}{|q|^2} & 1-\frac{3z^2}{|q|^2} \end{pmatrix}$$

for $|q| \geq c(N)$. For $|q| \leq c(N)$ we have

$$\frac{1}{|q|^3}\begin{pmatrix} x^2|q|r'(|q|)+r(|q|)(|q|^2-x^2) & xy(r'(|q|)|q|-r(|q|)) & xz(r'(|q|)|q|-r(|q|)) \\ xy(r'(|q|)|q|-r(|q|)) & y^2|q|r'(|q|)+r(|q|)(|q|^2-y^2) & yz(r'(|q|)|q|-r(|q|)) \\ xz(r'(|q|)|q|-r(|q|)) & yz(r'(|q|)|q|-r(|q|)) & z^2|q|r'(|q|)+r(|q|)(|q|^2-z^2) \end{pmatrix}$$

instead. Keeping in mind that $r' \leq C_r c(N)^{-3}$ as well as $r \leq C_r c(N)^{-2}$, it is easy to see that all these entries can be estimated by $\frac{3}{|q|^3}$ for $|q| \geq c(N)$ and by $8C_r c(N)^{-3}$ for $|q| \leq c(N)$. Hence, the matrix norm can be estimated by $C_{Df} c(N)^{-3}$ for a suitable constant.

We will also need to understand how $Df^N$ behaves if we have small changes in the argument. For this we will use the second derivative.

We start in the case where $|q| \geq c(N)$. Because of symmetry in coordinates, it is sufficient to look at the second derivatives of $\frac{x}{|q|^3}$. They have the form

$$\frac{d}{d[x^2,y^2,z^2]}\frac{x}{|q|^3} = \frac{1}{|q|^7}\begin{pmatrix} 3x\left(2x^2-3z^2-3y^2\right) & -3y\left(y^2+z^2-4x^2\right) & -3z\left(z^2+y^2-4x^2\right) \\ -3y\left(y^2+z^2-4x^2\right) & 3x\left(4y^2-z^2-x^2\right) & 15xzy \\ -3z\left(z^2+y^2-4x^2\right) & 15xzy & 3x\left(4z^2-y^2-x^2\right) \end{pmatrix}.$$

It is not to hard to see that this can be estimated by $\frac{15}{|q|^4}$.

For $|q| \leq c(N)$, up to symmetry, we need to consider the following derivatives.

$$\begin{aligned}
\frac{d}{dx^2}\frac{x}{|q|}r(|q|) &= \frac{x}{|q|^5}\left(x^2|q|^2r''(|q|) + 3(y^2+z^2)|q|r'(|q|) - 3(y^2+z^2)r(|q|)\right) \\
\frac{d}{dy^2}\frac{x}{|q|}r(|q|) &= \frac{x}{|q|^5}\left(y^2|q|^2r''(|q|) + (x^2-2y^2+z^2)|q|r'(|q|) - (x^2-2y^2+z^2)r(|q|)\right) \\
\frac{d}{dxdy}\frac{x}{|q|}r(|q|) &= \frac{1}{|q|^5}\Big(x^2y|q|^2r''(|q|) + y|q|(|q|^2-3x^2)r'(|q|) - y(|q|^2-3x^2)r(|q|)\Big) \\
\frac{d}{dydz}\frac{x}{|q|}r(|q|) &= \frac{1}{|q|^5}\Big(xyz|q|^2r''(|q|) + 3xyz|q|r'(|q|) + 3xyzr(|q|)\Big).
\end{aligned}$$

Note that our force does not need to be two times differentiable on the sphere with $|q| = c(N)$. This calculation still allows us to estimate all entries by $Cc(N)^{-4}$ for some constant $C$. We hence find a constant $c_h$ such that $Df^N$ is Lipschitz-continuous on its complete domain with Lipschitz-constant $c_h c(N)^{-4}$, and such that the matrix norm can be estimated by $\frac{1}{16} c_h \frac{1}{|q|^4}$ for $|q| \geq c(N)$.

The estimates for $Df^N$ can be combined and we get the following lemma.

**Lemma 6.1.** *There is a constant $C_{Df} > 0$ (independent of $N$ and the configuration) such that*

$$\left|Df^N(q)\right| \leq \left\{ \begin{matrix} C_{Df} c(N)^{-3} & \text{if} & |q| \leq c(N) \\ \frac{1}{8} C_{Df} |q|^{-3} & \text{if} & |q| \geq c(N) \end{matrix} \right\} \leq C_{Df} c(N)^{-3}.$$

This allows us to introduce

**Definition 6.2.** We define $g^N : \mathbb{R}^3 \to \mathbb{R}^+$ by

$$q \mapsto \begin{cases} C_{Df} c(N)^{-3} & \text{if } |q| \leq 2c(N), \\ C_{Df} |q|^{-3} & \text{else.} \end{cases}$$

This function can be used like the derivative of $f^N$ in many cases as the following lemma shows.

**Lemma 6.3.** *Let $f^N$ be our force from 1.1. The following estimates hold for all $q, q_0, q_1 \in \mathbb{R}^3$.*

$$\begin{aligned} \left|Df^N(q)\right| &\leq g(q) \\ \left|f^N(q_1) - f^N(q_0)\right| &\leq g^N(\min(q_0, q_1))|q_1 - q_0| \end{aligned}$$

*Further, if $\delta \in \mathbb{R}^6$ satisfies $|\delta| \leq c(N)$ or $|\delta| < \frac{1}{2}|q|$, it holds that*

$$\left|f^N(q) - f^N(q+\delta)\right| \leq g^N(q)|\delta|.$$

*Proof.* The first result follows directly from the definition and the lemma above. All the other results are consequences of the mean value theorem. For the second statement we estimate

$$\begin{aligned} |f^N(q_1) - f^N(q_0)| &= \left| \int_0^1 Df^N(q_0 + s(q_1 - q_0)) \, \mathrm{d}s \cdot (q_1 - q_0) \right| \\ &\leq \max_{0 \leq s \leq 1} |Df^N(q_0 + s(q_1 - q_0))| \cdot |q_1 - q_0| \leq g^N(\min(q_0, q_1))|q_1 - q_0|. \end{aligned}$$

Now let $\delta \in \mathbb{R}^6$ be given such that $|\delta| \leq c(N)$ or $|\delta| \leq \frac{1}{2}|q|$. The estimate is trivial for $|q| \leq 2c(N)$. If $|q| \geq 2c(N)$, then we can conclude that $|q + s\delta| \geq c(N)$ for all $s \in [0,1]$ in both cases. Hence, we can estimate

$$\begin{aligned} |f^N(q) - f^N(q+\delta)| &= \left| \int_0^1 Df^N(q + s\delta) \, \mathrm{d}s \cdot \delta \right| \\ &\leq \max_{0 \leq s \leq 1} |Df^N(q + s\delta)| \cdot |\delta| \leq \max_{0 \leq s \leq 1} \frac{C_{Df}}{2^3 |q + s\delta|^3} |\delta| \\ &\leq \left(\frac{1}{2}\right)^3 C_{Df} |\delta| \frac{1}{\left|q - \frac{q}{|q|}|\delta|\right|^3} \\ &\leq \left(\frac{1}{2}\right)^3 C_{Df} |\delta| \frac{1}{|q|^3} \cdot \frac{1}{\left(1 - \frac{|\delta|}{|q|}\right)^3} \leq \frac{1}{2^3} C_{Df} \frac{1}{|q|^3} \cdot 2^3 |\delta| = g^N(q)|\delta|. \end{aligned}$$

In the last estimate we used that $|\delta| \leq \frac{1}{2}|q|$. □

Using the bounds on $DDf^N$ from above, namely

$$|DDf^N(q)| \leq \begin{cases} c_h c(N)^{-4} & \text{if } |q| \leq c(N) \\ c_h |q|^{-4} & \text{else} \end{cases}$$

for a suitable constant $c_h > 0$, we can define another very helpful function, namely $h$.

**Definition 6.4.** For $\mathbf{c} \geq 1$ we define $h^N_{\mathbf{c}} : \mathbb{R}^3 \to \mathbb{R}^+$ by

$$q \mapsto \begin{cases} c_h c(N)^{-4} & \text{if } |q| \leq 2\,\mathbf{c} \cdot c(N) \\ (2\,\mathbf{c})^4 c_h |q|^{-4} & \text{else} \end{cases}.$$

The $\mathbf{c}$ will be chosen later and it will not depend on the number of particles $N$ but only on the time frame $T$ and our density $k$.

The following Lemma shows how $h_{\mathbf{c}}$ can be used and why the constant $\mathbf{c}$ in the cutoff is needed. It allows us to give good estimates as long as the disturbance $\delta$ satisfies $|\delta| \leq \mathbf{c}\, c(N)$.

**Lemma 6.5.** *Let $f^N$ be our force from 1.1. The following estimates hold for all $q \in \mathbb{R}^3$ for fixed $\mathbf{c} \geq 1$.*

$$\left|DDf^N(q)\right| \leq h^N_{\mathbf{c}}(q) \text{ for } |q| \neq c(N)$$
$$\left|Df^N(q_0) - Df^N(q_1)\right| \leq h^N_{\mathbf{c}}(\min(q_0, q_1))|q_0 - q_1|.$$

*Further, if $\delta \in \mathbb{R}^6$ satisfies $|\delta| \leq \mathbf{c}\, c(N)$ or $|\delta| < \frac{1}{2}|q|$, it holds that*

$$\left|Df^N(q) - Df^N(q+\delta)\right| \leq h^N_{\mathbf{c}}(q)|\delta|.$$

*Proof.* The first result follows directly from the definition and the estimates stated above. All the other results are consequences of the mean value theorem. For the second statement we estimate

$$\begin{aligned} \left|Df^N(q_1) - Df^N(q_1)\right| &= \left|\int_0^1 DDf^N(q_0 + s(q_1 - q_0))\,\mathrm{d}s \cdot (q_1 - q_2)\right| \\ &\leq \max_{0\leq s\leq 1} \left|DDf^N(q_1 + s(q_2 - q_1))\right| \mathrm{d}s \cdot |q_0 - q_1| \\ &\leq h^N(\min(q_0, q_1))|q_0 - q_1|. \end{aligned}$$

Now let $\delta \in \mathbb{R}^6$ be given such that $|\delta| \leq \mathbf{c}\, c(N)$ or $|\delta| \leq \frac{1}{2}|q|$. The estimate is trivial for $|q| \leq 2\,\mathbf{c}\, c(N)$. If $|q| \geq 2\,\mathbf{c}\, c(N)$, then we can conclude that $|q + s\delta| \geq \mathbf{c}\, c(N)$ for all $s \in [0,1]$ in both cases. Hence, we can estimate

$$\begin{aligned} \left|Df^N(q) - Df^N(q+\delta)\right| &= \left|\int_0^1 DDf^N(q + s\delta)\,\mathrm{d}s \cdot \delta\right| \\ &\leq \max_{0\leq s\leq 1} \left|DDf^N(q + s\delta)\right| \cdot |\delta| \leq \max_{0\leq s\leq 1} \frac{c_h}{|q + s\delta|^4}|\delta| \\ &\leq c_h|\delta| \frac{1}{\left|q - \frac{q}{|q|}|\delta|\right|^4} \\ &\leq c_h|\delta| \frac{1}{|q|^4} \cdot \frac{1}{\left(1 - \frac{|\delta|}{|q|}\right)^4} \leq c_h \frac{1}{|q|^4} \cdot 2^4|\delta| \leq h^N_{\mathbf{c}}(q)|\delta|. \end{aligned}$$

In the last estimate we used that $|\delta| \leq \frac{1}{2}|q|$. □

Before we can close this chapter, we also want to be able to estimate the mean-field versions of $Df$ and $h_{\mathbf{c}}$, i.e.

**Lemma 6.6.** *Let $q(s)$ be any point in $\mathbb{R}^3$ (here $s \in [0,T]$ is some fixed time). Then the following three inequalities hold.*

$$\int_{\mathbb{R}^3} \left|Df^N(q(s)-y)\right| \tilde{k}_s^N(y)\,\mathrm{d}y \leq C\ln(N), \tag{a}$$

$$\left|\int_{\mathbb{R}^3} Df^N(q(s)-y)\tilde{k}_s(y)\,\mathrm{d}y\right| \leq C_{Df}, \tag{b}$$

$$\int_{\mathbb{R}^3} h_{\mathbf{c}}^N(q(s)-y)\tilde{k}_s(y)\,\mathrm{d}y \leq C_h c(N)^{-1}, \tag{c}$$

*where $C_h$ **does** depend on the constant $\mathbf{c}$ chosen in $h$.*

*Proof.* All three proofs follow the same pattern. We divide the integral into two or three parts and then estimate. Only the estimate for (b) is a little more involved.

For (a) we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left|Df^N(q(s)-y)\right| \tilde{k}_s^N(y)\,\mathrm{d}y \\
\leq& C\left(\int\limits_{|q(s)-y|\leq c(N)} c(N)^{-3}|\tilde{k}_s^N|_\infty\,\mathrm{d}y + \int\limits_{c(N)\leq|q(s)-y|\leq 1} \frac{1}{|q(s)-y|^3}|\tilde{k}_s^N|_\infty\,\mathrm{d}y + \int\limits_{|q(s)-y|\geq 1} \tilde{k}_s^N(y)\,\mathrm{d}y\right) \\
\leq& C(c(N)^3\cdot c(N)^{-3} + \ln(N) + |\tilde{k}_s^N|_1) \leq C\ln(N).
\end{aligned}$$

The idea for (b) is the same and we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^3} Df(q(s)-y)\tilde{k}_s^N(y)\,\mathrm{d}y \\
\leq& \int_{|q(s)-y|\leq c(N)} c(N)^{-3}|\tilde{k}_s|_\infty\,\mathrm{d}y \\
&+ \left|\int_{c(N)\leq|q(s)-y|\leq 1} Df(q(s)-y)\tilde{k}_s^N(y)\,\mathrm{d}y\right| \\
&+ \int_{|q(s)-y|\geq 1} \tilde{k}_s^N(y)\,\mathrm{d}y \\
\leq& C\left(|\tilde{k}_s^N|_\infty c(N)^3\cdot c(N)^{-3} + c(N) + |\tilde{k}_s^N|_1\right) \leq C_{Df}.
\end{aligned}$$

Here we had to estimate $\left|\int_{c(N)\leq|y-q(s)|\leq 1} Df^N(q(s)-y)\tilde{k}_s^N(y)\,\mathrm{d}y\right|$ by $Cc(N)$:

$$
\begin{aligned}
&\left|\int_{c(N)\le|y-q(s)|\le 1} Df^N(q(s)-y)\tilde{k}_s^N(y)\,\mathrm{d}y\right| \\
=&\left|\int_{c(N)\le|y-q(s)|\le 1} Df^N(q(s)-y)\left(\tilde{k}_s^N(q(s))+\tilde{k}_s^N(y)-\tilde{k}_s^N(q(s))\right)\mathrm{d}y\right| \\
\le&\left|\int_{c(N)\le|y-q(s)|\le 1} Df^N(q(s)-y)\tilde{k}_s^N(q(s))\,\mathrm{d}y\right| \\
&+\left|\int_{c(N)\le|y-q(s)|\le 1} Df(q(s)-y)(\tilde{k}_s^N(y)-\tilde{k}_s^N(q(s)))\,\mathrm{d}y\right| \\
\le&0+\int_{c(N)\le|y-q(s)|\le 1}\left|Df^N(q(s)-y)\right|\left|\tilde{k}_s^N(y)-\tilde{k}_s^N(q(s))\right|\mathrm{d}y \\
\le&\int_{c(N)\le|y-q(s)|\le 1}\left|\frac{1}{(q(s)-y)^3}\right|\cdot C\cdot|q(s)-y|\,\mathrm{d}y \\
\le&Cc(N),
\end{aligned}
$$

where we used that the entries of $Df^N$ are anti-symmetric and hence $\int Df^N = 0$ for symmetric domains. The estimate for (c) is easier again:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} h_{\mathbf{c}}^N(q(s)-y)\tilde{k}_s^N(y)\,\mathrm{d}y \\
\le&c_h\left(\int_{|y-q(s)|\le 2\,\mathbf{c}\,c(N)} c(N)^{-4}\tilde{k}_s^N(y)\,\mathrm{d}y+\int_{|y-q(s)|>2\,\mathbf{c}\,c(N)}\frac{(2\,\mathbf{c})^4}{|q(s)-y|^4}\tilde{k}_s^N(y)\,\mathrm{d}y\right) \\
\le&Cc_h((\mathbf{c}\,c(N))^3\cdot c(N)^{-4}+(2\,\mathbf{c})^4c(N)^{-1})=C_hc(N)^{-1}.
\end{aligned}
$$

□

This concludes our discussion for $f^N$ and its derivatives.

The next lemma tells us that the mean-field trajectories behave Lipschitz-continuously. A result that will be useful many times.

**Lemma 6.7.** *For all $T>0$ there is a constant $C>0$ such that for all $x,y\in\mathbb{R}^6$, $N\in\mathbb{N}$ and $t_0,t_1\in[0,T]$ it holds that*

$$
|\varphi_{t_0,t_1}(x)-\varphi_{t_0,t_1}(y)|\le|x-y|e^{C|t_1-t_0|}.
$$

*Proof.* See [9] Lemma 2.1.2. . □

We will have to estimate $f^N, Df^N$ and $h^N$ quite often in our final proof. To this end the next lemma is very useful.

**Lemma 6.8.** *Let $1<\alpha\le 3$ and consider a function $g^N:\mathbb{R}^3\to$ such that*

$$
|g^N(q)|\le\begin{cases}Cc_N^{-\alpha} & \text{if}\quad |q|\le c_N,\\ \frac{C}{|q|^\alpha} & \text{if}\quad |q|>c_N.\end{cases}
$$

*Then the following estimates hold:*

*a) Let $y, z \in \mathbb{R}^6$, $t_{min}$ be a point in time where*

$$\begin{aligned} &\min |{}^1\widetilde{z}(s) - {}^1\widetilde{y}(s)| = |{}^1\widetilde{z}(t_{min}) - {}^1\widetilde{y}(t_{min})| =: \Delta q > 0 \\ &\text{and} \qquad |{}^2\widetilde{z}(t_{min}) - {}^2\widetilde{y}(t_{min})| =: \Delta p > 0. \end{aligned}$$

*Then there exists a constant $C > 0$ (independent of $y, z$ and $N$) such that*

$$\int_0^T \left| g\left({}^1\widetilde{z}(s) - {}^1\widetilde{y}(s)\right)\right| \mathrm{d}s \leq C \min\left(\frac{1}{\Delta q^\alpha}, \frac{1}{c_N^{\alpha-1}\Delta p}, \frac{1}{\Delta q^{\alpha-1}\Delta p}\right).$$

*b) Let $T > 0, z_1, z_2 \in \mathbb{R}^6$ be given and consider differentiable ${}^1y_1, {}^1y_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$. We denote the derivatives by ${}^2y_1, {}^2y_2$ and write $y_k = ({}^1y_k, {}^2y_k)$. Assume that for some $\sigma > 0$*

$$N^\sigma |{}^1\widetilde{z_1}(t_{min}) - {}^1\widetilde{z_2}(t_{min})| \leq |{}^2\widetilde{z_1}(t_{min}) - {}^2\widetilde{z_2}(t_{min})| = \Delta p \tag{A1}$$

*and*

$$\sup_{0\leq s\leq T} |\widetilde{z_1}(s) - y_1(s)| \leq N^{-\sigma}\Delta p \qquad \wedge \qquad \sup_{0\leq s\leq T} |\widetilde{z_2}(s) - y_2(s)| \leq N^{-\sigma}\Delta p. \tag{A2}$$

*Then there exists some $N_0$ such that for all $N \geq N_0$ we have*

$$\int_0^T \left| g\left({}^1y_1(s) - {}^1y_2(s)\right)\right| \mathrm{d}s \leq C \min\left(\frac{1}{c_N^{\alpha-1}\Delta p}, \frac{1}{\min_{0\leq s\leq T} |y_1(s) - y_2(s)|^{\alpha-1}\Delta p}\right).$$

Before we prove this lemma, let us briefly discuss the result. The important part in the estimate is that we can use $\Delta p$ to control the integral. This means that even if particles (or more precisely their mean-field trajectories) come close, we might still be able to estimate the effect of the force as long as their relative momentum is close. This is very intuitive, as if their relative momentum is very large, they will not be close for long, and hence the effect of the force can still be controlled. This is one of the key ideas in the later proof of our main result.

This was already used and proven by Graß in [9]. As we use a different notation and need a slightly modified version of the lemma in Graß's paper, we decided to give a proof for b) here as well.

*Proof.* We prove b). First, note that by A1 we know that $|\widetilde{z}_1(t_{min}) - \widetilde{z}_2(t_{min})|$ is of order $\Delta p$, i.e.

$$\Delta p \leq |\widetilde{z}_1(t_{min}) - \widetilde{z}_2(t_{min})| \leq 2\Delta p.$$

Second, we can apply 6.7 to obtain a constant $C_0 \geq 1$ depending on T such that

$$\begin{aligned} |\widetilde{z}_1(t_{min}) - \widetilde{z}_2(t_{min})| &\leq C_0 \min_{0\leq t\leq T} |\widetilde{z}_1(t) - \widetilde{z}_2(t)| \\ \text{and} \qquad \max_{0\leq t\leq T} |\widetilde{z}_1(t) - \widetilde{z}_2(t)| &\leq \frac{1}{2} C_0 |\widetilde{z}_1(t_{min}) - \widetilde{z}_2(t_{min})|. \end{aligned}$$

Combining these estimates, yields

$$\frac{\Delta p}{C_0} \leq \min_{0\leq t\leq T} |\widetilde{z}_1(t) - \widetilde{z}_2(t)| \leq \max_{0\leq t\leq T} |\widetilde{z}_1(t) - \widetilde{z}_2(t)| \leq C_0 \Delta p. \tag{i}$$

For large enough $N$ we use the second assumption A2 to obtain

$$\min_{0\le t\le T}|y_1(t)-y_2(t)| \ge \min_{0\le t\le T}|\widetilde{z}_1(t)-\widetilde{z}_2(t)| - 2N^{-\sigma}\Delta p \ge \frac{\Delta p}{C_0} - 2N^{-\sigma}\Delta p \ge \frac{\Delta p}{2C_0}. \qquad \text{(ii)}$$

For any $t' \in [0,T]$ and $C_1 > 1$ let $t'_{min}$ denote a point in time where $|{}^1y_1 - {}^1y_2|$ obtains its minimum on $[t', t' + \frac{1}{C_1}]$. The main step is to prove that

**Claim:**

$$\int_{t'}^{t'+\frac{1}{C_1}} \left|g({}^1y_1(s) - {}^1y_2(s))\right| \mathrm{d}s \le \frac{C}{\Delta p} \min\left(\frac{1}{c_N^{\alpha-1}}, \frac{1}{|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^{\alpha-1}}\right).$$

Note that after we have proven this we are basically done: As $\min|{}^1y_1(s) - {}^1y_2(s)| \le |{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|$, by definition we can add up these estimates for the whole interval $[0,T]$ to obtain

$$\begin{aligned}
&\int_0^T \left|g({}^1y_1(s) - {}^1y_2(s))\right| \mathrm{d}s \\
\le &\sum_{k=0}^{\lceil C_1\rceil} \int_{t'_k}^{t'_k+\frac{1}{C_1}} \left|g({}^1y_1(s) - {}^1y_2(s))\right| \mathrm{d}s \\
\le &C_1 \int_{t'}^{t'+\frac{1}{C_1}} \left|g({}^1y_1(s) - {}^1y_2(s))\right| \mathrm{d}s \\
\le &C_1 \frac{C}{\Delta p} \min\left(\frac{1}{c_N^{\alpha-1}}, \frac{1}{|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^{\alpha-1}}\right) \\
\le &C(T) \min\left(\frac{1}{c_N^{\alpha-1}\Delta p}, \frac{1}{\min_{0\le s\le T}|y_1(s)-y_2(s)|^{\alpha-1}\,\Delta p}\right).
\end{aligned}$$

It remains to prove the claim, and for the argument above to work we need to make sure that both constants, i.e. $C_1$ and the $C$ in the claim behave nicely. If $|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})| \ge \frac{\Delta p}{4C_0}$, the properties of our map imply that

$$\begin{aligned}
&\int_{t'}^{t'+\frac{1}{C_1}} \left|g({}^1y_1(s) - {}^1y_2(s))\right| \mathrm{d}s \\
\le &C \min\left(\frac{1}{c_N^{\alpha}}, \frac{1}{|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^{\alpha}}\right) \le \frac{C}{\Delta p} \min\left(\frac{1}{c_N^{\alpha-1}}, \frac{1}{|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^{\alpha-1}}\right),
\end{aligned}$$

where the last estimate is clear if the min takes on the value on the right, and otherwise we also know $c_N \ge |{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})| \ge \frac{\Delta p}{4C_0}$.

We now want to prove a similar result in the case $|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})| < \frac{\Delta p}{4C_0}$.

In this case we have

$$|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})| < \frac{\Delta p}{4C_0} \le |{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})|, \qquad \text{(iii)}$$

as otherwise

$$\begin{aligned}|y_1(t'_{min}) - y_2(t'_{min})| &= \sqrt{|{}^1y_1(t'_{min}) - {}^2y_2(t'_{min})|^2 + |{}^1y_1(t'_{min}) - {}^2y_2(t'_{min})|^2} \\ &< \sqrt{\left|\frac{\Delta p}{4C_0}\right|^2 + \left|\frac{\Delta p}{4C_0}\right|^2} \\ &\leq \frac{\sqrt{2}\Delta p}{4C_0} \\ &\leq \frac{\Delta p}{2C_0} \\ &\overset{(ii)}{\leq} \min_{0\leq t\leq T} |y_1(t) - y_2(t)|,\end{aligned}$$

which obviously yields a contradiction.

By [9] Lemma 2.1.3. it holds for all $|s - t'_{min}| \leq \frac{1}{C_1}$ that

$$\begin{aligned}&|{}^2\widetilde{z_1}(s) - {}^2\widetilde{z_2}(s) - ({}^2\widetilde{z_1}(t'_{min}) - {}^2\widetilde{z_2}(t'_{min}))| \\ &\leq C|s - t'_{min}| \left(|{}^1\widetilde{z_1}(t'_{min}) - {}^1\widetilde{z_2}(t'_{min})| + |{}^2\widetilde{z_1}(t'_{min}) - {}^2\widetilde{z_2}(t'_{min})||s - t'_{min}|\right)\end{aligned}$$

Combining this with our previous estimates, we obtain

$$\begin{aligned}&|{}^2\widetilde{z_1}(s) - {}^2\widetilde{z_2}(s) - ({}^2\widetilde{z_1}(t'_{min}) - {}^2\widetilde{z_2}(t'_{min}))| \\ &\leq C|s - t'_{min}| \left(|{}^1\widetilde{z_1}(t'_{min}) - {}^1\widetilde{z_2}(t'_{min})| + |{}^2\widetilde{z_1}(t'_{min}) - {}^2\widetilde{z_2}(t'_{min})||s - t'_{min}|\right) \\ &\overset{(i)}{\leq} C|s - t'_{min}| \left(|{}^1\widetilde{z_1}(t'_{min}) - {}^1\widetilde{z_2}(t'_{min})| + C_0\Delta p|s - t'_{min}|\right) \\ &\overset{A2}{\leq} C|s - t'_{min}| \left(|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})| + 2N^{-\sigma}\Delta p + C_0\Delta p|s - t'_{min}|\right) \\ &\overset{(iii)}{\leq} C|s - t'_{min}| \cdot |{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})|,\end{aligned} \tag{iv}$$

where the constant may depend on $C_0$ but importantly not on $N$.

This can be used to obtain the following estimate for large enough $N$ and $|t-t'_{min}| \leq \frac{1}{C_1}$

$$
\begin{aligned}
&\left|({}^1y_1(t) - {}^1y_2(t)) - ({}^1y_1(t'_{min}) - {}^1y_2(t'_{min})) - ({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t-t'_{min})\right| \\
=&\left|\int_{t'_{min}}^{t} ({}^2y_1(s) - {}^2y_2(s))\,\mathrm{d}s \quad - ({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t-t'_{min})\right| \\
\leq&\left|\int_{t'_{min}}^{t} ({}^2y_1(s) - {}^2y_2(s)) - ({}^2\widetilde{z_1}(s) - {}^2\widetilde{z_2}(s))\,\mathrm{d}s\right| \\
&+\left|\int_{t'_{min}}^{t} ({}^2\widetilde{z_1}(s) - {}^2\widetilde{z_2}(s)) - ({}^2\widetilde{z_1}(t'_{min}) - {}^2\widetilde{z_2}(t'_{min}))\,\mathrm{d}s\right| \\
&+\left|({}^2\widetilde{z_1}(t'_{min}) - {}^2\widetilde{z_2}(t'_{min})) - ({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))\right| |t-t'_{min}| \\
\overset{(iv)}{\leq}& 2\max_{k=1,2}\sup_{0\leq s\leq T} |\widetilde{z_k}(s) - y_k(s)||t-t'_{min}| \\
&+ C|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})||t-t'_{min}|^2 \\
&+ 2\max_{k=1,2}\sup_{0\leq s\leq T} |\widetilde{z_k}(s) - y_k(s)||t-t'_{min}| \\
\overset{A2}{\leq}& 4N^{-\sigma}\Delta p|t-t'_{min}| + C|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})||t-t'_{min}|^2 \\
\overset{(iii)}{\leq}& 4CN^{-\sigma}|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})||t-t'_{min}| + C|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})||t-t'_{min}|^2.
\end{aligned}
$$

Solving this estimate for $|({}^1y_1(t) - {}^1y_2(t))|$ yields that for $t \in [t', t' + \frac{1}{C_1}]$ we have

$$
\begin{aligned}
&|({}^1y_1(t) - {}^1y_2(t))| \\
\geq&\left|({}^1y_1(t'_{min}) - {}^1y_2(t'_{min})) + ({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t-t'_{min})\right| \\
&- C\left|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})\right| \cdot |t-t'_{min}|(4N^{-\sigma} + |t-t'_{min}|) \\
\overset{(\star)}{\geq}&|({}^1y_1(t'_{min}) - {}^1y_2(t'_{min})) + ({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t-t'_{min})| \\
&- \frac{1}{4}|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})| \cdot |t-t'_{min}| \\
\overset{(\star\star)}{\geq}&\max(|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|, |{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})||t-t'_{min}|) \\
&- \frac{1}{4}|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})| \cdot |t-t'_{min}| \\
\geq&\frac{3}{4}\max(|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|, |{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})| \cdot |t-t'_{min}|),
\end{aligned}
$$

where $(\star)$ holds for large enough $N$ if we choose $C_1$ so large that $|t-t'_{\min}| \ll \frac{1}{C}$.

The estimate at $(\star\star)$ stems from the fact that $t'_{\min}$ was defined such that $|({}^1y_1(t'_{min}) - {}^1y_2(t'_{min}))|$ is minimal on the interval $[t', t' + \frac{1}{C_1}]$. This yields three cases:

$\boxed{t'_{\min} = t'}$ This means that the derivative of $|({}^1y_1(t) - {}^1y_2(t))|$ which is easily calculated as

$$
\frac{1}{|({}^1y_1(t) - {}^1y_2(t))|} \cdot \langle {}^1y_1(t) - {}^1y_2(t), {}^2y_1 - {}^2y_2(t)\rangle
$$

needs to be non-negative at $t = t'_{min}$.

Hence, $\langle {}^1y_1(t) - {}^1y_2(t), {}^2y_1 - {}^2y_2(t)\rangle$ needs to be non-negative which leads to

$$\begin{aligned}
&\left|({}^1y_1(t'_{min}) - {}^1y_2(t'_{min})) + ({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})\right|^2 \\
=&\left|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^2 + |({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})\right|^2 \\
&+ 2(t - t'_{\min})\left\langle {}^1y_1(t'_{min}) - {}^1y_2(t'_{min}), {}^2y_1(t'_{min}) - {}^2y_2(t'_{min})\right\rangle \\
\geq&\left|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^2 + |({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})\right|^2 \\
\geq&\max\left(|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^2, |({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})|^2\right).
\end{aligned}$$

$\boxed{t'_{\min} = t' + \frac{1}{C_1}}$ In this case the same argument leads to

$$\left\langle {}^1y_1(t'_{min}) - {}^1y_2(t'_{min}), {}^2y_1(t'_{min}) - {}^2y_2(t'_{min})\right\rangle \leq 0.$$

This then implies

$$\begin{aligned}
&\left|({}^1y_1(t'_{min}) - {}^1y_2(t'_{min})) + ({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})\right|^2 \\
=&\left|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^2 + |({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})\right|^2 \\
&+ 2(t - t'_{\min})\left\langle {}^1y_1(t'_{min}) - {}^1y_2(t'_{min}), {}^2y_1(t'_{min}) - {}^2y_2(t'_{min})\right\rangle \\
\geq&\left|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^2 + |({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})\right|^2 \\
\geq&\max\left(|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^2, |({}^2y_1(t'_{min}) - {}^2y_2(t'_{min}))(t - t'_{min})|^2\right).
\end{aligned}$$

$\boxed{t' < t'_{\min} < t' + \frac{1}{C_1}}$ In this case we have

$$\left\langle {}^1y_1(t'_{min}) - {}^1y_2(t'_{min}), {}^2y_1(t'_{min}) - {}^2y_2(t'_{min})\right\rangle = 0,$$

which leads to the desired result. In conclusion we have proven that

$$\left|({}^1y_1(t) - {}^1y_2(t))\right| \geq \frac{3}{4}\max\left(\left|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})\right|, \left|{}^2y_1(t'_{min}) - {}^2y_2(t'_{min})\right||t - t'_{min}|\right), \quad \text{(v)}$$

for all $t$ in the interval. Using $r = |({}^1y_1(t'_{\min}) - {}^1y_2(t'_{\min}))|$ and $v = |({}^2y_1(t'_{\min}) - {}^2y_2(t'_{\min}))|$ as well as $\tilde{r} = \max(r, c_N)$, we get

$$\begin{aligned}
&\int_{t'}^{t'+\frac{1}{C_1}} |g({}^1y_1(s) - {}^1y_2(s))|\,\mathrm{d}s \\
\leq& C\int_{t'}^{t'+\frac{1}{C_1}} \frac{1}{\max(|({}^1y_1(t) - {}^1y_2(t))|, c_N)^\alpha} \\
\overset{(v)}{\leq}& \frac{4}{3}C\int_{t'}^{t'+\frac{1}{C_1}} \frac{1}{\max(\tilde{r}, v|t - t'_{\min}|)^\alpha} \\
=& C\int_{|t-t'_{min}|\leq\frac{\tilde{r}}{v}} \frac{1}{\tilde{r}^\alpha} + \int_{|t-t'_{min}|\geq\frac{\tilde{r}}{v}} \frac{1}{(v|t - t'_{\min}|)^\alpha} \\
\leq& C\left(\frac{1}{v}\frac{1}{\tilde{r}^{\alpha-1}} + v^{-\alpha}\left(\frac{v^{\alpha-1}}{\tilde{r}^{\alpha-1}} - C_1^{\alpha-1}\right)\right) \\
\overset{(iii)}{\leq}& \frac{C}{\Delta p}\min\left(\frac{1}{c_N^{\alpha-1}}, \frac{1}{|{}^1y_1(t'_{min}) - {}^1y_2(t'_{min})|^{\alpha-1}}\right).
\end{aligned}$$

This closes our case distinction from the very beginning of the proof.

□

In our estimates we will distinguish between "good" and "bad" particles. We call particles "bad" if their mean-field trajectories come very close to other particles' mean-field trajectories in phase-space, i.e they come close physically **and** they have a small relative velocity.

It might be tempting to say particles become bad if their real trajectories come close. However, it is (to our knowledge) basically impossible to get meaningful estimates with this approach, as the real system is too complicated.

With our mean-field approach on the other hand, we will be able to prove that there are typically only few bad interactions. This is a very important step towards our main goal, as good particles (i.e. the ones that are not bad) are often easier to work with and we can prove stronger estimates for them. For the remaining bad particles we can then leverage the fact that there are only very few of them to offset the fact that they are harder to estimate.

We now give a rigorous definition of good and bad particles.

**Definition 6.9.** Let $R, V > 0$. We say a particle $z \in \mathbb{R}^6$ has a bad collision with a particle $y \in \mathbb{R}^6$ in the time frame $[t_0, t_1]$ if

$$\exists t \in [t_0, t_1] : |^1\widetilde{z}(t) - {}^1\widetilde{y}(t)| \leq R \quad \wedge \quad |^2\widetilde{z}(t) - {}^2\widetilde{y}(t)| \leq V.$$

Most often we will work on the time interval $[0, T]$, and in this case we will simply say two particles have a bad collision. If the two particles do not have a bad collision, we write $z \in G^N_{R,V}(y)$. We call a particle $x_i$ good if all other $x_j \in G^N_{R,V}(x_i)$. Otherwise it is called bad. The set of good and bad particles in a configuration $X \in \mathbb{R}^{6N}$ is denoted by $\mathcal{M}^N_g(X)$ and $\mathcal{M}^N_b(X)$, respectively.

For the vast majority of this paper we will work in the setting $R = 6N^{-\frac{2}{9}-\sigma}, V = N^{-\frac{2}{9}}$. Whenever we talk about good and bad particles, it is meant with respect to this choice of $R$ and $V$, unless we specify it to be something else.

We also define the related sets $M^{N,(t_1,t_2)}_{(r,R)(v,V)}(y) \subseteq \mathbb{R}^6$, where $0 \leq r < R, 0 \leq v < V$ by

$$\begin{aligned} &z \in M^{N,(t_1,t_2)}_{(r,R)(v,V)}(y) \subseteq \mathbb{R}^6 \iff \exists t \in [t_1, t_2] : \\ &r \leq \min_{s \in [t_0,t_1]} |^1\widetilde{z}(s) - {}^1\widetilde{y}(s)| = |^1\widetilde{z}(t) - {}^1\widetilde{y}(t)| \leq R \quad \wedge \quad v \leq |^2\widetilde{z}(t) - {}^2\widetilde{y}(t)| \leq V. \end{aligned}$$

We will often leave out the time indicator if we consider the time frame $[0, T]$. We write

$$M^N_{R,V}(y) = M^N_{(0,R),(0,V)}(y).$$

These sets have a focus on the physical distance since the distance in space dictates the points we are interested in. Only for the points of minimal spacial distance we even consider the relative momentum.

**Remark 6.10.** Both sets, i.e. $G^N_{R,V}(y)$ and $M^N_{R,V}(y)$, measure how close particles come to each other or more precisely whether they satisfy a certain condition on their distance. One might be curious why we need both definitions. The first definition is more simple and hence arguably more natural. However, in some cases we want lower bounds on the relative position and momentum. It is hard to introduce such bounds in a meaningful way without changing the definition fundamentally. This is where the second definition is helpful, as the introduction of the minimal relative position allows us to introduce the lower bound in a sensible way.

This definition, however, becomes less stable as the point of minimal distance to another particle might jump drastically even if we only move one particle a tiny bit.

So, depending on the circumstances, both notions of distance are useful.

The sets $M^{N,(t_1,t_2)}_{(r,R)(v,V)}$ will mainly be used to define the following partition of $\mathbb{R}^{6N}$.

**Definition 6.11.** For each $r_0, v_0, \varepsilon > 0$ and $y \in \mathbb{R}^6$ we introduce a cover of $\mathbb{R}^6$:

(i) $M^N_{(0,r_0),(0,v_0)}(y)$

(ii) $M^N_{(0,r_0),(N^{l\varepsilon}v_0,N^{(l+1)\varepsilon}v_0)}(y), \quad 0 \le l \le \left\lfloor \frac{\ln(\frac{1}{v_0})}{\varepsilon \ln(N)} \right\rfloor$

(iii) $M^N_{(0,r_0),(1,\infty)}(y)$

(iv) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(0,v_0)}(y), \quad 0 \le k \le \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor$

(v) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(N^{l\varepsilon}v_0,N^{(l+1)\varepsilon}v_0)}(y), \quad 0 \le k \le \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor, \quad 0 \le l \le \left\lfloor \frac{\ln(\frac{1}{v_0})}{\varepsilon \ln(N)} \right\rfloor$

(vi) $M^N_{(N^{k\varepsilon}r_0,N^{(k+1)\varepsilon}r_0),(1,\infty)}(y), \quad 0 \le k \le \left\lfloor \frac{\ln(\frac{1}{r_0})}{\varepsilon \ln(N)} \right\rfloor$

(vii) $M^N_{(N^{-\varepsilon},\infty),(0,\infty)}(y)$

Here $\lfloor \cdot \rfloor$ is the floor operator.

Note that this is indeed a – not necessarily disjoint – cover of $\mathbb{R}^6$ as seen in 6.1.

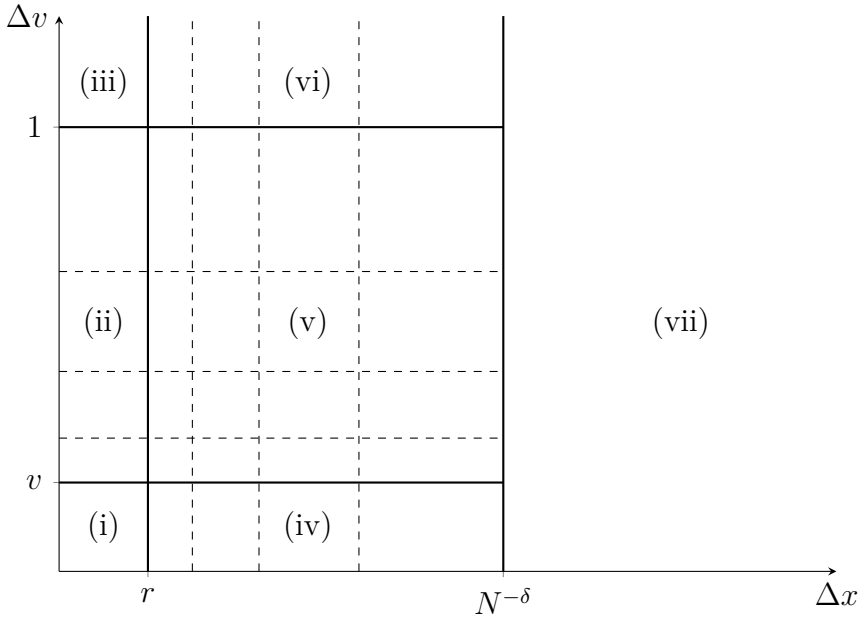


Figure 6.1: Collision classes

Here we see the seven classes depending on the relative position of a particle to $y$. The dotted lines are used to indicate the subdivision we use in (ii) and (iv)-(vi). For smaller $\varepsilon$ this subdivision becomes finer and can be considered as a discrete density.

The number of pieces used in the cover grows as $r_0, v_0$ and $\varepsilon$ get smaller. In our applications $r_0$ and $v_0$ will depend on $N$. Note that this is okay as long as they are bounded from below by some power of $N$, i.e. $r_0, v_0 \ge N^{-\alpha}$ with $\alpha > 0$. One easily checks that, as long as $\varepsilon$ does not depend on $N$, the number of $k$ and $l$ only depends on the $\alpha$ in the bound.

As mentioned, we want to estimate how many "bad" interactions and "bad" particles exist. The first step to this goal is the following lemma:

**Lemma 6.12.** *There exists a constant $C > 0$ such that for all $\Delta q, \Delta p > 0$, $N \in \mathbb{N}$, $y \in \mathbb{R}^6$ and $[t_0, t_1] \subseteq [0, T]$ it holds that*

*i)* $\mathbb{P}\left(x \in \mathbb{R}^6 : x \in (G^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y))^C\right) \leq C(\Delta q^2 \Delta p^4 \Delta t + \Delta q^3 \max(\Delta q, \Delta p)^3)$

*ii)* $\mathbb{P}\left(x \in \mathbb{R}^6 : x \in M^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y)\right) \leq C(\Delta q^2 \Delta p^4 \Delta t + \Delta q^3 \max(\Delta q, \Delta p)^3)$

*iii)* $\mathbb{P}\left(x \in \mathbb{R}^6 : \min_{0\leq s\leq T} |{}^1\widetilde{x}(s) - {}^1\widetilde{y}(s)| \leq \Delta q\right) \leq C\Delta q^2$.

*Proof.* (*ii*) follows directly from (*i*), as $M^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y) \subseteq (G^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y))^C$. For the proof of (*i*) and (*iii*) see Graß's paper [9, Lemma 1]. □

**Remark 6.13.** We will often apply the following combination of results. We can estimate

$$\begin{aligned}\mathbb{P}\left(x \in \mathbb{R}^6 : X \in M^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y)\right) &\leq C(\Delta q^2 \Delta p^4 \Delta t + \Delta q^3 \max(\Delta q, \Delta p)^3) \\ &\leq C(T)\Delta q^2 \max(\Delta q, \Delta p)^4.\end{aligned}$$

At the same time, if $x \in M^N_{\Delta q,\Delta p}(y)$, it implies that

$$\min_{0\leq s\leq T} |{}^1\widetilde{x}(s) - {}^1\widetilde{y}(s)| \leq \Delta q.$$

Hence, we can also estimate it using (iii) which yields

$$\mathbb{P}\left(x \in \mathbb{R}^6 : x \in M^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y)\right) \leq C\Delta q^2.$$

Combining these estimates, we get

$$\mathbb{P}\left(x \in \mathbb{R}^6 : x \in M^{N,(t_0,t_1)}_{\Delta q,\Delta p}(y)\right) \leq C(T)\Delta q^2 \min\left(\max(\Delta q, \Delta p), 1\right)^4.$$

While these estimates are helpful and give a first intuition how many bad particles we might expect, they only answer the question how probable it is that a particle $x$ is bad with respect to a given particle $y$. The question that we are more interested in is how many of our $x_i$ are typically bad. This may seem like an easy application of the law of large numbers at first. We know that for each $x_i$ the probability to be bad is approximately $N^{-\frac{1}{3}}$ by 6.12. Hence, the expected value of bad particles should be about $N^{\frac{2}{3}}$. However, one has to be careful as the question whether particles are bad is clearly correlated, i.e. if $x_0$ is bad it has to have a bad partner, so the probability for all other particles to be bad increases. This correlation, however, is very weak, and hence with some tricks we are still able to prove the following lemma:

**Lemma 6.14.** *Let $y \in \mathbb{R}^6, 0 \leq r < R, 0 \leq v < V$ and $a_N$ be any sequence s.t. $a_N \to \infty$. Then for any $\gamma > 0$ there exists a $C_\gamma$ (independent of $r, R, v, V, a_n, N$ and $y$) such that*

$$\mathbb{P}\left(|\mathcal{M}^N_b(X)| > a_N N^{\frac{2}{3}}\right) \leq C_\gamma N^{-\gamma} \qquad \text{and}$$

$$\mathbb{P}\left(\sum_{x_j \in \mathcal{M}^N_b(X)} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) > a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V, R), 1)^4 \rceil\right) \leq C_\gamma N^{-\gamma}.$$

The first part states that $|\mathcal{M}_b^N(X)| \leq a_N N^{\frac{2}{3}}$ holds almost surely. The second part gives an estimate how many bad particles come how close to a given particle $y$. Note that here the left side depends on $r, v$ while the "bound" does not. The result is at its strongest for $r = v = 0$, but we will often use it with $r, v \neq 0$ and hence stated it in this generality.

*Proof.* We do both estimates at the same time. For the first estimate we set $A = a_N N^{\frac{2}{3}}$ and for the second one we set $A = a_N \lceil N^{\frac{2}{3}} R^2 \min(\max(V, R), 1)^4 \rceil$. Also, let $b_N$ be a sequence with $b_N \to 0$ and $a_N \cdot b_N \to \infty$ (e.g. $\frac{1}{\sqrt{a_N}}$).

Before we can estimate how probable it is that $\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) \geq \lceil A \rceil$, we need to prove the following claim.

**Claim:** If $\sum\limits_{x_j \in \mathcal{M}_b} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) \geq \lceil A \rceil$ then one of the following three statements has to be true:

1.
$$\sum_{k=1}^{N} \mathbf{1}_{(G^N(y))^C}(x_k) \geq \lceil b_N \cdot a_N \rceil. \tag{1}$$

2. There is a $j \in \{1, \ldots, N\}$ such that
$$\sum_{k=1}^{N} \mathbf{1}_{(G^N(x_j))^C}(x_k) \geq \lceil b_N \cdot a_N \rceil. \tag{2}$$

3. There exists a subset $\mathcal{S} \subseteq \{1, \ldots, N\}^2 \setminus \bigcup\limits_{n=1}^{N} (n, n)$ satisfying the following three conditions:
$$\begin{aligned} &a) \quad |\mathcal{S}| = \left\lfloor \frac{A}{2 a_N b_N} \right\rfloor \\ &b) \quad \forall (j,k) \in \mathcal{S} : x_j \in (G^N(x_k))^C \cap M^N_{R,V}(y) \\ &c) \quad (j_1,k_1), (j_2,k_2) \in \mathcal{S} \implies \{j_1,k_1\} \cap \{j_2,k_2\} = \emptyset. \end{aligned} \tag{3}$$

The first inequality states that there are many particles making $y$ bad; the second statement states that there is at least one $x_i$ that has many bad interactions. The third statement is a bit more complicated. It states that one finds many (different) pairs of particles $x_j, x_k$ such that $x_j$ and $x_k$ have a bad interaction and $x_j$ comes $R, V$ close to $y$. For the estimate of all bad particles we set $R = V = \infty$ and hence work with $M^N_{\infty,\infty}(y) = \mathbb{R}^6$.

To prove the claim, we assume
$$\sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) \geq A. \tag{A}$$

We now additionally assume that neither (1) nor (2) are true, and under these assumptions we will construct a set $\mathcal{S}$ that satisfies all conditions we stated in (3).

For convenience we will say particles $z_1$ and $z_2$ have a collision if $z_1 \in M^N_{R,V}(z_2)$ while saying that they have a bad collision if $z_1 \in \left(G^N(z_2)\right)^C$.

We start with the set $B_0$ of bad particles that have a collision with $y$. By definition and assumption (A), we have that $|B_0| \geq A$. At the same time, because we assumed that (1) is not true, there are at most $\lfloor b_N a_N \rfloor$ particles that have a bad collision with $y$. We remove all of those particles from $B_0$ to obtain $B_1 \subseteq B_0$. It is easy to see that $|B_1| \geq A - \lfloor b_N a_N \rfloor \geq 1$ where the last inequality holds for large enough $N$ by choice of $b_N$.

Next, we take one of these remaining bad particles $x_{j_1}$ from $B_1$. Since $x_{j_1}$ is bad, it has to have a bad collision with another bad particle, which by definition of $B_1$ cannot be $y$ (if $y$ is not one of our particles $x_i$, this is trivial). Take one of those bad collision partners of $x_{j_1}$ and label it $k_1$. This gives the first tuple $(j_1, k_1)$.

Now, by our assumption that (2) is wrong, $x_{j_1}$ and $x_{k_1}$ each have at most $\lfloor b_n a_n \rfloor$ other bad collisions. After removing all of those collision partners, we obtain $B_2 \subseteq B_1$. So, as long as $B_2 \neq \emptyset$ we find a particle $x_{j_2}$ that came close to $y$ and by definition had to have a bad collision. This gives us our particle $x_{k_2}$. Note that $j_2 \neq j_1$ as otherwise $x_{j_2} = x_{j_1}$ would have had a bad collision with $x_{k_1}$ and we already would have removed it. Analogously, one can prove that $x_{j_2}$ and $x_{k_2}$ have to be completely new particles.

This means that by construction the tuples $(j_i, k_i)$ created in this way satisfy $b$) as well as $c$) in (3).

It now only remains to check that we can create enough tuples this way, i.e. we have to know how long this procedure can be repeated.

We know $|B_l| \geq |B_{l-1}| - 2\lfloor b_n a_n \rfloor$. We also have $|B_1| \geq A - \lfloor b_n a_n \rfloor$. Hence, by an easy induction we obtain $|B_l| \geq A - (2l-1)\lfloor b_n a_n \rfloor$. So if we ask for which $l$ we have $|B_l| \geq 1$, we have to find an $l$ such that $A + \lfloor b_n a_n \rfloor \geq 2l \lfloor b_n a_n \rfloor + 1$. Clearly, this works as long as

$$l \leq \frac{A + \lfloor b_N a_N \rfloor - 1}{2\lfloor b_N a_N \rfloor}.$$

But

$$\frac{A + \lfloor b_N a_N \rfloor - 1}{2\lfloor b_N a_N \rfloor} \geq \frac{A}{2 b_N a_N}.$$

This means one can construct the $l^{\text{th}}$ tuple as long as $l \leq \lfloor \frac{A}{2 b_N a_N} \rfloor$. This proves our claim.

So, instead of estimating the probability for

$$\sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) \geq A, \tag{A}$$

we can instead estimate the probabilities for (1), (2) and (3). Using Lemma 6.12, we can estimate

$$\sup_{z \in \mathbb{R}^6} \mathbb{P}\left(X \in G(z)^C\right) \leq C(T)(6N^{-\frac{2}{9}-\sigma})^2 (N^{-\frac{2}{9}})^4 \leq C(T) N^{-\frac{4}{3}-2\sigma}. \tag{$\star$}$$

With this we can now estimate the probability for the first two events.

$$\begin{aligned}
&\mathbb{P}\left(\exists j : \sum_{k=1}^{N} \mathbf{1}_{G(x_j)^C}(x_k) \geq \lceil b_N a_N \rceil \vee \sum_{k=1}^{N} \mathbf{1}_{G(y)^C}(x_k) \geq \lceil b_N a_N \rceil\right) \\
&\overset{iid}{\leq} N\,\mathbb{P}\left(\sum_{k=2}^{N} \mathbf{1}_{G(x_1)^C}(x_k) \geq \lceil b_N a_N \rceil\right) + \mathbb{P}\left(\sum_{k=1}^{N} \mathbf{1}_{G(y)^C}(x_k) \geq \lceil b_N a_N \rceil\right) \\
&\leq (N+1) \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(\sum_{k=1}^{N} \mathbf{1}_{G(z)^C}(x_k) \geq \lceil b_N a_N \rceil\right) \\
&\leq (N+1)\binom{N}{\lceil b_N a_N \rceil} \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(X \in \mathbb{R}^6 : X \in G(z)^C\right)^{\lceil b_N a_N \rceil} \\
&\leq (N+1)\frac{N^{\lceil b_N a_N \rceil}}{\lceil b_N a_N \rceil!}(CN^{-\frac{4}{3}-2\sigma})^{\lceil b_N a_N \rceil} \\
&\leq (N+1)N^{-2\sigma\cdot\lceil b_N a_N \rceil}(CN^{-\frac{1}{3}})^{\lceil b_N a_N \rceil} \\
&\leq (CN)^{-\frac{1}{3}\lceil b_N a_N \rceil}.
\end{aligned}$$

Here we used $a_N b_N \to \infty$ in the last step. This also implies that this is a suitable estimate.

Next, we want to estimate how probable the existence of a set $\mathcal{S}$ satisfying the three conditions in (3) is. For this, we define $c_N = \lfloor \frac{A}{2b_N a_N} \rfloor$ and estimate

$$\begin{aligned}
&\mathbb{P}\Bigg(\exists \mathcal{S} \subset \{1,\dots,N\}^2 \setminus \bigcup_{n=1}^{N}\{(n,n)\} : |\mathcal{S}| = c_N, \\
&\quad \forall (j,k) \in \mathcal{S} : x_j \in G(x_k)^C \cap M_{R,V}(y), \\
&\quad (j_1,k_1),(j_2,k_2) \in \mathcal{S} \implies \{j_1,k_1\} \cap \{j_2,k_2\} = \emptyset\Bigg) \\
&\leq \binom{N^2}{c_N}\mathbb{P}\Bigg(\forall (j,k) \in \{(1,2),(3,4)\dots,(2c_N-1,2c_N)\} : \\
&\qquad\qquad x_j \in (G(x_k)^C \cap M_{R,V}(y))\Bigg) \\
&\leq \frac{N^{2c_N}}{c_N!}\left(\sup_{z\in\mathbb{R}^6}\mathbb{P}\left(x_0 \in G(z)^C\right)\sup_{z\in\mathbb{R}^6}\mathbb{P}\left(x_0 \in M_{R,V}(z)\right)\right)^{c_N} \\
&\overset{(\star)}{\leq} C^{c_N}\frac{N^{2c_N}}{c_N^{c_N}}\left(C(T)N^{-\frac{4}{3}-2\sigma}\sup_{z\in\mathbb{R}^6}\mathbb{P}\left(x_0 \in M_{R,V}(z)\right)\right)^{c_N}.
\end{aligned}$$

Recall that we are trying to prove two statements at the same time. Here we have to consider the two possible choices of $A$ and hence $c_n$ separately. For the estimate of the number of all bad particles we set $A = a_N N^{\frac{2}{3}}$ and $M_{R,V}(z) = M_{\infty,\infty}(z) = \mathbb{R}^6$, so we simply estimate the remaining probability by 1. In this case $c_N = \lfloor \frac{N^{\frac{2}{3}}}{2b_N} \rfloor \geq N^{\frac{2}{3}}$ for large enough $N$ and hence we get

$$\begin{aligned}
&C(T)^{c_N}\left(\frac{N^2}{c_N}\cdot N^{-\frac{4}{3}-2\sigma}\right)^{c_N} \\
\leq &C(T)^{c_N}\left(\frac{N^2 N^{-\frac{4}{3}-2\sigma}}{N^{\frac{2}{3}}}\right)^{c_N} \\
\leq &(C(T)N^{-2\sigma})^{c_N}.
\end{aligned}$$

For the other estimate we have $A = N^{\frac{2\sigma}{3}}\lceil N^{\frac{2}{3}}R^2\min(\max(R,V),1)^4\rceil$ and can apply 6.12 to estimate the probability

$$\sup_{z\in\mathbb{R}^6}\mathbb{P}\left(X\in M_{R,V}(z)\right)\leq C(T)R^2(\min(\max(R,V),1)^4.$$

With this it now remains to estimate

$$\begin{aligned}
&\leq C(T)^{c_N}\frac{N^{2c_N}}{c_N^{c_N}}\left(N^{-\frac{4}{3}-2\sigma}C(T)R^2(\min(\max(R,V),1)^4\right)^{c_N}\\
&\leq C(T)^{2c_N}\left(N^{(\frac{2}{3}-2\sigma)}\cdot\frac{R^2(\min(\max(R,V),1)^4)}{\left\lfloor\frac{\lceil N^{\frac{2}{3}}R^2\min(\max(R,V),1)^4\rceil}{2b_N}\right\rfloor}\right)^{c_N}\\
&\leq C(T)^{2c_N}\left(N^{(\frac{2}{3}-2\sigma)}\cdot(R^2(\min(\max(R,V),1)^4))\cdot\frac{4b_N}{\lceil N^{\frac{2}{3}}R^2\min(\max(R,V),1)^4\rceil}\right)^{c_N}\\
&\leq C(T)^{2c_N}\left(N^{(\frac{2}{3}-2\sigma)}\cdot(R^2(\min(\max(R,V),1)^4))\cdot\frac{4b_N}{N^{\frac{2}{3}}R^2\min(\max(R,V),1)^4}\right)^{c_N}\\
&\leq C(T)^{2c_N}\left(N^{-2\sigma}b_N\right)^{c_N}\\
&\leq (C(T)^2N^{-2\sigma})^{c_N}.
\end{aligned}$$

As $c_N\geq\frac{1}{2b_N}\to\infty$ for $N\to\infty$ and as $C(T)N^{-2\sigma}\ll 1$ for large enough $N$, this proves our claim.

To make this precise, let $\gamma>0$. Then the probability

$$\mathbb{P}\left(\sum_{j\text{ bad}}\mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j)>a_N\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil\vee|\,\mathcal{M}_b^N(X)|>a_NN^{\frac{2}{3}}\right)$$

can be estimated by the sum of the two estimations we just did, i.e. by

$$2\cdot(C(T)N^{-2\sigma})^{c_N}=(C(T)N^{-\sigma})^{c_N}\cdot N^{-\sigma c_N}.$$

As $c_N\to\infty$, we find an $N\gg 0$ such that $c_N\sigma>\gamma$ and $(C(T)N^{-\sigma})^{c_N}\leq\frac{1}{2}$. This means for large enough $N$ the probability is bounded by $N^{-\gamma}$. Now we can choose $C_\gamma$ such that the estimate holds for all $N\in\mathbb{N}$. □

Setting $a_N=\ln(N)^{\frac{1}{4}}$, this lemma gives:

**Corollary 6.15.** *Let $0<r\leq R, 0<v\leq V$ and $y\in\mathbb{R}^6$. Then it holds for every $\gamma>0$ that*

$$\begin{aligned}
\mathbb{P}\Bigg(&\sum_{j\ bad}\mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j)>\ln(N)^{\frac{1}{4}}\lceil N^{\frac{2}{3}}R^2\min(\max(V,R),1)^4\rceil\\
&\vee|\,\mathcal{M}_b^N(X)|>\ln(N)^{\frac{1}{4}}N^{\frac{2}{3}}\Bigg)\leq C_\gamma N^{-\gamma}.
\end{aligned}$$

*Proof.* Simply set $a_N=\ln(N)^{\frac{1}{4}}$ in the lemma above. □

In one specific situation we will need a slightly modified version of this lemma that takes into account not only the relative position of the bad particles to $x_i$ but also their relative position to $x_0$. The proof works very similar and it can be found at the end of the appendix, i.e. 6.15.

It is not surprising that we will have to estimate the influence of bad particles on other particles. Given that there is sufficient control on the number of bad particles we will need to estimate the interaction force and some of its derivatives coming from bad particles. This can be done with the following lemma:

**Lemma 6.16.** *Let $0 < r \leq R, 0 < v \leq V$ and $y \in \mathbb{R}^6$. Let $X \in \mathbb{R}^{6N}$ be a configuration such that*

$$\sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) \leq \ln(N)^{\frac{1}{4}} \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \wedge |\mathcal{M}^N_b(X)| \leq \ln(N)^{\frac{1}{4}} N^{\frac{2}{3}}.$$

*Further, let $g^N : \mathbb{R}^3 \to \mathbb{R}^n$ be a family of functions such that*

$$|g^N(q)| \leq \begin{cases} Cc(N)^{-\alpha} & \text{if} \quad |q| \leq c(N) \\ \frac{C}{|q|^\alpha} & \text{if} \quad |q| > c(N) \end{cases}.$$

*Then*

$$\left| \frac{1}{N} \sum_{j \text{ bad}} \int_{t_0}^{t_1} g^N(\tilde{q}_i - \tilde{q}_j) \ \mathbf{1}_{M_{(6N^{-\frac{2}{9}-\sigma},\infty)(0,\infty)}(x_i)}(x_j) \, \mathrm{d}t \right| \leq \begin{cases} CN^{-\frac{1}{3}+3\sigma} & \text{if } \alpha = 2 \\ CN^{-\frac{1}{3}+4\sigma} & \text{if } \alpha = 3 \\ CN^{-\frac{1}{9}+5\sigma} & \text{if } \alpha = 4 \end{cases}$$

*and*

$$\left| \frac{1}{N} \sum_{j \text{ bad}} \int_{t_0}^{t_1} g^N(\tilde{q}_i - \tilde{q}_j) \mathbf{1}_{G^N(x_i)}(x_j) \, \mathrm{d}t \right| \leq \begin{cases} CN^{-\frac{1}{3}+4\sigma} & \text{if } \alpha = 2 \\ CN^{-\frac{1}{9}+5\sigma} & \text{if } \alpha = 3 \\ CN^{\frac{2}{9}+6\sigma} & \text{if } \alpha = 4 \end{cases}.$$

*Proof.* We use our partition 2.10 with $r_0 = 6N^{-\frac{2}{9}-\sigma}$, $v_0 = N^{-\frac{2}{9}}$, $\epsilon = \sigma$ and $y = x_i$. For the first estimate we only have to consider the classes with $r \geq 6N^{-\frac{2}{9}-\sigma}$.

We start with the set $M_{(N^{-\sigma},\infty),(0,\infty)}(x_i)$. In this case we know that the minimal distance between $\tilde{q}_i$ and $\tilde{q}_j$ is larger than $N^{-\sigma}$ and estimate using the bound on $|g^N|$

$$\begin{aligned} &\left| \int_{t_0}^{t_1} \frac{1}{N} \sum_{j \neq i \text{ bad}} g^N(\tilde{q}_i - \tilde{q}_j) \mathbf{1}_{M_{(N^{-\sigma},\infty),(0,\infty)}(x_i)}(x_j) \, \mathrm{d}t \right| \\ \leq &\int_{t_0}^{t_1} CN^{\alpha\sigma} \frac{1}{N} \sum_{j \neq i \text{ bad}} \mathbf{1}_{M_{(N^{-\sigma},\infty),(0,\infty)}(x_i)}(x_j) \, \mathrm{d}t \\ \leq &\frac{C(T)}{N} N^{\alpha\sigma} \ln(N)^{\frac{1}{4}} N^{\frac{2}{3}} \leq CN^{-\frac{1}{3}+(\alpha+1)\sigma}, \end{aligned}$$

where we used our assumption on the number of bad particles in the last line.

For the other classes with $r \geq 6N^{-\frac{2}{9}-\sigma}$ we know $R = N^\sigma r$ and can apply 6.8 to estimate $g$. Then we can use our assumption on the number of bad particles to estimate the sum.

This yields

$$\begin{aligned}
&\left|\int_{t_0}^{t_1} \frac{1}{N} \sum_{j\neq i \text{ bad}} g^N(\tilde{q}_i - \tilde{q}_j)\mathbf{1}_{M_{(r,R),(v,V)}(x_i)}(x_j)\,\mathrm{d}t\right| \\
&\leq \int_{t_0}^{t_1} \frac{C}{r^{\alpha-1}\max(r,v)} \frac{1}{N} \sum_{j\neq i \text{ bad}} \mathbf{1}_{M_{(r,R),(v,V)}(x_i)}(x_j)\,\mathrm{d}t \\
&\leq \frac{C(T)}{r^{\alpha-1}\max(r,v)} \frac{1}{N} \ln(N)^{\frac{1}{4}} \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\
&\leq C(T)\frac{1}{N}\frac{1}{r^{\alpha-1}\max(r,v)} \ln(N)^{\frac{1}{4}} (N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 + 1) \\
&\leq C(T)N^{(\alpha-3)(\frac{2}{9}+\sigma)-\frac{1}{3}+3\sigma} \frac{\min(\max(R,V),1)}{\max(r,v)} + C(T)N^{\alpha(\frac{2}{9}+\sigma)-1+\sigma} \\
&\leq C(T)N^{\alpha(\frac{2}{9}+\sigma)-1+\sigma},
\end{aligned}$$

where in the last step we used that we can either estimate $v \geq 1$ or know that $\max(R,V) = N^\sigma \max(r,v)$.

For the second group of estimates we also need to cover the cases where $r < 6N^{-\frac{2}{9}-\sigma}$. However, because we only need to consider particles that are good with respect to $x_i$, we know that $v \geq N^{-\frac{2}{9}}$ and can estimate

$$\begin{aligned}
&\left|\frac{1}{N} \sum_{j \text{ bad}} \int_{t_0}^{t_1} g(\tilde{q}_i - \tilde{q}_j)\mathbf{1}_{G^N(x_i)}(x_j)\mathbf{1}_{M_{(r,R),(v,V)}(x_i)}(x_j)\right| \\
&\leq C\ \frac{1}{N}\frac{1}{c(N)^{\alpha-1}v} \ln(N)^{\frac{1}{4}} \lceil N^{\frac{2}{3}} R^2 \min(\max(V,R),1)^4 \rceil \\
&\leq C\frac{1}{N}\frac{1}{c(N)^{\alpha-1}v} N^\sigma (N^{\frac{2}{3}} R^2 \min(V,1)^4 + 1) \\
&\leq CN^{(\alpha-1)(\frac{1}{3}+\sigma)+\frac{2}{9}+\sigma-1} \frac{\min(V,1)}{v} + CN^{(\alpha-1)(\frac{1}{3}+\sigma)+\frac{2}{9}+\sigma-1} \\
&\leq C(T)N^{\alpha(\frac{1}{3}+\sigma)-\frac{10}{9}+2\sigma},
\end{aligned}$$

where we again used that we are in a collision class where $v \geq 1$ or $V = N^\sigma v$. Combining these estimates, we have proven that

$$\left|\frac{1}{N} \sum_{j \text{ bad}} \int_{t_0}^{t_1} g(\tilde{q}_i - \tilde{q}_j)\ \mathbf{1}_{M_{(6N^{-\frac{2}{9}-\sigma},\infty)(0,\infty)}(x_i)}(x_j)\right| \leq C(T)N^{-\frac{1}{3}+(\alpha+1)\sigma} + CN^{\alpha(\frac{2}{9}+\sigma)-1+\sigma},$$

as well as

$$\begin{aligned}
&\left|\frac{1}{N} \sum_{j \text{ bad}} \int_{t_0}^{t_1} g(\tilde{q}_i - \tilde{q}_j)\mathbf{1}_{G^N(x_i)}(x_j)\right| \\
\leq& C(T)N^{-\frac{1}{3}+(\alpha+1)\sigma} + C(T)N^{\alpha(\frac{2}{9}+\sigma)-1+\sigma} + C(T)N^{\alpha(\frac{1}{3}+\sigma)-\frac{10}{9}+2\sigma}.
\end{aligned}$$

□

Lastly in this chapter, we will introduce a helpful version of the law of large numbers and than prove some related results in our setting.

We start with the following version of the law of large numbers that gives strong estimates for random variables that (can) depend on $N$.

**Lemma 6.17.** *Let $z_k$ be a sequence of i.i.d. random variables with $\mathbb{E}[z_i] = 0$, $\mathbb{E}[z_i^2] \leq b(N)$ and $|z_i| \leq C\sqrt{Nb(N)}$, where $b(N) : \mathbb{R} \to \mathbb{R}$ is our bounding function. Then for any $\gamma > 0$ the sample mean $\bar{z} = \frac{1}{N}\sum_{i=1}^{N} z_i$ satisfies*

$$\mathbb{P}\left(|\bar{z}| \geq \frac{C_\gamma \sqrt{b(N)}\ln(N)}{\sqrt{N}}\right) \leq N^{-\gamma},$$

*where $C_\gamma$ only depends on $C$ and $\gamma$.*

*Proof.* Can be found in [8, Lemma 1]. □

Next we will give a slightly weaker, more complicated looking version of the law of large numbers that is better fitted to our needs.

**Corollary 6.18.** *Let $\delta, C_0 > 0, N \in \mathbb{N}$ and let $(x_k)_k \in \mathbb{N}$ be a sequence of i.i.d. random variables $x_k : \Omega \to \mathbb{R}^6$ distributed with respect to a probability density $k \in \mathfrak{L}^1(\mathbb{R}^6)$. Moreover, let $(M_i^N)_{i\in I}$ be a family of (possibly $N$-dependent) sets in $\mathbb{R}^6$ fulfilling $\bigcup_{i\in I} M_i^N = \mathbb{R}^6$ where $|I| < C_0$ and $g_N : \mathbb{R}^6 \to \mathbb{R}$ be a family of measurable functions which fulfill $|g_N|_\infty \leq C_0 N^{1-\delta}$ as well as*

$$\max_{i\in I} \int_{M_i^N} g_N(y)^2 k(y)\,\mathrm{d}y \leq C_0 N^{1-\delta}.$$

*Then for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ such that for all $N \in \mathbb{N}$ it holds that*

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{k=1}^{N} g_N(x_k) - \int_{\mathbb{R}^6} g_N(y)k(y)\,\mathrm{d}y\right| \geq C_\gamma N^{-\frac{\delta}{3}}\right) \leq N^{-\gamma}.$$

*Proof.* This is a result of Lemma 6.17 above, with $z_i = g(x_i) - \mathbb{E}(g(x_i))$ and $b(N) = C_0^2 N^{1-\frac{\delta}{2}}$.
By assumption we have

$$|z_i| \leq 2|g_N|_\infty \leq 2C_0 N^{1-\delta} = 2C_0 b(N) \leq \sqrt{Nb(N)}.$$

It now remains to estimate

$$\mathbb{E}[z_i^2] \leq \mathbb{E}[g_N(x_i)^2] \leq C_0^2 N^{1-\delta} \leq C_0^2 N^{1-\frac{\delta}{2}} = b(N).$$

As $C_\gamma N^{-\frac{1}{2}}\sqrt{b(N)}\ln(N) = C_\gamma N^{-\frac{\delta}{2}}\ln(N) \leq C_\gamma N^{-\frac{\delta}{3}}$, the result follows. □

We will mainly need the following application:

**Lemma 6.19.** *Let $y \in \mathbb{R}^6$, $t \in [0,T], m \in \mathbb{N}$ and $j^N : \mathbb{R}^3 \to \mathbb{R}^m$ be given. If $|j^N| \leq |N^{\frac{4}{9}-\sigma} f^N|$, then*

$$g^N(z) = \int_0^t j^N({}^1\tilde{y} - {}^1\tilde{z})\,\mathrm{d}s \mathbf{1}_{G^N(y)}(z)$$

*satisfies the conditions of Corollary 6.18 and hence for any $\gamma > 0$ we find $C_\gamma > 0$ such that*

$$\mathbb{P}\left(\left|\frac{1}{N}\int_0^t \sum_{j=1}^{N} j^N({}^1\tilde{y} - {}^1\tilde{x}_j)\,\mathrm{d}s \mathbf{1}_{G^N(x_i)}(x_j) - \int_{\mathbb{R}^6} g^N(z)k(z)\,\mathrm{d}z\right| \geq N^{-\frac{\sigma}{6}}\right) \leq C_\gamma N^{-\gamma}.$$

*Proof.* Note that our law of large numbers required the function to map to $\mathbb{R}$. This can easily be resolved, as we can simply consider the $m$ coordinate functions $g_1^N, \dots, g_m^N$. Since all of these are bounded by $|g^N|$, we can estimate the function $|g^N|$ and thereby prove that all the coordinate functions satisfy the conditions for our law of large numbers 6.18. To estimate $|g^N|$, we use the partition defined in 6.11 with $r_0 = v_0 = c(N)$ and $\varepsilon = \sigma$. By abuse of notation we also write $r, v, R, V$ for the $r_i, v_i, R_i, V_i$ in the collision classes $M_i := M_{(r_i,R_i),(v_i,V_i)}$. With this notation and setting $\tilde{r} = \max(r, c(N))$ we can estimate:

$$
\begin{aligned}
&\int_{M_i^N(y)} g^N(z)^2 k_0(z)\,\mathrm{d}^6 z \\
\leq &\int_{M_i^N(y)} \left(\int_0^t |j^N({}^1\widetilde{y} - {}^1\widetilde{z})|\,\mathrm{d}s\right)^2 k_0(z)\,\mathrm{d}^6 z \\
\leq &N^{\frac{8}{9}-2\sigma} \int_{M_i^N(y)} \left(\int_0^t |f({}^1\widetilde{y} - {}^1\widetilde{z})|\,\mathrm{d}s\right)^2 k_0(z)\,\mathrm{d}^6 z \\
\leq &CN^{\frac{8}{9}-2\sigma} \left(\min\left(\frac{1}{\tilde{r}^2}, \frac{1}{\tilde{r}v}\right)\right)^2 \int_{M_i^N(y)} k_0(z)\,\mathrm{d}^6 z \\
\leq &CN^{\frac{8}{9}-2\sigma} \min\left(\frac{1}{\tilde{r}^4}, \frac{1}{\tilde{r}^2 v^2}\right) \min(1, R^2, R^2V^4 + R^3\max(V^3, R^3)) \\
\leq &CN^{\frac{8}{9}-2\sigma} \min\left(\frac{1}{\tilde{r}^2 v^2}, \frac{R^2}{\tilde{r}^2 v^2}, \frac{R^2V^4}{\tilde{r}^2\max(\tilde{r}, v)^2} + \frac{R^6}{\tilde{r}^4}\right),
\end{aligned}
$$

where we first used 6.8 followed by 6.12. Using our partition, the minimum can be estimated as follows. The enumeration is the same as in 6.11.

(i) $\leq \frac{c(N)^6}{c(N)^4} \leq c(N)^2$

(ii) $\leq \frac{c(N)^2(N^\sigma v)^4}{c(N)^2 v^2} \leq N^{2\sigma}(N^\sigma v)^2 \leq N^{4\sigma}$

(iii) $\leq \frac{c(N)^2}{c(N)^2 1^2} = 1$

(iv) $\leq \frac{(N^\sigma r)^6}{r^4} \leq N^{4\sigma}(N^\sigma r)^2 \leq N^{6\sigma}$

(v) $\leq \frac{(N^\sigma r)^2(N^\sigma v)^4}{r^2 v^2} + \frac{(N^\sigma r)^6}{r^4} \leq N^{4\sigma}\left((N^\sigma v)^2 + (N^\sigma r)^2\right) \leq 2N^{6\sigma}$

(vi) $\leq \frac{(N^\sigma r)^2}{r^2 1^2} = N^{2\sigma}$

(vii) $\leq \frac{1}{(N^{-\sigma})^4} = N^{4\sigma}$

Hence, we find a constant $C_0$ such that

$$\max_{i\in I} \int_{M_i^N} g_N(z)^2 k_0(z)\,\mathrm{d}^6 z \leq C_0 N^{1-\sigma}.$$

Next, we estimate $|g^N|_\infty$ using 6.8.

$$\begin{aligned}
&\int_0^t |j^N(\widetilde{y}-\widetilde{z})|\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}s\\
&\leq N^{\frac{4}{9}-\sigma}\int_0^t |f^N(\widetilde{y}-\widetilde{z})|\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}s\\
&\leq C(T)N^{\frac{4}{9}-\sigma}\min\left(\frac{1}{|{}^1\widetilde{z}(t_{min})-{}^1\widetilde{y}(t_{min})|^2},\frac{1}{c(N)|{}^2\widetilde{z}(t_{min})-{}^2\widetilde{y}(t_{min})|}\right)\mathbf{1}_{G^N(y)}(z)\\
&\leq C(T)N^{\frac{4}{9}-\sigma}\max((N^{\frac{2}{9}+\sigma})^2,c(N)^{-1}N^{\frac{2}{9}})\leq C(T)N^{\frac{4}{9}-\sigma}N^{\frac{5}{9}+\frac{\sigma}{16}}\leq C(T)N^{1-\frac{15}{16}\sigma}.
\end{aligned}$$

Hence, $|g|_\infty \leq CN^{1-\frac{15}{16}\sigma}$. So we can apply 6.18 to each $g_1^N,\dots,g_m^N$ and get a constant $\hat{C}_\gamma$ such that for all $l=1,\dots,m$

$$\mathbb{P}\left(\left|\int_0^t \frac{1}{N}\sum_{j=1}^N j_l^N({}^1\tilde{y}-{}^1\tilde{x}_j)\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}s-\int_{\mathbb{R}^6} g_l^N(z)k(z)\,\mathrm{d}^6z\right|\geq \hat{C}_\gamma N^{-\frac{3}{16}\sigma}\right)\leq N^{-\gamma}.$$

So by triangle inequality and as $N^{-\frac{3}{16}\sigma}\leq N^{-\frac{1}{6}\sigma}$, we obtain that for large enough $N$

$$\mathbb{P}\left(\left|\int_0^t \frac{1}{N}\sum_{j=1}^N j^N({}^1\tilde{y}-{}^1\tilde{x}_j)\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}s-\int_{\mathbb{R}^6} g^N(z)k(z)\,\mathrm{d}^6z\right|\geq N^{-\frac{1}{6}\sigma}\right)\leq N^{-\gamma}.$$

As this result holds only for large enough $N$, we have to choose a constant $C_\gamma$ such that

$$\mathbb{P}\left(\left|\int_0^t \frac{1}{N}\sum_{j=1}^N j^N({}^1\tilde{y}-{}^1\tilde{x}_j)\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}s-\int_{\mathbb{R}^6} g^N(z)k(z)\,\mathrm{d}^6z\right|\geq N^{-\frac{1}{6}\sigma}\right)\leq C_\gamma N^{-\gamma}$$

holds for all $N\in\mathbb{N}$. □

We also need this slightly modified version. Here the particles need to be better behaved but we get a better estimate in compensation, as one can see in the initial condition on $|j^N|$:

**Lemma 6.20.** *Let $y\in\mathbb{R}^6$, $t\in[0,T], m\in\mathbb{N}$ and $j^N:\mathbb{R}^3\to\mathbb{R}^m$ be given. If $|j^N|\leq|N^{\frac{1}{2}-\sigma}f^N|$, then*

$$g^N(z)=\int_0^t j^N({}^1\widetilde{y}-{}^1\widetilde{z})\,\mathrm{d}s\mathbf{1}_{M^N_{N^{-\frac{1}{6}+\sigma},N^{-\frac{1}{6}+\sigma}}(y)^C}(z)$$

*satisfies the conditions of Lemma 6.18 and hence for any $\gamma>0$ we find $C_\gamma>0$ such that*

$$\mathbb{P}\left(\left|\int_0^t \frac{1}{N}\sum_{j=1}^N j^N({}^1\tilde{y}-{}^1\tilde{x}_j)\,\mathrm{d}s\mathbf{1}_{G^N(x_i)}(x_j)-\int g^N(z)k(z)\,\mathrm{d}z\right|\geq N^{-\frac{\sigma}{6}}\right)\leq C_\gamma N^{-\gamma}.$$

*Proof.* Can be found at the very end of the appendix. □

There is one small problem with the result we have just proven. So far, we have only shown that at any time the probability can be controlled. However, we want to control the error uniformly in time, which still requires a little bit of extra work in form of the following corollary:

**Corollary 6.21.** *Let $j^N : \mathbb{R}^3 \to \mathbb{R}^m$ be given such that $|j^N| \leq |N^{\frac{4}{9}-\sigma} f^N|$. Then for any $y \in \mathbb{R}^6$ and $\gamma > 0$ there exists a $C_\gamma > 0$ such that*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N} \int_{t_1}^{t_2} j^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} j^N({}^1\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\; k(z)\,\mathrm{d}z \right| > 4N^{-\frac{\sigma}{6}} \right) \leq C_\gamma N^{-\gamma}.$$

*Proof.* We note that for any integrable function $a : \mathbb{R} \to \mathbb{R}^m$ and any $\delta_N$ we have

$$\begin{aligned}
&\left| \int_{t_1}^{t_2} a(s)\,\mathrm{d}t \right| \\
=&\left| \int_0^{t_2} a(s)\,\mathrm{d}t - \int_0^{t_1} a(s)\,\mathrm{d}t \right| \\
\leq&\left| \int_0^{\lfloor \frac{t_2}{\delta_N} \rfloor \delta_N} a(s)\,\mathrm{d}t \right| + \int_{\lfloor \frac{t_2}{\delta_N} \rfloor \delta_N}^{t_2} |a(s)|\,\mathrm{d}t + \left| \int_0^{\lfloor \frac{t_1}{\delta_N} \rfloor \delta_N} a(s)\,\mathrm{d}t \right| + \int_{\lfloor \frac{t_1}{\delta_N} \rfloor \delta_N}^{t_1} |a(s)|\,\mathrm{d}t \\
\leq&2 \max_{k \in [0, \lceil N^2 \max(t_1,t_2) \rceil]} \left( \left| \int_0^{k\delta_N} a(s)\,\mathrm{d}t \right| + \int_{k\delta_N}^{(k+1)\delta_N} |a(s)|\,\mathrm{d}t \right).
\end{aligned}$$

If we apply this to the function

$$\frac{1}{N} \sum_{j=1}^{N} j^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j) - \int_{\mathbb{R}^6} j^N({}^1\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z) k(z)\,\mathrm{d}z,$$

we see that for our desired estimate to fail there needs to be a $k$ such that

$$\left| \int_0^{k\delta_N} \left( \frac{1}{N} \sum_{j=1}^{N} j^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j) - \int_{\mathbb{R}^6} j^N({}^1\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z) k(z)\,\mathrm{d}z \right) \mathrm{d}t \right| > N^{-\frac{\sigma}{6}}$$

or

$$\int_{k\delta_N}^{(k+1)\delta_N} \left( \left| \frac{1}{N} \sum_{j=1}^{N} j^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j) + \int_{\mathbb{R}^6} j^N({}^1\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z) k(z)\,\mathrm{d}z \right| \mathrm{d}t \right) > N^{-\frac{\sigma}{6}}.$$

For the second case we can use triangle equality and the fact that $|j^N|_\infty \leq N^{\frac{4}{9}-\sigma} |f^N|_\infty \leq N$ for large enough $N$. Hence, we can choose $\delta_N$ as $N^{-2}$ and the second condition cannot be true. This means we have to be sure that for $k \in [0, \Delta t N^2]$ the first case cannot happen. But for this we simply apply 6.19 for $t_k = k\delta_N$ and all our $k$, with $\hat{\gamma} = \gamma + 2$.

We then have

$$\begin{aligned}
&\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N} \int_{t_1}^{t_2} j^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right. \\
&\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} j^N({}^1\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\; k(z)\,\mathrm{d}z \right| > 4N^{-\frac{\sigma}{6}} \right) \\
\leq&\mathbb{P}\left(\exists k \in [0, \lceil N^2 \max(t_1,t_2) \rceil] : \left| \frac{1}{N} \sum_{j=1}^{N} \int_0^{t_k} j^N({}^1\tilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right. \\
&\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} j^N({}^1\tilde{y} - {}^1\tilde{z}) \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\; k(z)\,\mathrm{d}z \right| > N^{-\frac{\sigma}{6}} \right) \\
\leq&\sum_{k=0}^{\lceil N^2 \max(t_1,t_2) \rceil} C_k N^{-\gamma-2} = C_\gamma N^{-\gamma}.
\end{aligned}$$

□

We stated this result with a general condition on $j^N$. We will only need this result to estimate four functions, and for better readability we restate the result in these cases.

**Corollary 6.22.** *Let $\gamma > 0$ and $y \in \mathbb{R}^6$. There exists a $C_\gamma > 0$ (not depending on $y$) such that for large enough $N$ the following estimates hold*

*a)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N}\sum_{j=1}^{N} \int_{t_1}^{t_2} f^N({}^1\widetilde{y} - \tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6}\int_{t_1}^{t_2} f^N({}^1\widetilde{y} - {}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > N^{-\frac{4}{9}+\sigma}\right) \leq C_\gamma N^{-\gamma}$$

*b)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N}\sum_{j=1}^{N} \int_{t_1}^{t_2} Df^N({}^1\widetilde{y} - \tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6}\int_{t_1}^{t_2} Df^N({}^1\widetilde{y} - {}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > N^{-\frac{1}{9}+\sigma}\right) \leq C_\gamma N^{-\gamma}$$

*c)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N}\sum_{j=1}^{N} \int_{t_1}^{t_2} g^N({}^1\widetilde{y} - \tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6}\int_{t_1}^{t_2} g^N({}^1\widetilde{y} - {}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > N^{-\frac{1}{9}+\sigma}\right) \leq C_\gamma N^{-\gamma}$$

*d)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N}\sum_{j=1}^{N} \int_{t_1}^{t_2} h^N_{\mathbf{c}}({}^1\widetilde{y} - \tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6}\int_{t_1}^{t_2} h^N_{\mathbf{c}}({}^1\widetilde{y} - {}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > N^{-\frac{1}{9}+\sigma} c(N)^{-1}\right) \leq C_\gamma N^{-\gamma}$$

*Proof.* One simply has to check that $4N^{\frac{4}{9}-\frac{7\sigma}{6}} f$, $4N^{\frac{1}{9}-\frac{7\sigma}{6}} Df$, $4N^{\frac{1}{9}-\frac{7\sigma}{6}} g$ and $4N^{\frac{1}{9}-\frac{7\sigma}{6}} c(N)h$ are all bounded by $4N^{\frac{4}{9}-\sigma} f^N$ for large enough $N$.

For example,

$$4N^{\frac{1}{9}-\frac{7\sigma}{6}} Df \leq N^{\frac{1}{9}-\frac{7\sigma}{6}} N^{\frac{1}{3}+\frac{\sigma}{6}} |f| = N^{\frac{4}{9}-\sigma} |f^N|.$$

Hence, we obtain

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N}\sum_{j=1}^{N} \int_{t_1}^{t_2} 4N^{\frac{1}{9}-\frac{7\sigma}{6}} Df({}^1\widetilde{y} - \tilde{q}_j)\mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6}\int_{t_1}^{t_2} 4N^{\frac{1}{9}-\frac{7\sigma}{6}} Df({}^1\widetilde{y} - {}^1\widetilde{z})\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > 4N^{-\frac{\sigma}{6}}\right) \leq C_\gamma N^{-\gamma}.$$

After multiplying both sides of the inequality by $\frac{1}{4}N^{-\frac{1}{9}+\frac{7\sigma}{6}}$, we obtain the desired result. □

Due to the special treatment the moved particle $x_0$ needs, it is helpful to state this slightly altered version of the result, where only $N-1$ particles are considered. We will need this version whenever we split off the effect of $x_0$ as then only $N-1$ i.i.d. particles remain.

**Corollary 6.23.** *Let $\gamma > 0$ and $y \in \mathbb{R}^6$. There exists a $C_\gamma > 0$ (not depending on $y$) such that for large enough $N$ the following estimates hold:*

*a)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N-1} \int_{t_1}^{t_2} f^N({}^1\widetilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} f^N({}^1\widetilde{y} - {}^1\widetilde{z}) \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > CN^{-\frac{4}{9}+\sigma} \right) \leq C_\gamma N^{-\gamma}$$

*b)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N-1} \int_{t_1}^{t_2} Df^N({}^1\widetilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} Df^N({}^1\widetilde{y} - {}^1\widetilde{z}) \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > CN^{-\frac{1}{9}+\sigma} \right) \leq C_\gamma N^{-\gamma}$$

*c)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N-1} \int_{t_1}^{t_2} g^N({}^1\widetilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} g^N({}^1\widetilde{y} - {}^1\widetilde{z}) \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > CN^{-\frac{1}{9}+\sigma} \right) \leq C_\gamma N^{-\gamma}$$

*d)*

$$\mathbb{P}\left(\exists t_1, t_2 \in [0,T] : \left| \frac{1}{N} \sum_{j=1}^{N-1} \int_{t_1}^{t_2} h_{\mathbf{c}}^N({}^1\widetilde{y} - \tilde{q}_j) \mathbf{1}_{G^N(y)}(x_j)\,\mathrm{d}t \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}^6} \int_{t_1}^{t_2} h_{\mathbf{c}}^N({}^1\widetilde{y} - {}^1\widetilde{z}) \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t \cdot k(z)\,\mathrm{d}z \right| > CN^{-\frac{1}{9}+\sigma} c(N)^{-1} \right) \leq C_\gamma N^{-\gamma}.$$

*Proof.* This follows from 6.22 and the fact that $|f^N| \leq C|\frac{N}{N-1} f^{N-1}|$, $|Df^N| \leq C|\frac{N}{N-1} Df^{N-1}|$, $|g^N \leq C\frac{N}{N-1} g^{N-1}|$ and $|h^N| \leq C|\frac{N}{N-1} h^{N-1}|$. □

Before we finally can start with the main proof, we need one last lemma, that gives interesting estimates as long as the law of large numbers holds.

**Lemma 6.24.** *Let $M \in \{N, N-1\}$, $y \in \mathbb{R}^6$ and $\beta_N : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a function that is bounded independently of $N$. Also assume that we work on a configuration $X \in \mathbb{R}^{6N}$ on which the law of large numbers holds for some $y \in \mathbb{R}^6$ and $Df^N$, $g^N$ and $h_{\mathbf{c}}^N$ in the sense of 6.22 and 6.23. In this situation the following estimates hold.*

$$\frac{1}{N} \sum_{j=1}^{M} \mathbf{1}_{G^N(y)}(x_j) \int_{t_1}^{t_2} |Df^N({}^1\widetilde{y} - \tilde{q}_j)| \beta_N(t)\,\mathrm{d}t$$
$$\leq CN^{-\frac{1}{9}+\sigma} \sup_{s \in [t_0, t_1]} \beta(s) + \int_{t_1}^{t_2} C \ln(N) \beta_N(t)\,\mathrm{d}t$$

*as well as*

$$\begin{aligned}&\frac{1}{N}\sum_{j=1}^{M}\mathbf{1}_{G^N(y)}(x_j)\int_{t_1}^{t_2}|g^N({}^1\widetilde{y}-\tilde{q}_j)|\beta_N(t)\,\mathrm{d}t\\ \leq& CN^{-\frac{1}{9}+\sigma}\sup_{s\in[t_0,t_1]}\beta_N(s)+\int_{t_1}^{t_2}C\ln(N)\beta_N(t)\,\mathrm{d}t\end{aligned}$$

*as well as*

$$\begin{aligned}&\frac{1}{N}\sum_{j=1}^{M}\mathbf{1}_{G^N(y)}(x_j)\int_{t_1}^{t_2}|h^N({}^1\widetilde{y}-\tilde{q}_j)|\beta_N(t)\,\mathrm{d}t\\ \leq& CN^{-\frac{1}{9}+\sigma}c(N)^{-1}\sup_{s\in[t_0,t_1]}\beta_N(s)+\int_{t_1}^{t_2}Cc(N)^{-1}\beta_N(t)\,\mathrm{d}t.\end{aligned}$$

*Proof.* We estimate

$$\begin{aligned}&\frac{1}{N}\sum_{j=1}^{M}\mathbf{1}_{G^N(y)}(x_j)\int_{t_1}^{t_2}|Df^N({}^1\widetilde{y}-\tilde{q}_j)|\beta_N(t)\,\mathrm{d}t\\ \leq&\Big|\frac{1}{N}\sum_{j=1}^{M}\mathbf{1}_{G^N(y)}(x_j)\int_{t_1}^{t_2}|Df^N({}^1\widetilde{y}-\tilde{q}_j)|\beta_N(t)\,\mathrm{d}t\\ &-\int_{\mathbb{R}^6}\int_{t_1}^{t_2}|Df^N({}^1\widetilde{y}-{}^1\widetilde{z})|\beta_N(t)\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\;k(z)\,\mathrm{d}z\Big|\\ &+\Big|\int_{\mathbb{R}^6}\int_{t_1}^{t_2}|Df^N({}^1\widetilde{y}-{}^1\widetilde{z})|\beta_N(t)\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\;k(z)\,\mathrm{d}z\Big|\\ \overset{6.22}{\leq}& CN^{-\frac{1}{9}+\sigma}\sup_{s\in[t_0,t_1]}\beta_N(s)+\Big|\int_{t_1}^{t_2}\beta_N(t)\int_{\mathbb{R}^6}\sup_{s\in[0,T]}|Df^N({}^1\widetilde{y}-{}^1\widetilde{z})|\mathbf{1}_{G^N(y)}(z)\,\mathrm{d}t\;k(z)\,\mathrm{d}z\Big|\\ \overset{6.6}{\leq}& CN^{-\frac{1}{9}+\sigma}\sup_{s\in[t_0,t_1]}\beta_N(s)+\int_{t_1}^{t_2}C\ln(N)\beta_N(t)\,\mathrm{d}t.\end{aligned}$$

This proves the first statement, and the other two estimate follow analogously. □

**Remark 6.25.** Corollary 6.22, 6.23 and Lemma 6.24 all are based on 6.19. If we instead base them on Lemma 6.20, we can strengthen the final estimates by $N^{-\frac{1}{18}}$ as one can see by comparing the conditions in Lemma 6.19 and 6.20. In the first one we need the function to be bounded by $N^{\frac{4}{9}+\sigma}f^N$; in the second one being bounded by $N^{\frac{1}{2}+\sigma}f^N$ suffices.

Of course, we need to replace the $G^N$ in the indicator functions by the more restrictive $\left(M^N_{N^{-\frac{1}{6}+\sigma},N^{-\frac{1}{6}+\sigma}}\right)^C$ for these results to hold, as this was exactly the leverage we used to strengthen the estimate.

We will only need this for a certain group of particles, namely particles that stay reasonably far from $x_0$. We make this precise in the next section.

## 6.2 Modified versions of previous lemmas

In this section we give two variations of previous lemmas. As we consider many different cases in the proof of our main result, it is not surprising that we also need to adapt our lemmas to the given situations. In the case of the two results in question, we only need the variation once, and the proof works very analogously to the original version. Hence, we decided to move those lemmas to the end of the appendix.

**Lemma 6.26.** *Let $x_0$ be a fixed point in $\mathbb{R}^6$. Let $0 \leq R_1, R_2, 0 \leq V_1, V_2$ and $a_N$ be any sequence s.t. $a_N \to \infty$. Then for any $\gamma > 0$ and any $x_i \in X = X^{\geq 1} \cup \{x_0\}$ there exists a $C_\gamma$ such that*

$$\mathbb{P}\Bigg( \sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j) > \\ a_N \left\lceil N^{\frac{2}{3}} R_1^2 R_2^2 \min(\max(V_1, R_1), 1)^4 \cdot \min(\max(V_2, R_2), 1)^4 \right\rceil \Bigg) \leq C_\gamma N^{-\gamma}$$

*Here the probability is taken with respect to $X^{\geq 1} \in \mathbb{R}^{6(N-1)}$, and hence $x_i$ and all the $x_j$ are random variables. As usual, $X = X^{\geq 1} \cup \{x_0\}$ and all the trajectories and definitions of good and bad interactions are with respect to $X$.*

*Proof.* We set $A = a_N \lceil N^{\frac{2}{3}} R_1^2 R_2^2 \min(\max(V_1, R_1), 1)^4 \cdot \min(\max(V_2, R_2), 1)^4 \rceil$. Also let $b_N$ be a sequence with $b_N \to 0$ and $a_N \cdot b_N \to \infty$ (e.g. $\frac{1}{\sqrt{a_N}}$).

Before we can estimate how probable it is that $\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j) \geq \lceil A \rceil$, we need to prove the following claim.

**Claim:** If $\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j) \geq \lceil A \rceil$, then one of the following two statements has to be true:

1. There is a $j \in \{0, 1, \ldots, N\}$ such that

$$\sum_{k=1}^{N} \mathbf{1}_{(G^N(x_j))^C}(x_k) \geq \lceil b_N \cdot a_N \rceil$$

2. There exists a subset $\mathcal{S} \subseteq \{1, \ldots, N\}^2 \setminus \bigcup_{n=1}^{N} (n, n)$ satisfying the following three conditions.

$$a) \quad |\mathcal{S}| = \lfloor \frac{A-1}{2 a_N b_N} \rfloor$$
$$b) \quad \forall (j,k) \in \mathcal{S} : x_j \in (G^N(x_k))^C \cap M^N_{R_1,V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)$$
$$c) \quad (j_1,k_1), (j_2,k_2) \in \mathcal{S} \implies \{j_1,k_1\} \cap \{j_2,k_2\} = \emptyset.$$

The first statement states that there is at least one $x_i$ that has many bad interactions. The second statement is slightly more complicated. It states that one finds many (different) pairs of particles $x_j, x_k$ such that $x_j$ and $x_k$ have a bad interaction and $x_j$ comes $R, V$ close to $x_0$.

To prove the claim, we assume

$$\sum_{x_j \in \mathcal{M}_b^N(X)} \mathbf{1}_{M^N_{R_1,V_1}(x_0) \cap M^N_{R_2,V_2}(x_i)}(x_j) \geq \lceil A \rceil.$$

We now additionally assume that the first condition of the claim is false, i.e. for all $j \in \{0, 1, \dots, N\}$ we have $\sum_{k=1}^N \mathbf{1}_{(G^N(x_j))^C}(x_k) < \lceil b_N \cdot a_N \rceil$. Under these assumptions we will construct a set $\mathcal{S}$ that satisfies all conditions we stated.

For convenience, we will say particles $z_1$ and $z_2$ have a collision if $z_1 \in M^N_{R,V}(z_2)$, and write that they have a bad collision if $z_1 \in G^N(z_2)^C$.

We start with the set $B_0$ of bad particles that have a collision with $x_0$ and $x_i$. By definition and assumption we have that $|B_0| \geq A$. At the same time, because of our second assumption, there are at most $\lfloor b_N a_N \rfloor$ particles that have a bad collision with $x_0$ or $x_i$. We remove all of those particles from $B_0$ to obtain $B_1 \subseteq B_0$. It is easy to see that $|B_1| \geq A - 2\lfloor b_N a_N \rfloor \geq 1$, where the last inequality holds for large enough $N$ by choice of $b_N$.

Next, we take one of these remaining bad particles $x_{j_1}$ from $B_1$. Since $x_{j_1}$ is bad, it has to have a bad collision with another bad particle, which by definition of $B_1$ cannot be $x_0$ or $x_i$. Take one of those bad collision partners of $x_{j_1}$ and label it $k_1$. This gives the first tuple $(j_1, k_1)$.

Now, by our second assumption, we know that $x_{j_1}$ and $x_{k_1}$ each have at most $\lfloor b_n a_n \rfloor$ other bad collisions. After removing all of those collision partners, we obtain $B_2 \subseteq B_1$. So, as long as $B_2 \neq \emptyset$, we find a particle $x_{j_2}$ that came close to $x_i$ and $x_0$ and by definition, had to have a bad collision. This gives us our particle $x_{k_2}$. Note that $j_2 \neq j_1$ as otherwise $x_{j_2} = x_{j_1}$ would have had a bad collision with $x_{k_1}$ and we already would have removed it. Analogously, one can prove that $x_{k_2}$ has to be a completely new particle.

This means that by construction the tuples $(j_i, k_i)$ created in this way satisfy $b$) as well as $c$) in our claim.

It now only remains to check that we can create enough tuples this way, i.e. we have to know how long this procedure can be repeated.

We know $|B_l| \geq |B_{l-1}| - 2\lfloor b_n a_n \rfloor$. We also have $|B_1| \geq A - 2\lfloor b_n a_n \rfloor$. Hence, by an easy induction we obtain $|B_l| \geq A - (2l)\lfloor b_n a_n \rfloor$. So if we ask for which $l$ we have $|B_l| \geq 1$, we have to find an $l$ such that $A \geq 2l\lfloor b_n a_n \rfloor + 1$. Clearly this works as long as

$$l \leq \frac{A-1}{2\lfloor b_N a_N \rfloor}.$$

But

$$\frac{A-1}{2\lfloor b_N a_N \rfloor} \geq \frac{A-1}{2 b_N a_N},$$

and hence we can construct the $l^{\text{th}}$ tuple as long as $l \leq \lfloor \frac{A-1}{2 b_N a_N} \rfloor$. This proves our claim.

So instead of estimating the probability for

$$\sum_{j \text{ bad}} \mathbf{1}_{M^N_{(r,R),(v,V)}(y)}(x_j) \geq A$$

we can instead estimate the probabilities for the two events in our claim. Using Lemma 2.11, we can estimate

$$\sup_{z \in \mathbb{R}^6} \mathbb{P}\left(X \in G(z)^C\right) \leq C(T)(6N^{-\frac{2}{9}-\sigma})^2 (N^{-\frac{2}{9}})^4 \leq C(T) N^{-\frac{4}{3}-2\sigma}. \qquad (\star)$$

With this we can now estimate the probability for the first event:

$$\begin{aligned}
&\mathbb{P}\left(\exists j : \sum_{k=1}^{N} \mathbf{1}_{G(x_j)^C}(x_k) \geq \lceil b_N a_N \rceil\right) \\
&\overset{iid}{\leq} N\,\mathbb{P}\left(\sum_{k=2}^{N} \mathbf{1}_{G(x_0)^C}(x_k) \geq \lceil b_N a_N \rceil\right) \\
&\leq N \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(\sum_{k=1}^{N} \mathbf{1}_{G(z)^C}(x_k) \geq \lceil b_N a_N \rceil\right) \\
&\leq N \binom{N}{\lceil b_N a_N \rceil} \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(X \in \mathbb{R}^6 : X \in G(z)^C\right)^{\lceil b_N a_N \rceil} \\
&\leq N \frac{N^{\lceil b_N a_N \rceil}}{\lceil b_N a_N \rceil!} (CN^{-\frac{4}{3}-2\sigma})^{\lceil b_N a_N \rceil} \\
&\leq N N^{-2\sigma\cdot\lceil b_N a_N \rceil} (CN^{-\frac{1}{3}})^{\lceil b_N a_N \rceil} \\
&\leq (CN)^{-\frac{1}{3}\lceil b_N a_N \rceil}.
\end{aligned}$$

Here we used $a_N b_N \to \infty$ in the last step, which also implies that this is a suitable estimate.

Next, we want to estimate how probable the existence of a set $\mathcal{S}$ satisfying the three conditions is. For this we define $c_N = \lfloor \frac{A-1}{2b_N a_N} \rfloor$ and estimate

$$\begin{aligned}
&\mathbb{P}\Bigg(\exists \mathcal{S} \subset \{1,\dots,N\}^2 \setminus \bigcup_{n=1}^{N} \{(n,n)\} : |\mathcal{S}| = c_N, \\
&\quad \forall (j,k) \in \mathcal{S} : x_j \in G(x_k)^C \cap M_{R_1,V_1}(x_0) \cap M_{R_2,V_2}(x_i), \\
&\quad (j_1,k_1),(j_2,k_2) \in \mathcal{S} \implies \{j_1,k_1\} \cap \{j_2,k_2\} = \emptyset\Bigg) \\
&\leq \binom{N^2}{c_N} \mathbb{P}\Bigg(\forall (j,k) \in \{(2,3),(4,5)\dots,(2c_N,2c_N+1)\} : \\
&\qquad\qquad x_j \in \left(G(x_k)^C \cap M_{R_1,V_1}(x_0) \cap M_{R_2,V_2}(x_i)\right)\Bigg).
\end{aligned}$$

Before we can move on, we need to rearrange the intersection a bit.

We start with the probability of $x_j \in M_{R_1,V_1}(x_0)$ which can be estimated by

$$\sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_j \in M_{R_1,V_1}(z)\right).$$

Next, we estimated the probability that under this condition, $x_j$ also lies in $M_{R_2,V_2}(x_i)$. By symmetry of the set, this is the same as the probability of $x_i \in M_{R_2,V_2}(x_j)$ which can be estimated by

$$\sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_i \in M_{R_2,V_2}(z)\right),$$

where we have taken care of the effect of the condition on $x_j$ by introducing the supremum.

With a similar trick we can estimate the last probability by $\sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_k \in G(z)^C\right)$ where we again used the symmetry and the supremum to deal with the conditional probabilities.

For simplicity and as all the particles are i.i.d., we use $x_2$ instead of $x_j$, $x_i$ and $x_k$ and obtain the following estimate:

$$\begin{aligned}
&\leq \binom{N^2}{c_N} \left( \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_2 \in G(z)^C\right) \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_2 \in M_{R_1,V_1}(z)\right) \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_2 \in M_{R_2,V_2}(z)\right)\right)^{c_N} \\
&\leq \frac{N^{2c_N}}{c_N!} \left( \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_2 \in G(z)^C\right) \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_2 \in M_{R_1,V_1}(z)\right) \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_2 \in M_{R_2,V_2}(z)\right)\right)^{c_N} \\
&\leq C^{c_N} \frac{N^{2c_N}}{c_N^{c_N}} \left( C(T) N^{-\frac{4}{3}-2\sigma} \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_0 \in M_{R_1,V_1}(z)\right) \sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_0 \in M_{R_2,V_2}(z)\right)\right)^{c_N}
\end{aligned}$$

Using that

$$\sup_{z\in\mathbb{R}^6} \mathbb{P}\left(x_2 \in M_{R_1,V_1}(z)\right) \leq C(T) R_1^2 \Big( \min(\max(R_1, V_1), 1)\Big)^4$$

as well as the corresponding estimate for $R_2, V_2$, we obtain

$$\begin{aligned}
&\leq C(T)^{c_N} \frac{N^{2c_N}}{c_N^{c_N}} \left( N^{-\frac{4}{3}-2\sigma} C(T) R_1^2 \Big( \min(\max(R_1, V_1), 1)\Big)^4 R_2^2 \Big( \min(\max(R_2, V_2), 1)\Big)^4 \right)^{c_N} \\
&\leq C(T)^{2c_N} \left( N^{(\frac{2}{3}-2\sigma)} \cdot \frac{R_1^2 \Big( \min(\max(R_1, V_1), 1)\Big)^4 R_2^2 \Big( \min(\max(R_2, V_2), 1)\Big)^4}{\left\lfloor \frac{\lceil N^{\frac{2}{3}} R_1^2 R_2^2 \min(\max(V_1,R_1),1)^4 \cdot \min(\max(V_2,R_2),1)^4 \rceil}{2b_N} \right\rfloor} \right)^{c_N} \\
&\leq C(T)^{2c_N} \left( N^{-2\sigma} b_N \right)^{c_N} \\
&\leq \left( C(T)^2 N^{-2\sigma} \right)^{c_N}.
\end{aligned}$$

As $c_N \geq \frac{a_N - 1}{2 a_N b_n} \geq \frac{1}{b_N} \to \infty$ for $N \to \infty$ and as $C(T) N^{-2\sigma} \ll 1$ for large enough $N$, this proves our claim. □

We need to prove the stronger version of our law of large numbers for nice enough particles, i.e.

**Lemma 6.27.** *Let $y \in \mathbb{R}^6$, $t \in [0,T], m \in \mathbb{N}$ and $j^N : \mathbb{R}^3 \to \mathbb{R}^m$ be given. If $|j^N| \leq |N^{\frac{1}{2}-\sigma} f^N|$, then*

$$g^N(z) = \int_0^t j^N({}^1\widetilde{y} - {}^1\widetilde{z}) \, \mathrm{d}s \mathbf{1}_{M^N_{N^{-\frac{1}{6}+\sigma}, N^{-\frac{1}{6}+\sigma}}(y)^C}(z)$$

*satisfies the conditions of Lemma 6.18, and hence for any $\gamma > 0$ we find $C_\gamma > 0$ such that*

$$\mathbb{P}\left( \left| \int_0^t \sum_{j=1}^N j^N({}^1\tilde{y} - {}^1\tilde{x}_j) \, \mathrm{d}s \mathbf{1}_{G^N(x_i)}(x_j) - \int g^N(z) k(z) \, \mathrm{d}z \right| \geq N^{-\frac{\sigma}{10}} \right) \leq C_\gamma N^{-\gamma}.$$

*Proof.* Note that our law of large numbers required the function to go to $\mathbb{R}$. This can be resolved as we can simply consider the, in this case $m$, coordinate functions $g_1^N, \ldots, g_m^N$. As all of them are bounded by $|g^N|$, we can estimate the function $|g^N|$ and thereby prove that all the coordinate functions satisfy the conditions for our law of large numbers 6.18. To estimate $|g^N|$, we use our partition 2.10 with $r_0 = v_0 = c(N)$ and $\varepsilon = \frac{\sigma}{6}$. By abuse of notation we

also write $r, v, R, V$ for the $r_i, v_i, R_i, V_i$ in the collision classes $M_i := M_{(r_i,R_i),(v_i,V_i)}$. With this notation and setting $\tilde{r} = \max(r, c(N))$, we can estimate:

$$\begin{aligned}
&\int_{M_i^N(y)} g^N(z)^2 k_0(z)\,\mathrm{d}^6 z \\
&\le \int_{M_i^N(y)} \left(\int_0^t |j^N({}^1\widetilde{y} - {}^1\widetilde{z})|\,\mathrm{d}s\right)^2 k_0(z)\,\mathrm{d}^6 z \\
&\le N^{1-2\sigma} \int_{M_i^N(y)} \left(\int |f({}^1\widetilde{y} - {}^1\widetilde{z})|\,\mathrm{d}s\right)^2 k_0(z)\,\mathrm{d}^6 z \\
&\le CN^{1-2\sigma} \left(\min\left(\frac{1}{\tilde{r}^2}, \frac{1}{\tilde{r}v}\right)\right)^2 \int_{M_i^N(y)} k_0(z)\,\mathrm{d}^6 z \\
&\le CN^{1-2\sigma} \min\left(\frac{1}{\tilde{r}^4}, \frac{1}{\tilde{r}^2 v^2}\right) \min(1, R^2, R^2V^4 + R^3\max(V^3, R^3)) \\
&\le CN^{1-2\sigma} \min\left(\frac{1}{\tilde{r}^2 v^2}, \frac{R^2}{\tilde{r}^2 v^2}, \frac{R^2V^4}{\tilde{r}^2\max(\tilde{r}, v)^2} + \frac{R^6}{\tilde{r}^4}\right),
\end{aligned}$$

where we first used 2.7 followed by 2.11. Using our partition, the minimum can be estimated as follows. The enumeration is the same as in the definition of the partition in 2.10.

(i) $\le \frac{c(N)^6}{c(N)^4} \le c(N)^2$

(ii) $\le \frac{c(N)^2 (N^\varepsilon v)^4}{c(N)^2 v^2} \le N^{2\delta}(N^\delta v)^2 \le N^{4\varepsilon}$

(iii) $\le \frac{c(N)^2}{c(N)^2 1^2} = 1$

(iv) $\le \frac{(N^\varepsilon r)^6}{r^4} \le N^{4\delta}(N^\varepsilon r)^2 \le N^{6\varepsilon}$

(v) $\le \frac{(N^\varepsilon r)^2 (N^\varepsilon v)^4}{r^2 v^2} + \frac{(N^\varepsilon r)^6}{r^4} \le N^{4\varepsilon}((N^\varepsilon v)^2 + (N^\varepsilon r)^2) \le 2N^{6\varepsilon}$

(vi) $\le \frac{(N^\varepsilon r)^2}{r^2 1^2} = N^{2\varepsilon}$

(vii) $\le \frac{1}{(N^{-\varepsilon})^4} = N^{4\varepsilon}$.

Hence, we find a constant $C_0$ such that

$$\max_{i\in I} \int_{M_i^N} g_N(z)^2 k_0(z)\,\mathrm{d}X \le C_0 N^{1-\sigma}.$$

Next, we estimate $|g^N|_\infty$ using 2.7.

$$\begin{aligned}
&\int_0^t |j^N(\widetilde{y} - \widetilde{z})| \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}s \\
&\le N^{\frac{1}{2}-\sigma} \int_0^t |f^N(\widetilde{y} - \widetilde{z})| \mathbf{1}_{G^N(y)}(z)\,\mathrm{d}s \\
&\le C(T) N^{\frac{1}{2}-\sigma} \min\left(\frac{1}{|{}^1\widetilde{z}(t_{min}) - {}^1\widetilde{y}(t_{min})|^2}, \frac{1}{c(N)|{}^2\widetilde{z}(t_{min}) - {}^2\widetilde{y}(t_{min})|}\right) \mathbf{1}_{G^N(y)}(z) \\
&\le C(T) N^{\frac{1}{2}-\sigma} \max((N^{\frac{1}{6}-\sigma})^2, c(N)^{-1} N^{\frac{1}{6}-\sigma}) \le C(T) N^{\frac{1}{2}-\sigma} N^{\frac{1}{2}+\frac{\sigma}{16}} \le C(T) N^{1-\frac{15}{16}\sigma}.
\end{aligned}$$

Hence, $|g|_\infty \leq C(T)N^{1-\frac{15}{16}\sigma}$ and we can apply 6.18 to each $g_1^N, \dots, g_m^N$ to get a constant $\hat{C}_\gamma$ such that for all $l = 1, \dots, m$

$$\mathbb{P}\left(\left|\int_0^t \sum_{j=1}^N j_l^N({}^1\tilde{y} - {}^1\tilde{x}_j)\,\mathrm{d}s\mathbf{1}_{G^N(x_i)}(x_j) - \int_{\mathbb{R}^6} g_l^N(z)k(z)\,\mathrm{d}z\right| \geq \hat{C}_\gamma N^{-\frac{3}{16}\sigma}\right) \leq N^{-\gamma}.$$

By triangle inequality and as $N^{-\frac{3}{16}\sigma} \leq N^{-\frac{1}{6}\sigma}$, we obtain that for large enough $N$

$$\mathbb{P}\left(\left|\int_0^t \frac{1}{N}\sum_{j=1}^N j^N({}^1\tilde{y} - {}^1\tilde{x}_j)\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}s - \int_{\mathbb{R}^6} g^N(z)k(z)\,\mathrm{d}^6z\right| \geq N^{-\frac{1}{6}\sigma}\right) \leq N^{-\gamma}.$$

As this result holds only for large enough $N$, we have to choose a constant $C_\gamma$ such that

$$\mathbb{P}\left(\left|\int_0^t \frac{1}{N}\sum_{j=1}^N j^N({}^1\tilde{y} - {}^1\tilde{x}_j)\mathbf{1}_{G^N(x_i)}(x_j)\,\mathrm{d}s - \int_{\mathbb{R}^6} g^N(z)k(z)\,\mathrm{d}^6z\right| \geq N^{-\frac{1}{6}\sigma}\right) \leq C_\gamma N^{-\gamma}.$$

holds for all $N \in \mathbb{N}$. □

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